On the Oscillation of Third Order Linear Neutral Delay Differential Equations

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ABSTRACT

Sufficient conditions for oscillation of solutions of Third order linear neutral delay differential equations of the type

$$\frac{d}{dt}\left(r(t)\,\frac{d^2}{dt^2}\big(y(t)+p(t)y(t-\tau)\big)\right)+f(t)y(t-\sigma)=0$$

are obtained, where

$$p(t), f(t) \in C([t_0\,,\infty\,), R) \text{ and } f(t) \geq 0, r(t) \in C'\big([t_0,\infty),(0,\infty)\big), \;\; r'(t) \geq 0,$$

KEY WORDS: Oscillation, Third Order, Neutral Differential Equation.

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1. INTRODUCTION

In this paper we consider the linear neutral delay differential Equation

$$\frac{d}{dt}\left(r(t)\frac{d^2}{dt^2}\big(y(t)+p(t)y(t-\tau)\big)\right)+f(t)(y(t-\;\sigma\;)=0 \qquad (1)$$

where

$p(t), f(t) \in C([t_0,\infty), R)$, and $f(t) \geq 0$,

$r(t) \in C'([t_0,\infty),(0,\infty)), \quad r'(t) \geq 0 \; and \; \int_{t_0}^{\infty} \frac{1}{r(s)} ds = \infty$,

When $p(t) \equiv 0$ the above equation reduces to the third order delay differential equation

$$\frac{d}{dt}\left(r(t)\frac{d^2}{dt^2}y(t)\right) + f(t)(y(t-\sigma)) = 0 \qquad (2)$$

The study of behavior of solutions of differential equations has been a subject of interest for several researches. We mention the works of [3,8,10].

Oscillatory behaviour of delay differential equations is extensively studied by [1,2,4,5,6,7,9,15] The authors studied the nonoscillatory behavior of solutions of certain first and second order neutral delay differential equations in [12,13,14], however we find less work concerning the oscillation criteria of third order neutral delay differential equations. We investigate the conditions under which the solutions of equation (1) are oscillatory.

By a solution of equation (1) we mean a function $y(t) \in C\left([T_y,\infty)\right)$, where $T_y \geq t_o$ which satisfies (1) on $[T_y, \infty)$. We consider only those solutions of $y(t)$ of (1) which satisfy $\sup\ \{|\ y(t)\ |: t \geq \mathrm{T}\} > 0$ for all $T \geq T_y$ and assume that (1) possesses such solutions.

A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions oscillate. Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

2. MAIN RESULTS

We need the following in our discussion.

(H_1): $r(t) \in C'([t_0,\infty),(0,\infty))$,

$r'(t) \geq 0$ for $t \geq t_0$.

(H_2): $p(t) \in (C[t_0,\infty), R)$

(H_3): $f(t) \in C([t_0,\infty),[0,\infty))$

(H_4): *There exists a positive decreasing function* $q(t)$ *such that* $f(t) \geq q(t)$ for $t \in [t_0,\infty)$.

(H_5) $\int_{t_0}^{\infty} \int_{v}^{\infty} [\frac{1}{r(u)} \int_{u}^{\infty} f(s)\, ds] du dv = \infty$,

(H_6) $\lim_{t\to\infty} sup \int_{t_1}^{t} [\, 2sq(s)(1-p(s-\sigma))]KM(s-\sigma)^2 \; - r(s)\,]ds = \infty$, for some $K, M \in (0,1)$ for sufficiently large $t_1 \geq t_0$,

(H_7): $\lim_{t\to\infty} sup \int_{t_2}^{t} \{[2q(s)(1-p(s-\sigma))]KM\frac{(s-\sigma)^2}{s} - \frac{1}{R(S)\, r(s)}\}ds = \infty$ holds for some $K, M \in (0,1)$ and for sufficiently large $t_2 \geq t_0$,

where , $$R(t) = \int_t^{\infty} \frac{1}{r(s)} ds.$$

We set

$$z(t) = y(t) + p(t)y(t-\tau) \tag{3}$$

Then we have the following:

