Products of Conjugate K-Normal Matrices

S.Krishnamoorthy And K.Gunasekaran
Department of Mathematics, Government Arts College (Autonomous),
Kumbakonam, Tamilnadu, India 612 001.
K.Arumugam
Department of Mathematics,
A.V.C College (Autonomous), Mannampandal, Mayiladuthurai,
Tamilnadu, India 609305.

Abstract: In this paper, we discussed properties of conjugate k-normal matrices. The product of k-normal and conjugate k-normal matrices are also discussed..

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1. INTRODUCTION

A k-normal matrix A = < a_{ij} > with complex elements is a matrix such that $AA^* k = KA^* A$, where A^* denotes the complex conjugate transpose of A. A conjugate k-normal matrix is defined to be a complex matrix A which is such that AA^* $K=\overline{KA^*A}$. Here, we developed further properties of conjugate k-normal matrices, their relation, in a sense; to k-normal matrices in considered and further results concerning k-normal products are obtained including an analogous for conjugate k-normal matrices.

2. PROPERTIES OF CONJUGATE k-NORMAL MATRICES

Theorem 2.1:

A matrix A in conjugate k-normal if and only if there exists a k-unitary matrix U such that UAU^T is a direct sum of non-negative real numbers and of 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where a and b are non-negative real numbers.

Proof:

Let A be conjugate k-normal, where A = M+N, where $M=M^T$ and $N= -N^T$.

Then, $AA^*K = \overline{KA^*A}$

$\Rightarrow \quad AA^*K = K\overline{A^*}\,\overline{A}$

$\Rightarrow \quad AA^*K = K(\overline{A^{-T}})\,\overline{A}$

$\Rightarrow \quad AA^*K = K\,A^T\,\overline{A}$

$$(M+N)(M+N)^*K = K(M+N)^T\,\overline{(M+N)}$$

$$(M+N)(M^*+N^*)K = K(M^T+N^T)(\overline{M}+\overline{N})$$

$$(M+N)(\overline{M}^T+\overline{N}^T)K = K(M-N)(\overline{M}+\overline{N}), \quad \text{since } M = M^T \text{ and } N = -N^T$$

$$(M+N)((\overline{M^T})+(\overline{N^T}))K = K(M-N)(\overline{M}+\overline{N})$$

$$(M+N)(\overline{M}-\overline{N})K = K(M-N)(\overline{M}+\overline{N})$$

$$(M\overline{M}-M\overline{N}+N\overline{M}-N\overline{N})K = K(M\overline{M}+M\overline{N}-N\overline{M}-N\overline{N})$$

$$M\overline{M}K-M\overline{N}K+N\overline{M}K-N\overline{N}K = KM\overline{M}+KM\overline{N}-KN\overline{M}-KN\overline{N}$$

$$-M\overline{N}K+N\overline{M}K = KM\overline{N}-KN\overline{M}$$

$$-M\overline{N}K-KM\overline{N} = -KN\overline{M}-N\overline{M}K$$

Since A is conjugate k-normal. Therefore M and N is also a conjugate k-normal.

Therefore, $-M\overline{N}K-M\overline{N}K = -N\overline{M}K-N\overline{M}K$

$$-2M\overline{N}K = -2N\overline{M}K$$

$$M\overline{N} = N\overline{M}$$

There exists a k-unitary matrix U such that $UMU^T = D$ is a k-diagonal matrix with real, non-negative elements.

Therefore, $UNU^T\,\overline{U}\overline{M}U^* = UMU^T\,\underline{U}\underline{N}U^*$

$\Rightarrow \quad WD = D\overline{W}$, where W= -W^T

Let U be chosen so that D is such that $d_{k(i)} \geq d_{k(j)} \geq 0$ for i< j, where $d_{k(i)}$ in the ith k-diagonal element of D.

If $W=(t_{k(i)k(j)})$, where $(t_{k(i)k(j)})=-(t_{k(i)k(j)})$,then $t_{k(i)k(j)}d_{k(j)}=d_{k(i)}t_{k(i)k(j)}$, for j > i and three possibilities may occur: if $d_{k(i)}=d_{k(j)}\neq 0$, then $t_{k(i)k(j)}$ is real ; $d_{k(i)}=d_{k(j)}=0$, then $t_{k(i)k(j)}$ is arbitrary(though $W=-W^T$ still holds); and if $d_{k(i)}\neq d_{k(j)}$, then $t_{k(i)k(j)}=0$ for if $t_{k(i)k(j)}=a+ib$, then $(a+ib)d_{k(j)}=d_{k(i)}(a-ib)$ and $a(d_{k(j)}-d_{k(i)})=0$ implies a=0 and $b(d_{k(i)}+d_{k(j)})=0$ implies $d_{k(i)}=-d_{k(j)}$ (which is not possible since $d_{k(i)}$ are real and non-negative and $d_{k(i)}\neq d_{k(j)}$) or b=0 so $t_{k(i)k(j)}=0$.

So if $UMU^T=d_{k(1)}I_{k(1)}+d_{k(2)}I_{k(2)}+d_{k(3)}I_{k(3)}+\ldots+d_{k(n)}I_{k(n)}$, where + denotes the direct sum , then $UNU^T=N_{k(1)}+N_{k(2)}+N_{k(3)}\ldots+N_{k(n)}$, where $N_{k(i)}=-N^T_{k(i)}$ is real and $N_{k(n)}=-N^T_{k(n)}$ is complex if and only if $d_{k(n)}=0$. For each real $T_{k(i)}$ there exists a real orthogonal matrix $V_{k(i)}$ so that $V_{k(i)}N_{k(i)}V^T_{k(i)}$ is a direct sum of zero matrices and matrices of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$, where b is real. If $N_{k(n)}=-N^T_{k(n)}$ is complex ,there exists a complex k-unitary matrix $V_{k(n)}$ such that $V_{k(n)}N_{k(n)}V_{k(n)}$, N is a direct sum of matrices of the some form ,so that $V=V_{k(1)}+V_{k(2)}+V_{k(3)}+\ldots+V_{k(n)}$, then $VUMU^TV^T=D$ and $VUNU^TV^T=F$ =the direct sum described.

