On the exponential Diophantine equation $x^{x^n} y^{y^m} z^{z^n} = w^w$

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Abstract: The exponential Diophantine equation in four variables given by $x^{x^n} y^{y^m} z^{z^n} = w^w \ (m,n \succ 0)$ **is considered and analyzed for finding its non-zero integer solutions for different choices of m and n. A few numerical illustrations are presented for the values of m and n given by** $(m,n) = (1,1),(2,2),(2,1),(1,2),(\frac{1}{2},1)$ **.A few relations between solutions and the special numbers are also presented.**

***Keywords*: Exponential Diophantine equation, integral solutions.**

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Notations:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

PR_n - Pronic number of rank n

OH_n - Octahedral number of rank n

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

I. INTRODUCTION

The exponential Diophantine equation $a^x + b^y = c^z$ in positive integers x,y,z has been studied by number of authors [1-5].In [6-11] the existence and the process of determining some positive integer solutions to a few special cases of an exponential Diophantine equation are studied. In this paper an interesting form of an exponential Diophantine equation is considered ,that is, we present the non-zero distinct integral solutions to the exponential Diophantine equation $x^{x^n} y^{y^m} z^{z^n} = w^w$.

II METHOD OF ANALYSIS

The exponential Diophantine equation with four variables to be solved for its non-zero distinct integral solution is

$$x^{x^n} y^{y^m} z^{z^n} = w^w \qquad (1)$$

where m, n are positive and x, y, z and w are unknowns.

Taking

$$x = a^{\frac{1}{n}} w^{\frac{1}{n}}, y = b^{\frac{1}{m}} w^{\frac{1}{m}}, z = c^{\frac{1}{n}} w^{\frac{1}{n}} \qquad (2)$$

In (1) and raising to the power $\frac{1}{w}$, it gives

$$w = a^{\frac{\frac{a}{n}}{1-\frac{a}{n}-\frac{b}{m}-\frac{c}{n}}} b^{\frac{\frac{b}{m}}{1-\frac{a}{n}-\frac{b}{m}-\frac{c}{n}}} c^{\frac{\frac{c}{n}}{1-\frac{a}{n}-\frac{b}{m}-\frac{c}{n}}} \qquad (3)$$

Setting

$$\frac{\frac{a}{n}}{1-\frac{a}{n}-\frac{b}{m}-\frac{c}{n}} = -n_1$$

$$\frac{\frac{b}{m}}{1-\frac{a}{n}-\frac{b}{m}-\frac{c}{n}} = -n_2$$

$$\frac{\frac{c}{n}}{1-\frac{a}{n}-\frac{b}{m}-\frac{c}{n}} = -n_3$$

And solving for a, b and c, we get

$$a = \frac{nn_1}{n_1+n_2+n_3-1}$$
$$b = \frac{mn_2}{n_1+n_2+n_3-1} \qquad (4)$$
$$c = \frac{nn_3}{n_1+n_2+n_3-1}$$

in which n_1, n_2, n_3 are natural numbers.

Substituting (4) in (2) and (3), the corresponding solutions of (1) are given by,

$$x=\left(\frac{n_1+n_2+n_3-1}{nn_1}\right)^{\frac{n_1-1}{n}}\left(\frac{n_1+n_2+n_3-1}{mn_2}\right)^{\frac{n_2}{n}}\left(\frac{n_1+n_2+n_3-1}{nn_3}\right)^{\frac{n_3}{n}}$$

$$y=\left(\frac{n_1+n_2+n_3-1}{nn_1}\right)^{\frac{n_1}{m}}\left(\frac{n_1+n_2+n_3-1}{mn_2}\right)^{\frac{n_2-1}{m}}\left(\frac{n_1+n_2+n_3-1}{nn_3}\right)^{\frac{n_3}{m}} \quad (5)$$

$$z=\left(\frac{n_1+n_2+n_3-1}{nn_1}\right)^{\frac{n_1}{n}}\left(\frac{n_1+n_2+n_3-1}{mn_2}\right)^{\frac{n_2}{n}}\left(\frac{n_1+n_2+n_3-1}{nn_3}\right)^{\frac{n_3-1}{n}}$$

$$w=\left(\frac{n_1+n_2+n_3-1}{nn_1}\right)^{n_1}\left(\frac{n_1+n_2+n_3-1}{mn_2}\right)^{n_2}\left(\frac{n_1+n_2+n_3-1}{nn_3}\right)^{n_3}$$