Lemma 2.1. Suppose $x(t)$ is twice continuously differentiable real valued function on $[t_0, \infty)$ such $x(t) > 0$, $x'(t) \geq 0$, and $x''(t) \leq 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$ .Then for each k with $0 < k < 1$,there exists $t_2 \geq t_1$ such that

$$\frac{x(t-\sigma)}{x(t)} \geq K\frac{(t-\sigma)}{t} \qquad t \geq t_2 . \tag{4}$$

Proof: From the Lagrange's Mean Value Theorem we have for $t \geq t_1$,

$$\frac{x(t)-x(t-\sigma)}{t-(t-\sigma)} = x'(\xi), \text{for some } \xi \text{ such that } t-\sigma < \xi < t .$$

$$x''(t) \leq 0 \quad => \quad x'(t) \text{ is non increasing}$$

$$=> x'(\xi) < x'(t-\sigma) ,$$

and hence

$$x(t) - x(t-\sigma) \leq x'(t-\sigma)(t-(t-\sigma)) \tag{5}$$

ie $$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{x'(t-\sigma)}{x(t-\sigma)}(t-(t-\sigma)) \tag{6}$$

Applying Lagrange's Mean Value Theorem once again for $x(t)$ on $[t_1 , t-\sigma]$ for $t \geq t_1 + \sigma$

$$\frac{x(t-\sigma)-x(t_1)}{t-\sigma-t_1} = x'(\eta) \text{ for some } \eta \text{ such that } t_1 < \eta < t-\sigma .$$

$$=> x'(\eta) > x'(t-\sigma)$$

$$x(t-\sigma) \geq x'(t-\sigma)(t-\sigma-t_1)$$

$$\frac{x(t-\sigma)}{x'(t-\sigma)} \geq (t-\sigma) - t_1$$

Given $K \in (0,1)$. Then we can find $t_2 \geq t_1 + \sigma$ such that,

$$\frac{x(t-\sigma)}{x'(t-\sigma)} \geq K(t-\sigma) \text{ for } t \geq t_2 \tag{7}$$

From (5) and for all $t \geq t_2$, we have ie $$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{x'(t-\sigma)}{x(t-\sigma)}(t-(t-\sigma))$$

$$\leq 1 + \frac{1}{K(t-\sigma)}(t-(t-\sigma)), \text{ in view of (6)}$$

or

$$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{t}{K(t-\sigma)} - \frac{(t-\sigma)}{K(t-\sigma)},$$

$$= \left(1-\frac{1}{K}\right) + \frac{t}{K(t-\sigma)}$$

$$\leq \frac{t}{K(t-\sigma)}, \text{ since } 0 < K < 1.$$

Hence

$$\frac{x(t-\sigma)}{x(t)} \geq \frac{K(t-\sigma)}{t} \qquad \text{for } t \geq t_2 \tag{8}$$

Lemma 2.2. Let $z(t) \in C^3([t_0,\infty),R)$ and suppose that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ on $[t_1,\infty)$ for some $t_1 \geq t_0$. Then there exists $t_3 \geq t_1$ such that

$$z(t) \geq \frac{1}{2}Mt\, z'(t), t \geq T_2$$

for each M ; $0 < M < 1$.

We define

$$Q(t) = \left\{\frac{KM}{2}\right\}\left\{\frac{(t-\sigma)^2}{t}\right\}(1-p(t-\sigma))q(t),\ t \geq t_0 \text{ for each K and M}$$

with $0 < K < 1$, and $0 < M < 1$.

Proof : We define a function $H(t)$ for $t \geq t_2 \geq t_1$ as

$$H(t) = (t-t_2)z(t) - \frac{M(t-t_2)^2}{2} z'(t). \tag{9}$$

and $$H'(t) = (t-t_2)z'(t) + z(t) - \frac{M}{2}\{2(t-t_2)z'(t) + (t-t_2)^2 z''(t)\}$$

$$H'(t) = z(t) + (t-t_2)(1-M)z'(t) - \frac{M(t-t_2)^2}{2} z''(t) \tag{10}$$

Now we have to prove that $H'(t) > 0$.