Therefore, $VUAU^TV^T=D+F$ which is the desired form.

Properties of conjugate k-normal matrices:

Let A and B are two conjugate k-normal matrices such that $A\overline{B}=B\overline{A}$, then A and B can be simultaneously brought in to the above k-normal form under the same U (with generalization to a finite number) but not conversely; if A is conjugate k-normal, $A\overline{A}$ is k-normal in the usual sense, but not conversely and if A is conjugate k-normal and $A\overline{A}$ is real, there is real orthogonal matrix which gives the above form.

Properties of con k-normal matrices not obtained in this section but of subsequent use are the following:

(a) A is conk-normal iff $A=HU=UH^T$ Where H is k-hermitian and U is k-unitary.

For if $A=HU$ is a polar form of A, then $U^*HU=L$ is such that $A=HU=UL$ and if $AA^*=A^T\overline{A}$ then $H^2=(L^T)^2$ and since this is a k-hermitian matrix with non-negative roots, $H=L^T$ and $A=HU=UH^T$ .The converse is immediate.

This same result may be seen as follows. If $UAU^{T} = F$ is the k-normal form in theorem 1, $F = D_{K(r)}, V = VD_{k(r)}$, where $D_{K(r)}$ is real K- diagonal and V is a direct sum of 1's or block in the form $(a^2+b^2)^{-1\backslash 2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ which are k- unitary.

Therefore, $A = U^{*}D_{k(r)}UU^{*}V\overline{U} = U^{*}VUU^{T}D_{k(r)}\overline{U}$ this exhibits the polar form in another guise.

(b) A is both k-normal & con k-normal if and only if $A = HU = UH = UH^{T}$. So $H = H^{T} = H^{*}$ so that H is real.

(c) if $A = HU=UH^{T}$ is con k-normal , then UH is con k-normal , if and only if $HU^{2} = U^{2}H$,(i.e.) if and only if HU^{2} is k-normal . For if UH is con k-normal .$UH = H^{T}U$ so that $HU^{2} = UH^{T}U = U^{2}H$, and if $HU^{2} = U^{2}H$, then $HUU = UH^{T}U = UUH$ or $H^{T}U = UH$.

(d) A matrix A is con k- normal if and only if A can be written $A = SW = \overline{W}S$ where $M = M^{T}$ and W is k-unitary. If A is con k-normal, from the above

$A = U^{*}F\overline{U} = U^{*}D_{k(r)}\overline{U}V^{T}V\overline{U} = MW = U^{*}VUU^{*}D_{k(r)}\overline{U} = \overline{W}M$, where $M = U^{*}D_{k(r)}\overline{U}$

is symmetric and $W = U^{T}V\overline{U}$ is k-unitary. Conversely,

if $A = MW = \overline{W}M$, $AA^{*} = MWW^{*}M^{*} = A^{T}\overline{A} = M^{T}W^{*}W\overline{M}$.

Remarks 2.2:

If B is con k-normal and if B=MU where $M=M^{T}$ and U ia k-unitary, it does not necessarily follow that $B=\overline{U}S$, but it is possible to find an M_1 and U_1 such that $B = M_1U_1 = \overline{U_1}M_1$ holds. This may be seen as follows .If B=MU is con k-normal, let V be k-unitary such that $VMV^{T}=D$ is k-diagonal, real and non-negative, so that $VBV^{T} = VSV^{T}\overline{V}UV^{T} = DW$ is con k-normal from which $DWW^{*}\overline{D} = W^{T}D^{T}DW$ or since D is real, $WD^{2}=D^{2}W$ and WD=DW since D is non-negative.

Then $B = (V^{*}D\overline{V})(V^{T}W\overline{V}) = MV = (V^{*}WV)(V^{*}DV)$ which is not necessarily = $\overline{U}S = (V^{*}WV)(V^{*}D\overline{V})$. However, if $D = r_1I_{1+} r_2I_{2+} r_3I_{3+\dots} + r_nI_n$,$r_i > r_j$ for i>j, then $w= w_{1+} w_{2+} w_3+\dots+w_n$. Since each W_i is k-unitary, it is con k-normal and hence there exist k-unitary Xi so that $X_i\ W_i\ X_i^T = F_i$ is in the k-normal form of theorem 1. If $x = x_1 + x_2 + x_3 + \ldots + x_n$, then

$$XVBU^T X^T = XDWX^T = DXWX^T = DF = FD, \text{ where } F = F_1 + F_2 + \square + F_n.$$

$$\text{So, } B = (V^* X^* D\overline{X}\overline{V})(V^T X^T F\overline{X}\overline{V}) = (V^* XFXV)(V^* X^* D\overline{X}\overline{V}) = M_1 U_1$$

$$\Rightarrow B = \overline{U}_1 M_1$$

and $$M_1 = V^* X^* D\overline{X}\overline{V} \neq V^* D\overline{V} = M$$

$$U_1 = V^T X^T F\overline{X}\overline{V} \neq V^T W\overline{V} = U.$$

3. k-NORMAL PRODUCTS OF MATRICES:

In this section, if A, B and AB are k-normal matrices, the BA is k-normal, a necessary and sufficient condition that the products, AB of two k-normal matrices A and B be k-normal is that each commute with the k-hermitian polar matrix of each other .First a generalization of this theorem is obtained here and then an analogous for the con k-normal case is developed.

Theorem 3.1: Let A and B be a k-normal matrices and AB and BA are k-normal.
Then $K(A^*A)B = B(AA^*)K$ and $K(B^*B)A = A(BB^*)K$.

Proof:

If AB and BA are k-normal. Let U be a k-unitary matrix such that
$UAU^*K = D$ is diagonal, $d_{k(i)}\overline{d}_{k(i)} \geq d_{k(j)}\overline{d}_{k(j)} \geq 0$ for i < j.