The numbers n_1, n_2 and n_3 are chosen in such a way that the solutions (5) are integers. We present below infinitely many integer solutions to (1) by considering $n_1 = n_2$ and $n_3 = 1.$

A.Choice I Let m=1,n=1.The corresponding non-zero integral solutions to (1) are given by

$$x = 2^{2n_1} n_1, y = 2^{2n_1} n_1, z = 2^{2n_1}, w = 2^{2n_1+1} n_1 \quad (6)$$

TABLE: 1

n_1	x	y	z	W
1	2^2	2^2	2^2	2^3
2	2^5	2^5	2^4	2^6
3	$2^6 3$	$2^6 3$	2^6	$2^7 3$
4	2^{10}	2^{10}	2^8	2^{11}

Properties:

1. $Z^2 + 2Z - Ky_{2n_1} = 1$

2. $(y^2 + w)z - yKy_{2n_1} - x = 0$

3. $6[x^2 + 2y - w]$ is a nasty number.

B Choice :II Take $m = 2, n = 2, n_1 = \alpha^2$,the non-zero integral solutions to (1) are obtained as

$x = \alpha, y = \alpha, z = 1, w = \alpha^2$

Table: 2

α	x	y	z	W
2	2	2	1	4
3	3	3	1	9
4	4	4	1	16
5	5	5	1	25

Properties:

1. $xy = w$

2. $w + kxy - 8t_{3,k} - 8P_k^5 \equiv 1(\bmod k)$

3. $k^2 xy - w - 96Pt_k + 144P_k^3 - 48t_{3,k} = 0$

C Choice III Take $m = 2, n = 1, n_1 = (2\alpha + 1)^2$,the non-zero distinct integral solutions to (1) are given by,

$x = 2^{(2\alpha+1)^2}(2\alpha+1)^2, y = 2^{2\alpha^2+2\alpha+1}(2\alpha+1), z = 2^{(2\alpha+1)^2}, w = 2^{(2\alpha+1)^2+1}(2\alpha+1)^2$

Table: 3

α	x	y	z	W
1	$2^9 9$	$2^5 3$	2^9	$2^{10} 9$
2	$2^{25} 25$	$2^{13} 5$	2^{25}	$2^{26} 25$
3	$2^{49} 49$	$2^{25} 7$	2^{49}	$2^{50} 49$
4	$2^{81} 81$	$2^{41} 9$	2^{81}	$2^{82} 81$

Properties:

1. $w = 2x$

2. $2z[x+w]$ is a nasty number.

3. $x+z+w-2^{(2\alpha+1)^2+1}S_{\alpha} \equiv 0 \pmod 2$

D Choice IV Choosing $m=1, n=2, n_1=4k^2$,the non-zero distinct integral solutions to (1) are given by,

$$x=2^{2k^2+1}k, y=2^{4k^2+1}k^2, z=2^{2k^2}, w=2^{4k^2+2}k^2$$

Table: 4

k	x	y	z	W
1	2^3	2^5	2^2	2^6
2	2^{10}	2^{19}	2^8	2^{20}
3	$2^{19}3$	$2^{37}3^2$	2^{18}	$2^{38}9$
4	2^{35}	2^{69}	2^{32}	2^{70}

Properties:

1. $2y=x^2$

2. $x=2kz$

3. x^2+y-w is a nasty number.

4. $y+w-2^{4k^2}(S_k-1)=0$

E Choice V Choosing $m=\frac{1}{2}, n=2, n_1=\alpha^2$,the non-zero distinct integral solutions to (1) are given by,

$$x=2^{\alpha^2}\alpha, y=2^{4(\alpha^2-1)}\alpha^4, z=2^{\alpha^2}, w=2^{2\alpha^2}\alpha^2$$

Table: 5

α	x	y	z	W
2	2^5	2^{16}	2^4	2^{10}
3	$2^9 3$	$2^{32} 3^4$	2^9	$2^{18} 9$
4	2^{18}	2^{68}	2^{16}	2^{36}
5	$2^{25} 5$	$2^{96} 5^4$	2^{25}	$2^{50} 25$

Properties:

1. $16y = x^4 = w^2$

2. When $\alpha = 2^{2k}$,

$$\frac{x}{z} = (j_{2k} - 1)$$

3. $16y = x^2 zw$

4. Each of the following expressions is a nasty number.

i) $96y$

ii) $\dfrac{96y}{x^2 z^2}$

III CONCLUSION

One may search for the integral solutions of the exponential Diophantine equation under consideration for other choices of n_1, n_2 and n_3 and search for corresponding properties.

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