By Taylors Theorem,

we have

$$z(t) = z(t_2) + (t-t_2)z'(t_2) + \frac{(t-t_2)^2}{2} z''(t_2)) + \frac{(t-t_2)^3}{6} z'''(t_2+\theta(t-t_2))$$

$$z(t) \geq z(t_2) + (t-t_2)z'(t_2) + \frac{(t-t_2)^2}{2} z''(t_2)$$

$$\geq z(t_2) + (t - t_2)z'(t_2) + \frac{(t - t_2)^2}{2} z''(t)$$

From (10)

$$H'(t) = z(t_2) + (t - t_2)z'(t_2) + \frac{(t - t_2)^2}{2} z''(t)$$
$$+(t - t_2)(1 - M)z'(t) - \frac{M(t - t_2)^2}{2} z''(t)$$

Hence $H'(t) > 0$,

$H(t_2) = 0$. we have $H(t) > H(t_2) = 0$ for every $t \geq t_0$.

Then from (9)

$$(t - t_2)z(t) - \frac{M(t - t_2)^2}{2} z'(t)$$

ie $$z(t) > \frac{M(t - t_2)^2}{2} z'(t)$$

ie $$\frac{z(t)}{z'(t)} \geq \frac{Mt}{2} - \frac{M t_2}{2}$$

or $$\frac{z(t)}{z'(t)} \geq \frac{Mt}{2} \quad \text{for } t \geq t_2 . \tag{11}$$

Hence the lemma.

Theorem 2.3. *Assume that, $(H_1) - (H_7)$ hold, then equation (1) is oscillatory*

Proof. Suppose, if possible that the equation (1) has a nonoscillatory solution. Without loss of generality supposes that $y(t)$ is a positive solution of equation (1). Then there exists three possible cases for $z(t)$.

(i) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$; $z'''(t) \leq 0$,

(ii) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$; $z'''(t) \leq 0$,

(iii) $z(t) > 0$, $z'(t) > 0$, $z''(t) < 0$; $(r(t)(z''(t)))' \leq 0$

for $t \geq t_l \geq t_0$.

Case I: $\boldsymbol{z(t) > 0, z'(t) < 0, z''(t) > 0; z'''(t) \leq 0}$,

Since $z(t) > 0$ and$z'(t) < 0$, then there exists finite limit $\lim_{t\to\infty} z(t) = k$

We shall prove that $k = 0$.

Assume that $k > 0$. Then for any $\varepsilon > 0$, we have $k + \varepsilon > z(t) > k$, Let $0 < \varepsilon < \frac{k(1-p)}{p}$.

From (3), we have

$$\begin{aligned} y(t) &= z(t) - p(t)y(t-\tau) \\ &> k - p(t)z(t-\tau) \\ &> k - p(k+\varepsilon) \\ &= m(k+\varepsilon) \\ &> mz(t). \end{aligned} \tag{12}$$

where $m = \frac{k-p(k+\epsilon)}{k+\epsilon}$

Now from the equation (1) we have

$$\frac{d}{dt}\left(r(t)\frac{d^2}{dt^2}\big(y(t)+p(t)y(t-\tau)\big)\right) = -\,f(t)y(t-\sigma) \tag{13}$$

ie.
$$\left(r(t)\left(z''(t)\right)\right)' \le -f(t)mz(t-\sigma) \tag{14}$$

$$-\left(r(t)\left(z''(t)\right)\right)' \ge -[-f(t)mz(t-\sigma)]$$

$$-\left(r(t)\left(z''(t)\right)\right)' \ge [f(t)mz(t-\sigma)]$$

Integrating the above inequality from t to ∞ we get

$$-\int_t^\infty \left(r(t)\left(z''(t)\right)\right)' dt \ge m\int_t^\infty f(s)z(s-\sigma)ds$$

$$-[r(\infty)(z''(\infty)) - r(t)\left(z''(t)\right)] \ge m\int_t^\infty f(s)z(s-\sigma)ds$$

$$r(t)\left(z''(t)\right) \ge m\int_t^\infty f(s)z(s-\sigma)ds$$

Using the fact that, $z(t-\sigma) \ge k$,

we obtain

$$r(t)z''(t) \ge mk\,[\int_t^\infty f(s)ds].$$

$$z''(t) \ge mk[\frac{1}{r(t)}\int_t^\infty f(s)ds]. \tag{15}$$

Integrating between t to ∞ we have

$$\int_t^\infty z''(t)\,dt \ge mk\int_t^\infty [\frac{1}{r(u)}\int_u^\infty f(s)ds]\,du.$$

$$z'(\infty) - z'(t) \ge mk\int_t^\infty [\frac{1}{r(u)}\int_u^\infty f(s)ds]\,du.$$

$$-z'(t) \geq mk \int_t^{\infty} [\frac{1}{r(u)} \int_u^{\infty} f(s)ds]\, du . \qquad (16)$$