Let $UBU^*K = B_1 = b_{k(i)\,k(i)}$ .since AB and BA are k-normal.
Then $ABB^*A^*K = KB^*A^*AB \Rightarrow DB_1B_1^*D^*K = KB_1^*D^*DB_1$

By equating diagonal elements it follows that

$$\sum_{j=1}^{n} d_{k(i)}\overline{d}_{k(i)}\ b_{k(i)\,k(j)}\overline{b}_{k(i)\,k(j)} = \sum_{j=1}^{n} d_{k(j)}\overline{d}_{k(j)}\ b_{k(j)\,k(i)}\overline{b}_{k(j)\,k(i)} \quad \rightarrow (1) \text{ for i = 1, 2…n.}$$

Similarly, $BAA^*B^*K = KA^*B^*BA \Rightarrow B_1DD^*B_1^*K = KD^*B_1^*B_1D$

$$\Rightarrow \sum_{j=1}^{n} d_{k(j)}\overline{d}_{k(j)}\ b_{k(i)\,k(j)}\overline{b}_{k(i)\,k(j)} = \sum_{j=1}^{n} \overline{d}_{k(i)} d_{k(i)}\ \overline{b}_{k(j)\,k(i)} b_{k(j)\,k(i)} \rightarrow (2).$$

Let i=1, from (1) and (2)

$$\Rightarrow \sum_{j=1}^{n} d_{k(1)}\overline{d}_{k(1)}\ b_{k(1)\,k(j)}\overline{b}_{k(1)\,k(j)} = \sum_{j=1}^{n} d_{k(j)}\overline{d}_{k(j)}\ b_{k(j)\,k(1)}\overline{b}_{k(j)\,k(1)} \quad \rightarrow (3)$$

$$\sum_{j=1}^{n} d_{k(j)}\bar{d}_{k(j)}\ b_{k(1)\,k(j)}\,\bar{b}_{k(1)\,k(j)} = \sum_{j=1}^{n}\bar{d}_{k(1)}d_{k(1)}\ \bar{b}_{k(j)\,k(1)}\,b_{k(j)\,k(1)} \rightarrow (4)$$

Now (3)-(4), we

get,

$$\sum_{j=1}^{n}(d_{k(1)}\bar{d}_{k(1)} - d_{k(j)}\bar{d}_{k(j)})\,b_{k(1)\,k(j)}\,\bar{b}_{k(1)\,k(j)} = \sum_{j=1}^{n}(d_{k(j)}\bar{d}_{k(j)} - \bar{d}_{k(1)}d_{k(1)})\,b_{k(j)\,k(1)}\,\bar{b}_{k(j)\,k(1)}$$

$$\sum_{j=1}^{n}(d_{k(1)}\bar{d}_{k(1)} - d_{k(j)}\bar{d}_{k(j)})(b_{k(1)\,k(j)}\,\bar{b}_{k(1)\,k(j)} + b_{k(j)\,k(1)}\,\bar{b}_{k(j)\,k(1)}) = 0$$

$$d_{k(1)}\,\bar{d}_{k(1)} = d_{k(2)}\,\bar{d}_{k(2)} = \ldots\ldots = d_{k(t)}\,\bar{d}_{k(t)} > d_{k(t+1)}\bar{d}_{k(t+1)};$$

Then $b_{k(1)\,k(j)}\,\bar{b}_{k(1)\,k(j)} + b_{k(j)\,k(1)}\,\bar{b}_{k(j)\,k(1)} = 0$, for j = t+1, t+2, … n.

Since $d_{k(1)}\,\bar{d}_{k(1)} - d_{k(j)}\,\bar{d}_{k(j)} = 0$ or positive value and is the latter for j >t
So $b_{k(1)k(j)} = 0$ and $b_{k(j)\,k(1)} = 0$ for j = t+1, t+2… n.

For i = 2,3,… t is turn it follows that $b_{k(i)\,k(j)} = 0$ and $b_{k(j)\,k(i)} = 0$, for i = 1,2…t and

j = t+1, t+2, … n.

Let $UAU^*K = D = r_1 D_1 + r_2 D_2 + \ldots\ldots + r_s D_s$, where the r_i are real, $r_i > r_j$ for

$i < j$ and the D_i are k-unitary.

Then by repeating the above process it follows that

$UBU^*K = B_1 = C_1 + C_2 + \ldots\ldots + C_s$ is conformal to D. it follows from the given

condition that $(r_i D_i)C_i C_i^*(D_i^* r_i)K = K C_i^*(r_i D_i^*)(D_i r_i)C_i$

and $C_i(r_i D_i)(D_i^* r_i)C_i^* K = K(r_i D_i^*)C_i^* C_i(D_i r_i)$
$\Rightarrow D_i C_i C_i^* K = K C_i^* C_i D_i$ and $D_i C_i C_i^* K = K C_i^* C_i D_i$ if $r_i > 0$

If $r_s = 0$, D_s is arbitrary insofar as D is concerned and so may be chosen so that

$D_s C_s C_s^* K = K C_s^* C_s D_s$ in which case D_s may not be diagonal. But whether or not

this is done, it follows that $DB_1B_1^*K = KB_1^*B_1D$ and $B_1DD^*K = KD^*DB_1$ so that

$K(A^*A)B = B(AA^*)K$ and $K(B^*B)A = A(BB^*)K$.

Theorem 3.2:
Let $A = PW = WP$ both polar form of the k-normal matrix A. Then AB & BA are k-normal iff $B = NW^*$, where N is k-normal and $PN = NP$

Proof:

Let $C_{k(i)} = H_{k(i)}U_{k(i)} = U_{k(i)}L_{k(i)}$ be the polar form of the $C_{k(i)}$.

Then $U^*_{k(i)}H_{k(i)}U_{k(i)} = L_{k(i)}$.

