

# Some Studies on Semirings

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**Abstract:** In this paper, we study the structures of Boolean Semirings and multiplicatively semirings. We proved that, Let  $S$  be a multiplicatively subidempotent semiring which contains multiplicative identity 1 which is also additive identity. Then (i)  $S$  is viterbi semiring. (ii)  $(S, +)$  is commutative. We framed an example for this theorem by considering three element set.

**Keywords:** Absorbing, Positive Rational Domain (PRD), Viterbi semiring, Weak commutative, Zeroid.

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1. **INTRODUCTION:** The theory of rings and the theory of semigroups have considerable impact on the developments of theory of semirings. During the last three decades, there is considerable impact of semigroup theory and semiring theory. However, in semirings it is possible to derive the additive structures from their special multiplicative structures. In this paper, we have two sections. Section one deals with properties of multiplicatively subidempotent semirings. S.Gosh studied on the class of idempotent semiring. He proved that an idempotent commutative semiring  $S$  is distributive lattice if and only if it satisfies the absorption equality  $a + ab = a$  for all  $a, b$  in  $S$ . Coming to section two we observe the Boolean Semiring and its associations with other semirings. A triple  $(S, +, \cdot)$  is called a semiring if  $(S, +)$  is a semigroup;  $(S, \cdot)$  is semigroup;  $a(b + c) = ab + ac$  and  $(b + c)a = ba + ca$  for every  $a, b, c$  in  $S$ .

2. **MULTIPLICATIVELY SUB IDEMPOTENT SEMIRING:**

**Definition 2.1:** A semiring is said to be multiplicatively sub idempotent semiring. If it satisfies the following conditions.

- (1.)  $(S, +)$  is semigroup
- (2.)  $(S, \cdot)$  is semigroup

$$(3.) \quad a(b + c) = ab + ac, (b + c) a = ba + ca \text{ and}$$

$$(4.) \quad a + a^2 = a \text{ for all } a, b \text{ in } S$$

**Definition 2.2:** A semiring is said to be b – lattice if  $S$  is an idempotent semiring and  $(S, +)$  is commutative.

**Definition 2.3:** Zeroid of a semiring  $(S, +, \cdot)$  is the set of all  $x$  in  $S$  such that  $a + b = b$  or  $b + a = b$  for some  $b$  in  $S$  we may also term this as the zeroid of  $(S, +)$ .

**Theorem 2.3:** Let  $S$  be a multiplicatively sub idempotent semiring. If  $S$  contains the multiplicative identity which is also additive identity, then (i)  $S$  is b-lattice (ii)  $(S, +)$  is zeroid if  $(S, \cdot)$  left cancellative.

**Proof: (i)** Suppose  $S$  is a multiplicatively sub idempotent semiring, i.e,  $a + a^2 = a$ , for all  $a$  in  $S$

$$a + a^2 = a \Rightarrow a(e + a) = a \Rightarrow a^2 = a \text{ for all } a \text{ in } S \dots\dots(1)$$

$$\text{Substitute (1) in } a = a + a^2 \text{ which implies } a = a + a \dots\dots(2)$$

From (1) and (2) we can say that  $S$  is an Idempotent semiring

$$\text{Again, assume } a + a^2 = a$$

$$\Rightarrow a + a + a^2 = a + a \Rightarrow a(e + a) + a = a$$

$$\Rightarrow a^2 + a = a \dots\dots(3)$$

$$\text{From above equation we can say that, } a^2 + a = a + a^2$$

$(S, +)$  is commutative

Hence  $S$  is b – lattice

**(ii)** Consider  $a + a^2 = a$

$$\text{Adding } ab \text{ on both sides, we obtain } ab + a^2 = ab \Rightarrow a(b + a) = ab$$

$$\text{Using left cancellative, we get } b + a = b$$

Therefore  $(S, +)$  is zeroid

Hence the theorem

**Example:**  $S = \{e, a, b\}$  the following is the example for the above theorem 2.3.

+	e	a	b
e	e	a	b

.	e	a	b
e	e	a	b

a	a	a	a
b	b	a	b

a	a	a	a
b	b	a	b

**Definition 2.4:** A semiring  $(S, +, \cdot)$  is said to be a Positive Rational Domain (PRD) if and only if  $(S, \cdot)$  is an abelian group.