Integrating from t_1 to ∞ , we get

$$-\int_{t_1}^{\infty} z'(t)dt \geq mk \int_{t_1}^{\infty} \int_v^{\infty} [\frac{1}{r(u)} \int_u^{\infty} f(s)ds]\, dudv .$$

$$-[z(\infty) - z(t_1)] \geq mk \int_{t_1}^{\infty} \int_v^{\infty} [\frac{1}{r(u)} \int_u^{\infty} f(s)ds]\, dudv .$$

$$-z(\infty) + z(t_1) \geq mk \int_{t_1}^{\infty} \int_v^{\infty} [\frac{1}{r(u)} \int_u^{\infty} f(s)ds]\, dudv .$$

$$z(t_1) \geq mk \int_{t_1}^{\infty} \int_v^{\infty} [\frac{1}{r(u)} \int_u^{\infty} f(s)ds] dudv . \qquad (17)$$

This contradicts (H_5) .

The inequality $0 \leq y(t) \leq z(t)$

implies $\lim_{t\to\infty} y(t) = 0$

Case II: $\boldsymbol{z(t) > 0, z'(t) > 0, z''(t) > 0; z'''(t) \leq 0}$,

If we set

$$z(t) = y(t) + p(t)y(t-\tau)$$

we obtain, further,

$$y(t) = z(t) - p(t)y(t-\tau)$$

$$\begin{aligned} y(t-\sigma) &= z(t-\sigma) - p(t-\sigma)y(t-\sigma-\tau) \\ &\geq z(t-\sigma) - p(t-\sigma)y(t-\sigma) \\ &\geq [1 - p(t-\sigma)]z(t-\sigma) \qquad (18) \end{aligned}$$

From (1) we have

$$\left[r(t)[z''(t)]\right]' + f(t)y(t-\sigma) = 0,$$

$$\left[r(t)[z''(t)]\right]' \leq -q(t)y(t-\sigma),$$

$$\left[r(t)[z''(t)]\right]' \leq -q(t)\,[1 - p(t-\sigma)]z(t-\sigma) , \qquad (19)$$

so that $[r(t)[z''(t)]]' \leq 0$

We define the function φ by

$$\varphi(t) = t\,\frac{r(t)(\,z''(t)\,)}{z'(t)}\,, \qquad t \geq t_1\,. \tag{20}$$

$$\varphi'(t) = \frac{r(t)(z''(t))}{z'(t)} + t\Big(\frac{r(t)(z''(t)}{z'(t)}\Big)'$$

$$= \frac{r(t)(z''(t))}{z'(t)} + t\Big(\frac{r(t)(z''(t)}{z'(t)}\Big)' - t\,\frac{r(t)(z''(t))^2}{(z'(t))^2}$$

$$= \frac{\varphi(t)}{t} + t\Big(\frac{r(t)(z''(t)}{z'(t)}\Big)' - \varphi(t)\,\frac{(z''(t))}{(z'(t))} \tag{21}$$

$$\leq \frac{\varphi(t)}{t} - tq(t)(1-p(t-\sigma))\,\frac{z(t-\sigma)}{z'(t)} - \varphi(t)\frac{(z''(t))}{(z'(t))}$$

$$\varphi'(t) \leq \frac{\varphi(t)}{t} - tq(t)(1-p(t-\sigma))\,\frac{z(t-\sigma)}{z'(t)} - \frac{\varphi^2(t)}{tr(t)} \tag{22}$$