So that $U^*_{k(i)}C_{k(i)}C^*_{k(i)}U_{k(i)} = C^*_{k(i)}C_{k(i)}$ or $U^*_{k(i)}C_{k(i)}C^*_{k(i)} = C^*_{k(i)}C_{k(i)}U^*_{k(i)}$.

Also from the above $D_{k(i)}C_{k(i)}C^*_{k(i)} = C^*_{k(i)}C_{k(i)}D_{k(i)}$.

Let $R_{k(i)} = \bar{D}_{k(i)}U^*_{k(i)}$.

Then $R_{k(i)}C_{k(i)}C^*_{k(i)} = \bar{D}_{k(i)}U^*_{k(i)}C^*_{k(i)}C_{k(i)}$

$= \bar{D}_{k(i)}C^*_{k(i)}C_{k(i)}U^*_{k(i)}$

$= C^*_{k(i)}C_{k(i)}\bar{D}_{k(i)}U^*_{k(i)}$

$= C^*_{k(i)}C_{k(i)}R_{k(i)}$, where $R_{k(i)}$ is k-unitary ($r_{k(s)} = 0$, $D_{k(s)}$ may be chosen $= {U^*}_{k(s)}$ as describe above). So $R_{k(i)}H^2_{k(i)} = H^2_{k(i)}R_{k(i)}$ and since $H_{k(i)}$ has positive or zero roots, $R_{k(i)}H_{k(i)} = H_{k(i)}R_{k(i)}$ and so $H_{k(i)}R^*_{k(i)} = R^*_{k(i)}H_{k(i)}$.

Then, $A = U^*DU = U^*D_{k(i)}UU^*D_{k(i)}U = PW = PW$ and

$$B = U^*B_{k(i)}U = U^*\left(c_{k(1)} + c_{k(2)} + ... + c_{k(s)}\right)U$$

$$= U^*\left(H_{k(1)}U_{k(1)} + H_{k(2)}U_{k(2)} + ... + H_{k(s)}U_{k(s)}\right)U$$

$$= U^*\left(H_{k(1)}R^*_{k(1)}\bar{D}_{k(1)} + H_{k(2)}R^*_{k(2)}\bar{D}_{k(2)} + ... + H_{k(s)}R^*_{k(s)}\bar{D}_{k(s)}\right)U$$

$= NW^*$, where $N = U^*\left(H_{k(1)}R^*_{k(1)} + H_{k(2)}R^*_{k(2)} + ... + H_{k(s)}R^*_{k(s)}\right)U$

(which is k-normal since the k-hermition $H_{k(i)}$ and k-unitary $R^*_{k(i)}$ commute) and $W^* = U^*\left(\bar{D}_{k(1)} + \bar{D}_{k(2)} + ... + \bar{D}_{k(3)}\right)U$ it is evident that $PN = NP$

Conversely, if $A = PW = WP$ and $B = NW^*$ an described, then $AB = WPNW^*$ which is obviously k-normal is $BA = NW^*WP = NP$.

It is early seen that $B = NW^*$ is k-normal iff $NW^* = W^*N$ if $B = NW^* = (HR)W^*$ is can k-normal then $B = H(RW^*) = (RW^*)H^T = RHW^*$ (from property a) so $W^*H^T = HW^*$ or $WH = H^TW$ and $W(BB^*) = (B^*B)W$.

Remark 3.3:

If A is k-normal .if B is conk-normal and if AB is k-normal, if does not necessarily follow that BA is k-normal though it can occur.

For example 3.4:

If $B = HU = UH^T$ is con k-normal and if $A = U^*$, then $AB = U^*UH^T = H^T$ and $BA = HUV^* = H$ are both k-normal. But the following is an example in which AB is k-normal but not BA .Let $B = HU = UH^T$ be conk-normal but not k-normal (ie H is not real by property (b)) and let H be non-singular.

Let $A = H^{-1}$ which is k-hamitian (so k-normal) and not conk-normal (since H^{-1} is not real). Then $AB = H^{-1}HU = U$ is k-normal. If BA were also k-normal, then by the above theorem $(A^*A)B = B(AA^*)$ and $(B^*B)A = A(BB^*)$ but $(B^*B)A = (H^T)^2H^{-1}$ and $A(BB^*) = (H)^{-1}(H^2)$ and if there were equal, $(H^T)^2 = H^2$ would follow which means that $H^2 = (H^T)^2 = (H^*)^2$ so that H^2 is real. But this is not possible for if $H = VDV^*$ where D is k-diagonal with the real elements (since H is non-singular), then $H^2 = VD^2V^* = \bar{V}D^2V^T$ if H^2 is real so that $V^TVD^2 = D^2V^TV$ so $V^TVD = DV^TV$ so $VDV^* = \bar{V}DV^T = H$ is real which contradicts the above consumption. But the following theorem result when A and B are both con k- normal.

Theorem 3.5:

If A and B are con k-normal and if AB is k–normal, then BA is k-normal.

Proof:

Let U be a k-unitary matrix such that $UAU^T = F$ is the k-normal form described in theorem 1 and where $FF^* = FF^T = r_{k(1)}^2 I_{k(1)} + r_{k(2)}^2 I_{k(2)} + ... + r_{k(n)}^2, I_{k(n)}$

which is real k-diagonal with $r_{k(1)}^2 > r_{k(2)}^2 > ... > r_{k(n)}^2 \geq 0$.

These $r^2_{k(i)}$ may be either the squares of k-diagonal elements of F or they may arise when matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are squared. Assume that any of the letter where $r^2_{k(i)}$ are equal are arranged first in a given block followed by any k-diagonal elements whose square is the same $r^2_{k(i)}$.

Let $\bar{U}BU^* = B_{k(1)}$ which is conk-normal and then $UAU^T\bar{U}BU^* = FB_{k(1)}$ is k-normal.