**Theorem 2.5:** Let  $S$  be a multiplicatively sub idempotent semiring and PRD. Then  $b + ab = b$  for all  $a, b$  in  $S$

**Proof:** Consider  $a + a^2 = a \Rightarrow a(1 + a) = a.1$

$\Rightarrow (1 + a) = 1 \Rightarrow (1 + a)b = 1.b \Rightarrow b + ab = b$

Therefore,  $b + ab = b$  for all  $a, b$  in  $S$

**Example:**  $S = \{ 1, a \}$  the following is the example for the above theorem 2.5.

+	1	a
1	1	1
a	1	a

.	1	a
1	1	a
a	a	a

**Theorem 2.6:** Let  $S$  be a semiring in which 1 is multiplicative identity. If  $(S, +)$  is zeroid then  $a^n + ab = ab$  for all  $n \geq 1$

**Proof:** let  $a \in Z$  where  $Z$  is a zeroid of  $(S, +, \cdot)$  then there exists  $b \in S$  such that  $a + b = b$  or  $b + a = b$  ----I

Since 1 contains multiplicative identity

Let  $1 \in Z$  where  $Z$  is a zeroid of  $S$  then there exists  $b \in S$  such that  $1 + b = b$  or  $b + 1 = b$  -----(I)

We take  $b + 1 = b$

$\Rightarrow a(b + 1) = ab \Rightarrow ab + a = ab$  -----(II)

$\Rightarrow a(a + b) + a = ab$

$\Rightarrow a^2 + ab + a = ab$

Substitute (II) in above equation  $\Rightarrow a^2 + ab = ab$

Continuing like this, we get  $a^n + ab = ab$

### 3. **BOOLEAN SEMIRING:**

**Definition 3.1:** A Semiring  $S$  is said to be a Boolean Semiring provided  $a^2 = a$  for every  $a \in S$ . (or) A semiring  $S$  is said to be a Boolean Semiring provided every element of  $S$  is an idempotent.

**Definition 3.2:** A Semiring  $S$  is said to be an ordering on boolean semiring provided a relation " $\leq$ " on  $S$  by  $a \leq b$  if  $a = ba$ .

**Theorem 3.3 :** If  $S$  is a weak commutative boolean Semiring then  $(S, \leq)$  is a partially ordered set.

**Proof:** For any  $a \in S$ , we have by the definition of Boolean Semiring  $a^2 = a, \Rightarrow a.a = a$

$\Rightarrow a \leq a$ , thus " $\leq$ " is reflexive

Suppose  $a \leq b$  and  $b \leq a$  then  $a = ba$  and  $b = ab$

Now  $a = ba \Rightarrow ab = bab (\because b = ab)$

$ab = b(ba)$  (since  $S$  is weak commutative)

$\Rightarrow ab = ba$

$\Rightarrow a = b$  thus " $\leq$ " is anti symmetric.

Suppose  $a \leq b$  and  $b \leq c$  then  $a = ba$  and  $b = cb$

Now  $a = ba = cba = ca \Rightarrow a = ca$  which implies  $a \leq c$ , thus ' $\leq$ ' is transitive.

Therefore,  $(S, \leq)$  is partially ordered set.

**Note:** If  $S$  is a boolean semiring defined above is called the usual ordering. Hence further we consider this ordering on boolean semiring.