Also from lemma 2.1 with $x(t) = z'(t)$, we have

$$\frac{x(t-\sigma)}{x(t)} \geq \frac{k(t-\sigma)}{t} \qquad t-\sigma \geq t_1$$

$$\frac{z'(t-\sigma)}{z'(t)} \geq \frac{k(t-\sigma)}{t}$$

$$\Rightarrow \qquad \frac{1}{z'(t)} \geq \frac{k(t-\sigma)}{t}\cdot\frac{1}{z'(t-\sigma)} \qquad for \qquad t-\sigma \geq t_1 \geq t_2 \tag{23}$$

and by Lemma 2.2 for M and by (22),

$$\frac{z(t-\sigma)}{z'(t)} \geq \frac{K(t-\sigma)}{t}\cdot\frac{z(t-\sigma)}{z'(t-\sigma)}$$

$$\geq \frac{K(t-\sigma)}{t}\cdot\frac{M(t-\sigma)}{2}$$

$$\geq \frac{KM}{2}\frac{(t-\sigma)^2}{t}$$

(24)

Combining (22) and (24) we get

$$\varphi'(t) \leq \frac{\varphi(t)}{t} - tq(t)(1-p(t-\sigma))\frac{KM}{2}\frac{(t-\sigma)^2}{t} - \frac{\varphi^2(t)}{tr(t)} \tag{25}$$

Using the inequality $\qquad Vx - Ux^2 \leq \frac{1}{4}\frac{V^2}{U} \qquad U > 0$

with $$x = \varphi(t)\,;U = \frac{1}{tr(t)}\;\;;V = \frac{1}{t}.$$

we have $$\varphi'(t) \le -tq(t)(1-p(t-\sigma))\frac{KM}{2}\frac{(t-\sigma)^2}{t} + \frac{1}{4}\frac{r(t)}{t}$$

Integrating the last inequality from t_2 to t , we obtain

$$\int_{t_2}^{t}[\;q(s)(1-p(s-\sigma))\frac{1}{2}KM(s-\sigma)^2 - \frac{1}{4}\frac{r(S)}{S}]ds \le \varphi(t_2)\,, \tag{26}$$

which contradicts (H_6).

We now consider

Case III: $\boldsymbol{z(t) > 0, z'(t) > 0,\; z''(t) < 0; (r(t)(z''(t)))' \le 0}$

We define the function $\emptyset$ by

$$\emptyset(t) = \frac{r(t)(z''(t))}{z'(t)}\,,\qquad t \ge t_1\,. \tag{27}$$

Then $\emptyset(t) < 0$. Noting that $r(t)(z''(t))$ is nonincreasing, we get

$$r(s)z''(s) \le r(t)z''(t),\qquad s \ge t \ge t_1.$$

Dividing the above inequality by $r(s)$, and integrating it from t to l , we get

$$\int_t^l \frac{r(s)z''(s)}{r(s)}ds \le \int_t^l \frac{r(t)z''(t)}{r(s)}ds$$

$$\int_t^l z''(s)ds \le r(t)z''(t)\int_t^l \frac{1}{r(s)}ds$$

$$z'(l) - z'(t) \le r(t)z''(t)\int_t^l \frac{1}{r(s)}ds$$

$$z'(l) \le z'(t) + r(t)z''(t)\int_t^l \frac{1}{r(s)}ds$$

on letting $l \to \infty$, we have

$$0 \le z'(t) + r(t)z''(t)R(t).$$

$$-r(t)z''(t)R(t) \le z'(t)$$

$$-R(t)\frac{r(t)z''(t)}{z'(t)} \le 1.$$

Hence by (27) we get $$-R(t)\,\emptyset(t) \le 1. \tag{28}$$

Differentiating (27) we get

$$\emptyset'(t) = \frac{z'(t)(r(t)(z''(t)))' - (r(t)(z''(t)))z''(t)}{(z'(t))^2}$$

$$\emptyset'(t) = \frac{z'(t)(r(t)(z''(t)))'}{z'(t)z'(t)} - \frac{(r(t)(z''(t)))z''(t)}{(z'(t))^2}$$

$$\emptyset'(t) = \frac{(r(t)(z''(t)))'}{z'(t)} - \frac{(r(t)(z''(t)))z''(t)}{z'(t)z'(t)}$$

$$\emptyset'(t) = \frac{(r(t)(z''(t)))'}{z'(t)} - \emptyset(t)\frac{z''(t)}{z'(t)}$$

Thus
$$\emptyset'(t) = \frac{(r(t)(z''(t)))'}{z'(t)} - \frac{(\emptyset(t))^2}{r(t)} \tag{29}$$

Since $z(t) > 0,\ z'(t) > 0,\ z''(t) < 0$.