Let V be the k – unitary matrix $\sqrt{2^{-1}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ then the following matrix relation holds, independent of a and b.

$$V = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad V^* = \begin{bmatrix} a-b^i & 0 \\ 0 & a+b^i \end{bmatrix}$$

Let $F = F_{k(1)} + F_{k(2)} + ... + F_{k(n)}$ where the direct sum is conformable to that of FF^* given above (i.e., $F_{k(i)}F^*_{k(i)} = r^2_{k(i)}I_{k(i)}$) and consider $F_{k(1)} = G_{k(1)} + G_{k(2)} + ... + G_{k(t)} + r_{k(t)}I$ where each $G_{k(i)}$ is 2 x 2 as described above and I is an identity matrix of proper size.

Let $W_{k(1)} = V + V + ... + V + I$ be conformable to $F_{k(i)}$, define $W_{k(i)}$ for each $F_{k(i)}$, in like manner and let $W = W_{k(1)} + W_{k(2)} + ... + W_{k(n)}$ if $r_{k(n)} = 0$, $W_{k(n)} = I$. Then $WFW^* = D$ is complex k-diagonal, where $d_{k(i)}$ is the ith diagonal element $d_{k(i)}\bar{d}_{k(i)} \geq d_{k(i+1)}\bar{d}_{k(i+1)}$. Then $W(UAU^T)W^*W(\bar{U}BU^*)W^* = (WFW^*)(WB_{k(1)}{}^{W^*}) = DB_{k(2)}$ is k–normal for $B_{k(2)} = WB_{k(1)}W^*$ or $B_{k(1)} = W^*B_{k(2)}W$.

Since B_1 is con k – normal, $B_{k(1)}B^*_{k(1)} = B^T{}_{k(1)}B_{k(1)}$,

so that $W^*B_{k(2)}WW^*B^*_{k(2)}W = W^TB^*_{k(2)}\bar{W}W^T\underline{B}_{k(2)}\bar{W}$
or that $B_{k(2)}B^*_{k(2)}WW^T = WW^TB^T_{k(2)}\underline{B}_{k(2)}$.

Now VV^T is a matrix of the form $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, so that WW^T is a direct sum of matrices of this form and 1's .

Let $B_{k(2)} = \left(b_{k(i)k(j)}\right)$ and consider $(WW^T)^* B_{k(2)} B^*_{k(2)} (WW^T) = B^T_{k(2)} \bar{B}_{k(2)}$. Let $B_{k(2)} B^*_{k(2)} = \left(c_{k(i)k(j)}\right), B^T_{k(2)} \bar{B}_{k(2)} = \left(f_{k(i)k(j)}\right), C_{k(i)k(j)}$ and $f_{k(i)k(j)}$ are identifiable with the $b_{k(i)k(j),}$ both matrices being k-hermitian.

Consider two cases:

(a) if $d_{k(1)} \bar{d}_{K(1)} = d_{k(j)} \bar{d}_{K(j)}$ for all j (where $d_{k(j)}$ is the j $_{th}$ k-diagonal element of D), then $D = nD_{K(u)}$, where $D_{K(u)}$ is k-unitary k-diagonal. Since $WFB_{k(1)} W^* = DB_{k(2)} = nD_{k(u)} B_{k(2)} = D_{k(u)} (nB_{k(2)})$ is k-normal, and then $\bar{D}_{k(u)} (D_{k(u)} B_{k(2)} n) D_{k(u)} = B_{k(2)} D = WB_{k(1)} FW^*$ is k-normal as is $B_{k(1)} F = \bar{U} BU^* UAU^T$ so BA is k-normal.

(b) If $l_{k(i)} \bar{d}_{k(i)} \neq d_{k(j)} \bar{d}_{k(j)}$ for some j, let $d_{k(1)} d_{k(1)} = d_{k(2)} d_{k(2)} = ... = d_{k(1)} \bar{d}_{k(1)}$ for $1 \le l \le n$ (so that $d_{k(l)} \bar{d}_{k(l)} > d_{k(l+1)} \underline{d}_{k(l+1)}$). $-$ $-$

Suppose $F_{k(1)} = G_{k(1)} + G_{k(2)} + r_{k(1)} I_{k(1)}$ where $I_{k(1)}$ is the 2×2 identify matrix. From $(WW^T)^* B_{k(2)} B^*_{k(2)} (WW^T) = B^T_{k(2)} \bar{B}_{k(2)}$ and fact that $W_{k(1)} = V + V + I_{k(1)}$, it follow that

$$c_{k(1)k(1)} = \sum b_{k(1)k(i)} \bar{b}_{k(1)k(i)} = \sum b_{k(i)k(2)} \bar{b}_{k(i)k(2)} = f_{k(2)k(2)}$$

$$c_{k(2)k(2)} = \sum b_{k(2)k(i)} \bar{b}_{k(2)k(i)} = \sum b_{k(i)k(1)} \bar{b}_{k(i)k(1)} = f_{k(1)k(1)}$$

$$c_{k(3)k(3)} = \sum b_{k(3)k(i)} \bar{b}_{k(3)k(i)} = \sum b_{k(i)k(4)} \bar{b}_{k(i)k(4)} = f_{k(4)k(4)}$$

$$c_{k(4)k(4)} = \sum b_{k(4)k(i)} \bar{b}_{k(4)k(i)} = \sum b_{k(i)k(3)} \bar{b}_{k(i)k(3)} = f_{k(3)k(3)}$$

$$c_{k(5)k(5)} = \sum b_{k(5)k(i)} \bar{b}_{k(5)k(i)} = \sum b_{k(i)k(5)} \bar{b}_{k(i)k(5)} = f_{k(5)k(5)}$$

$$c_{k(6)k(6)} = \sum b_{k(6)k(i)} \bar{b}_{k(6)k(i)} = \sum b_{k(i)k(6)} \bar{b}_{k(i)k(6)} = f_{k(6)k(6)}$$

$DB_{k(2)}$ is k-normal so that the following relations also hold.