**Theorem 3.4:** Let  $S$  be a boolean Semiring and  $a, b \in S$  and weak commutative  $a \leq b$  then  $ab = ba$ .

**Proof:** Let  $S$  be a boolean semiring and  $a, b \in S$ .

$a \leq b \Rightarrow a = ba$

Now  $a = ba$

Multiply  $b$  on both sides,  $ab = bab$

Since weak commutative,  $ab = bba = ba$

Therefore  $ab = ba$

**Theorem 3.5:** A boolean Semiring  $S$  and  $ab = 0$  then  $ba = 0$  and  $asb = 0 \quad \forall s \in S$ .

**Proof:** consider  $S$  is a boolean semiring.

Every boolean Semiring has no non-zero nilpotent elements. i.e.,  $a^n = 0 \Rightarrow a = 0$

$$\text{Let } (ba)^2 = (ba)(ba) = b(ab)a = b(0)a = b \cdot 0 = 0$$

$$(ba)^2 = 0 \Rightarrow ba = 0$$

Let  $s \in S$  and consider  $asb$

$$(asb)^2 = (asb)(asb) = (as)(ba)(sb) = (as)0(sb) = 0$$

$$(asb)^2 = 0 \Rightarrow asb = 0$$

Thus  $ab = 0 \Rightarrow asb = 0, \quad \forall s \in S$ .

**Definition 3.6:** Zeroid of a semiring  $(S, +, \cdot)$  is the set of all  $x$  in  $S$  such that  $a + b = b$  or  $b + a = b$  for some  $b$  in  $S$  we may also term this as the zeroid of  $(S, +)$ .

**Theorem 3.7:** Let  $(S, +, \cdot)$  be a zeroid semiring with multiplicative identity 1 and cancellative then  $S$  is boolean Semiring and  $(a + b)^n = (a + b)$

**Proof:** Assume that  $S$  is a zeroid Semiring i.e.,  $a + b = b$  or  $b + a = b$

Since  $S$  contain multiplicative identity 1 which implies  $a + 1 = 1$  and  $a + a = a$

$$\text{Suppose } a = a \cdot 1 \Rightarrow a + a = a(a + 1) \Rightarrow a + a = a^2 + a$$

$$\text{Using cancellative property, we have } \Rightarrow a^2 = a$$

we say that  $S$  is a boolean semiring.

$$\begin{aligned} \text{Consider } (a + b)^2 &= (a + b)(a + b) = a(a + b) + b(a + b) \\ &= a^2 + ab + ba + b^2 = a \cdot 1 + ab + ba + b \cdot 1 \\ &= a \cdot (1 + b) + b(a + 1) = a \cdot 1 + b \cdot 1 \end{aligned}$$

$$\therefore (a + b)^2 = a + b$$

$$\begin{aligned} \text{Again } (a + b)^3 &= (a + b)^2 \cdot (a + b) \\ &= (a + b)(a + b) = (a + b) \end{aligned}$$

Continuing like this  $(a + b)^n = (a + b)$

Hence the theorem.

**Definition 3.8:** Suppose  $p$  is prime number. A Semiring  $S$  is called  $p$ -semiring provided that for all  $a \in S$ ;  $a^p = a$  and  $pa = 0$

**Theorem 3.9:** Let  $S$  be a ordering on boolean semiring. If  $1$  is multiplicative identity and it has the least element then  $0$  is the least element.

**Proof:** Suppose  $S$  has the least element  $l$ , then  $l \leq a, \forall a \in S$

$$\Rightarrow l = a.l, \forall a \in S. \text{ In particular, } l \leq 0, \text{ (since } l = 0.l \Rightarrow l = 0)$$

$$\Rightarrow 0 \leq a, \forall a \in S.$$

Hence  $0$  is least element

**Theorem 3.10:** Let  $S$  be an ordering on boolean semiring contains  $1$ . Suppose  $S$  has the greatest element then  $1$  is the multiplicative identity.

**Proof:** Suppose  $S$  has the greatest element say  $1$ .

$$\text{then } a \leq 1, \forall a \in S \Rightarrow a = 1.a$$

$$\therefore a . 1 = 1 . a = a, \forall a \in S$$

Thus ' $1$ ' is the identity element of  $S$

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