It follows from (1), (19), (24) and (29) that there exists a $t_3 \geq t_1$ such that

$$\emptyset'(t) = \frac{(r(t)(z''(t)))'}{z'(t)} - \frac{(\emptyset(t))^2}{r(t)}$$

$$\emptyset'(t) = \frac{-q(t)(1-p(t-\sigma))z(t-\sigma)}{z'(t)} - \frac{(\emptyset(t))^2}{r(t)}$$

$$\emptyset'(t) \leq -q(t)(1-p(t-\sigma))\frac{KM}{2}\frac{(t-\sigma)^2}{t} - \frac{\emptyset^2(t)}{r(t)} \tag{30}$$

Multiplying (30) by $R(t)$ and integrating it from t_3 to t we have

$$\int_{t_3}^{t} R(s)\emptyset'(s)ds \leq -\int_{t_3}^{t} q(s)(1-p(s-\sigma))\frac{KM}{2}\frac{(s-\sigma)^2}{s}ds - \int_{t_3}^{t}\frac{\emptyset^2(s)}{r(s)}R(s)ds$$

$$R(t)\emptyset(t) - R(t_3)\emptyset(t_3) + \int_{t_3}^{t}\frac{1}{r(s)}\emptyset(s)ds + \int_{t_3}^{t} q(s)(1-p(s-\sigma))\frac{KM}{2}\frac{(s-\sigma)^2}{s}ds$$

$$+\int_{t_3}^{t}\frac{\emptyset^2(s)}{r(s)}R(s)ds \leq 0$$

set with
$$x = -\emptyset(s)\,;\, U = \frac{1}{r(s)}R(s)\,;\, V = \frac{1}{r(s)}.$$

Using the inequality
$$U x^2 - Vx \geq -\frac{1}{4}\frac{V^2}{U} \qquad U > 0$$

we have

$$\int_{t_3}^{t}\left[q(s)(1-p(s-\sigma))\frac{KM}{2}\frac{(s-\sigma)^2}{s} - \frac{1}{4}\frac{1}{R(s)r(s)}\right]ds \leq R(t_3)\emptyset(t_3) + 1. \tag{31}$$

Letting $t \to \infty$, we obtain a contradiction to (H_7). Therefore all the solutions of the equation (1) are oscillatory.

Example2.4:

Consider the neutral delay differential equation,

$$\frac{d}{dt}\left(e^t\left(\frac{d^2}{dt^2}\left(y(t)+\frac{1}{2}y(t-2\pi)\right)\right)\right)+e^t y\left(t-\frac{49\pi}{4}\right)=0 \qquad (32)$$

Here, $r(t)=e^t$, $p(t)=\frac{1}{2}$, $\tau=2\pi$, $f(t)=e^t$

$$\sigma=\frac{49\pi}{4} \ \ and\ \ q(t)=\frac{1}{t^7}$$

and

$$R(t)=\int_t^\infty \frac{1}{e^s}ds \ =e^{-t}$$

Let $K=0.4 \ \ and\ M=0.4$ Then we observe that all the conditions of Theorem 2.3 are satisfied. Hence all solutions of the equation (32) are oscillatory.

3. Summary:

In this work, an attempt is made to establish the conditions under which the solutions of neutral delay differential equation (1) are oscillatory.

We wish to mention here, under suitable conditions one can show that all solutions of a more general equation

$$\frac{d}{dt}[r_2(t)\left[\frac{d}{dt}r_1(t)\left\{\frac{d}{dt}\left(y(t)+p(t)y(t-\tau)\right)\right\}\right]]+f(t)(y(t-\sigma)=0 \qquad (33)$$

are oscillatory following the lines of Theorem 2.3.

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