$$d_{k(1)} \bar{d}_{k(1)} \sum b_{k(1)k(i)} \bar{b}_{k(1)k(i)} = \sum d_{k(i)} \bar{d}_{k(i)} b_{k(i)k(1)} \bar{b}_{k(i)k(1)}$$

$$d_{k(2)} \bar{d}_{k(2)} \sum b_{k(2)k(i)} \bar{b}_{k(2)k(i)} = \sum d_{k(i)} \bar{d}_{k(i)} b_{k(i)k(2)} \bar{b}_{k(i)k(2)}$$

$$d_{k(3)} \bar{d}_{k(3)} \sum b_{k(3)k(i)} \bar{b}_{k(3)k(i)} = \sum d_{k(i)} \bar{d}_{k(i)} b_{k(i)k(3)} \bar{b}_{k(i)k(3)}$$

$$d_{k(4)}\bar{d}_{k(4)}\sum b_{k(4)k(i)}\bar{b}_{k(4)k(i)} = \sum d_{k(i)}\bar{d}_{k(i)}b_{k(i)k(4)}\bar{b}_{k(i)k(4)}$$

$$d_{k(5)}\bar{d}_{k(5)}\sum b_{k(5)k(i)}\bar{b}_{k(5)k(i)} = \sum d_{k(i)}\bar{d}_{k(i)}b_{k(i)k(5)}\bar{b}_{k(i)k(5)}$$

$$d_{k(6)}\bar{d}_{k(6)}\sum b_{k(6)k(i)}\bar{b}_{k(6)k(i)} = \sum d_{k(i)}\bar{d}_{k(i)}b_{k(i)k(6)}\bar{b}_{k(i)k(6)} .$$

Since $d_{k(1)}\bar{d}_{k(1)} = d_{k(2)}\bar{d}_{k(2)}$, can combining the first two relations in each of these sets,

$$d_{k(1)}\bar{d}_{k(1)}\left(\sum b_{k(1)k(i)}\bar{b}_{k(1)k(i)} + \sum b_{k(2)k(i)}\bar{b}_{k(2)k(i)}\right) = d_{k(1)}\bar{d}_{k(1)}$$
$$\left(\sum b_{k(i)k(1)}\bar{b}_{k(i)k(1)} + b_{k(i)k(2)}\bar{b}_{k(2)}\right) \text{ so that}$$

$$\left(\sum d_{k(1)}\bar{d}_{k(1)} - d_{k(i)}\bar{d}_{k(i)}\right)\left(b_{k(i)k(1)}\bar{b}_{k(i)k(1)} + b_{k(i)k(2)}\bar{b}_{k(i)k(2)}\right) = 0$$

$d_{k(1)}\bar{d}_{k(1)} = d_{k(j)}\bar{d}_{k(j)}$ for $j = 1,2,....,6$ but for j beyond 6,

$d_{k(1)}\bar{d}_{k(1)} - d_{k(j)}\bar{d}_{k(j)} > 0$ so that $b_{k(i)k(1)}\bar{b}_{k(i)k(1)} + b_{k(i)k(2)}\bar{b}_{k(i)k(2)} = 0$ or $b_{k(i)k(1)} = 0$ and $b_{k(i)k(2)} = 0$ for $i = 7,8,...,n.$

Similarly, $b_{k(i)k(3)} = 0$ and $b_{k(i)k(4)} = 0$ for $i > 6$

The third relations in each set give $b_{k(i)k(5)} = 0$ and $b_{k(i)k(6)} \geq 0$ for $i > 6$.

On adding all 6 relation in the first set

$$\sum_{i,j=1}^{6} b_{k(i)k(j)}\bar{b}_{k(i)k(j)} + \sum_{i=1}^{6}\sum_{j=7}^{n} b_{k(i)k(j)}\bar{b}_{k(i)k(j)} = \sum_{i,j=1}^{6} b_{k(i)(j)}\bar{b}_{k(i)k(j)} + \sum_{i=7}^{n}\sum_{j=1}^{6} b_{k(i)k(j)}\bar{b}_{k(i)k(j)}$$

and on cancelling the first summations on each side,

$$\sum_{i=1}^{6}\sum_{j=7}^{n} b_{k(i)k(j)}\bar{b}_{k(i)k(j)} = \sum_{i=7}^{n}\sum_{j=1}^{6} b_{k(i)k(j)}\bar{b}_{k(i)k(j)} .$$

But the right side is 0 from the above, so the left side is 0 and so $b_{k(i)k(j)} = 0$ for $i = 1,2,...,6$.

From this it is evident that this procedure may be repeated, and that if $D = r_{k(1)}D_{k(1)} + r_{k(2)}D_{k(2)} + ... + r_{k(n)}D_{k(n)}$,where $D_{k(i)}$ are unitary and the $r_{k(i)}$ non-negative real, as above then $B_{k(2)} = C_{k(1)} + C_{k(2)} + ... + C_{k(n)}$ conformable to D.

Then, $r_{k(i)}D_{k(i)}C_{k(i)}$ is k-normal so $D^*_{k(i)}(D_{k(i)}C_{k(i)}r_{k(i)})D_{k(i)} = C_{k(i)}r_{k(i)}D_{k(i)}$ is k-normal. So $B_{k(2)}$D is k-normal. So $B_{k(i)}F$ and so $\bar{U}BU^*UAU^T$ and BA.

Theorem 3.6:

If A and B are con k-normal, then AB is k-normal if and only if $A^*AB = BAA^*$ and $ABB^* = B^*BA$ (i.e. if and only if each is k-normal relative to the other).

Proof:

Let AB is k-normal, from the above $D^*DB_{k(2)} = B_{k(2)}DD^*$ so that $F^*FB_{k(1)} = B_{k(1)}FF^*$ or $A^*AB = BAA^*$.

Similarly, since $DB_{k(2)}$ is k-normal, $DB_{k(2)}B^*_{k(2)}\bar{D} = B^*_{k(2)}, \bar{D}DB_{k(2)}$ so $DB_{k(2)}B^*_{k(2)} = B^*_{k(1)}B_{k(1)}F$ or $ABB^* = B^*BA$.

The converse is directly verifiable.

Theorem 3.7:

Let A and B be con k-normal. If AB is k-normal, then $\mathrm{A} = \mathrm{LW} = \mathrm{WL}^T$ (with L is k-hermition & W is k-unitary) and $L^T N = NL^T$ and conversely.

Proof:

Let $UAU^T = F = W^*DW = W^*D_{k(r)}\ \ WW^*D_{k(u)}\ \ W$ (where $D_{k(r)}$ and $D_{k(u)}$ are the k-hermition and k-unitary polar matrices of D) and $\bar{U}BU^* = B_{k(1)} = W^*B_{k(2)}W = W^*(C_{k(1)} + C_{k(2)} + ... + C_{k(n)})W$.

As in the proof of theorem 3 it follows that for all i,
$D^*_{k(i)}C_{k(i)}C^*_{k(i)} = C^*_{k(i)}C_{k(i)}D_{k(i)}$ *and* $U^*_{k(i)}C_{k(i)}C^*_{k(i)} = C^*_{k(i)}C_{k(i)}U^*_{k(i)}$, with $U_{k(i)}$ is defined there, so that when $R_{k(i)} = \bar{D}_{k(i)}U^*_{k(i)}$

(where , D here, $= r_{k(1)}D_{k(1)} + r_{k(2)}D_{k(2)} + ... + r_{k(n)}D_{k(n)}$ as earlier),
then $C_{k(i)} = H_{k(i)}U_{k(i)} = H_{k(i)}R^*_{k(i)}\underline{D}_{k(i)}$ with $H_{k(i)}R_{k(i)} = R_{k(i)}H_{k(i)}$.

Then, since $WD_{k(r)} = D_{k(r)}W$, $UAU^T = W^*DK_{(r)}WW^*D_{k(u)}W = D_{k(r)}(W^*D_{k(u)}W)$

and $A = (U^* D_{k(r)} U)(U^* W^* D_{k(u)} W\bar{U}) = LX$

$A = (U^* W^* D_{k(u)} W\bar{U})(U^T D_{k(r)} \bar{U}) = XL^T$

With $L = U^* D_{k(v)} U$ k-hermitian and $X = U^* W^* D_{k(u)} W\bar{U}$ k-unitary.

Also, $\bar{U}BU^* = W^*(H_{k(1)}R^*_{k(1)}\bar{D}_{k(1)} + H_{k(2)}R^*_{k(2)}\bar{D}_{k(2)} + ... + H_{k(n)}R^*_{k(n)}\bar{D}_{k(n)})W = N_{k(1)}\mathrm{Y}$,

where $N_{k(1)} = W^*(H_{k(1)}R^*_{k(1)} + H_{k(2)}R^*_{k(2)} + ... + H_{k(n)}R^*_{k(n)})W$ is k-normal and $\mathrm{Y} = W^*(\bar{D}_{k(1)} + \bar{D}_{k(2)} + ... + \bar{D}_{k(n)})W$ is k-unitary, then $B = U^T N_{k(1)} yU = (U^T N_{k(1)} \bar{U})(U^T yU) = NX^*$, where $N = U^T N_{k(1)} \bar{U}$ is k-normal and $X^* = U^T yU = U^T W^* \bar{D}_{k(u)} WU$.

Also, $L^T N = NL^T$ since $D_{k(r)} N_{k(1)} = N_{k(1)} D_{k(v)}$, $\bar{D}_{k(v)} N_{k(1)} = N_{k(1)} D_{k(v)}$.

So $(\bar{U}\,\bar{L}U^T)(\bar{U}NU^T) = (\bar{U}NU^T)(\bar{U}\,\bar{L}U^T)$.

So $L^T N = NL^T$. The converse is immediate.

4. CON k-NORMAL PRODUCT OF MATRICES

It is possible if A is k-normal and B is can K-normal that AB us cab k-normal, for example, any can k-normal matrix $C = HU = UH^T$ and $A = H$, then $AC = H^2 U = HUH^T = U(H^T)^2$ is con k-normal, the following theorem clarify this matter.

Theorem 4.1:

If A is k-normal and B is con-k-normal, then AB is con k-normal if and only if $ABB^* = BB^* A$ and $\bar{B}AA^* = A^T \bar{A}\bar{B}$ or $B\bar{A}A^* = A^* AB$.

Proof:

By the condition, then $(AB)(AB)^* = ABB^* A^* = BB^* AA^*$ and $(AB)^T (\overline{AB}) = B^T A^T \bar{A}\bar{B} = B^T \bar{B}AA^*$ which are equal. Conversely, let AB be can k-normal and let $UAU^* = D = d_{k(1)} I_{k(1)} + d_{k(2)} I_{k(2)} + ... + d_{k(n)} I_{k(n)}$, where $d_{k(i)}\bar{d}_{k(i)} > d_{k(j)}\bar{d}_{k(j)}$, $i > j$ .let $UB^T U^T = B_{k(1)} = (b_{k(i)k(j)})$.

If $(AB)(AB)^{*} = ABB^{*}A^{*} = AB^{T}\bar{B}A^{*} = (AB)^{T}(\overline{AB}) = B^{T}A^{T}\bar{A}\bar{B} = B^{T}\bar{A}A^{T}\bar{B}$, then

$$(UAU^{*})(UB^{T}U^{T}\bar{U}\bar{B}U^{*})(UA^{*}U^{*}) = (UB^{T}U^{T})(\bar{U}\bar{A}U^{T}\bar{U}A^{T}U^{T})(\bar{U}\bar{B}U^{*})$$

So that $DB_{k(1)}B_{k(1)}^{*}D^{*} = B_{k(1)}\bar{D}DB_{k(1)}^{*}$.

Equating k-diagonal elements an each side of this relation,

$$\sum_{j=1}^{n} d_{k(i)}\bar{d}_{k(i)}b_{k(i)k(j)}\bar{b}_{k(i)k(j)} = \sum_{j=1}^{n} d_{k(j)}\bar{d}_{k(j)}b_{k(i)k(j)}\bar{b}_{k(i)k(j)}.\ i=1,2,...,n \text{ (or)}$$

$$\sum_{j=1}^{n}\left(d_{k(i)}\bar{d}_{k(i)} - d_{k(j)}\bar{d}_{k(j)}\right)b_{k(i)k(j)}\bar{b}_{k(i)k(j)} = 0$$

Let $d_{k(1)}\bar{d}_{k(1)} = d_{k(2)}\bar{d}_{k(2)} = ...d_{k(l)}d_{k(l)} > d_{k(l+1)}d_{k(l+1)}$ .Then $b_{k(i)(j)} = 0$ for $i=1,2,...l$ and $j = l+1, l+2,..., n.$

Since $B_{k(1)}$ is con k-normal,

$$\sum_{j=1}^{n} b_{k(i)k(j)}\bar{b}_{k(i)k(j)} = \sum_{j=1}^{n} b_{k(j)k(i)}\bar{b}_{k(j)k(i)} \text{ for } i=1,2,..n$$

On adding the first $'l'$ of these equations and cancelling, $b_{k(i)k(j)} = 0$ for $i = l+1, l+2,...n$ and $i = l+1, l+2,...n$ in this manner if $D = r_{k(1)}D_{k(i)} + ... + r_{k(t)}D_{k(t)}$ with $r_{k(i)} > r_{k(i+!)}$ and $D_{k(i)}$ is k-unitary, then $B_{k(1)} = C_{k(1)} + C_{k(2)} + ... + c_{k(t)}$ conformal to D. since $r_{k(i)}D_{k(i)}D_{k(i)}^{*}r_{k(i)}C_{k(i)}^{T} = r_{k(i)}^{2}C_{k(i)}^{T} = C_{k(i)}^{T}r_{k(i)}^{2} = C_{k(i)}^{T}r_{k(i)}D_{k(i)}D_{k(i)}^{*}r_{k(i)}$ for all i, $DD^{*}B_{k(1)}^{T} = B_{k(1)}^{T}DD^{*}$ and so $U^{*}DD^{*}UU^{*}B_{k(i)}^{T}\bar{U} = U^{*}B_{k(1)}^{T}\bar{U}U^{T}DD^{*}U$. $A^{*}AB = BA^{T}\bar{A}$ or $AA^{*}B = BA^{T}\bar{A}$ or $A^{T}\bar{A}\bar{B} = \bar{B}AA^{*}$

Also $D(B_{k(1)}B_{k(1)}^{*}D^{*}) = B_{k(1)}\bar{D}DB_{k(1)}^{*} = \bar{D}DB_{k(1)}^{*} = D(\bar{D}B_{k(1)}\bar{B}_{k(1)}^{*})$ so that $C_{k(i)}C_{k(i)}^{*}(r_{k(i)}\bar{D}_{k(i)}) = (r_{k(i)}\bar{D}_{k(i)})C_{k(i)}C_{K(i)}^{*}$ for $i=1,2,..t$

(If $r_{k(t)} = 0$, this is stile true and $D_{k(t)}$ may be chosen to be the identify matrix.)

Therefore, $B_{k(1)}B_{k(1)}^{*}D^{*} = D^{*}B_{k(1)}B_{k(1)}^{*}$ and

$UB^{T}U^{T}\bar{U}\bar{B}U^{*}UA^{*}U^{*} = UA^{*}U^{*}UB^{T}U^{T}\bar{U}\bar{B}_{k(1)}U^{*}$ so $B^{T}\bar{B}A^{*} = A^{*}B^{T}B$ or $AB^{T}\bar{B} = B^{T}\bar{B}A$.

Corollary 4.2:

Let A be k-normal, B can k-normal, if AB is con-k-normal, then $B\bar{A}$ is con k-normal and conversely.

Proof:

By theorem7, $UAU^{*}UBU^{T} = DB_{k(1)}^{T}$ is con k-normal, and if $D = D_{k(r)}D_{k(u)}, D_{k(r)}$ real and $D_{k(u)}$ is k-unitary, then since $\bar{D}_{k(u)} = D_{k(u)}^{*}, D_{k(u)}^{*}(DB_{k(1)}^{T})\bar{D}_{k(u)} = D_{k(r)}B_{k(1)}^{T}\bar{D}_{k(u)} = B_{k(1)}^{T}D_{k(r)}\underline{D}_{k(u)} = B_{k(1)}^{T}\underline{D}$ is con k-normal as are $UBU^{T}\bar{U}\ \bar{A}U^{T}$ and $B\bar{A}$. Conversely

If A is k-normal and B is con k-normal, $B\bar{A}$ is con k-normal if and only if AB is con k-normal if and only if $(B^{T}\bar{B})A = A(BB^{*})$ and $(A^{T}\bar{A})\bar{B} = \bar{B}(AA^{*})$.

Therefore, if A is k-normal and B is con k-normal, BA is con k-normal if and only if $(B^{T}\bar{B})\bar{A} = \bar{A}(BB^{*})$ and $(A^{*}A)\bar{B} = \bar{B}(\bar{A}A^{T})$ is replace A by $\bar{A}$ in the preceding, or $(B^{*}B)A = A(\bar{B}B^{T})$ and $(A^{*}A)\bar{B} = \bar{B}(\bar{A}A^{T})$, thus exhibiting the fact that when AB is con k-normal, BA is not necessary so.

Theorem 4.3:

If $A = PW = WP$ is k-normal and $B = LV = VL^{T}$ is con k-normal (where P & L and k-hermitian and W and V are k-unitary) then AB is con k-normal if and only if $PL = LP$, $PV = VP^{T}$ and $WL = LW$.

Proof:

If three relations hold, then $AB = PWLV = PLWV$ on one hand, and $AB = WPLV = WLPV = WLVP^{T} = WVL^{T}P^{T} = WV(PL)^{T}$ con k-normal since PL is k-hermitian and WV is k-unitary.

Conversely, let $A = U^{*}DU = (U^{*}D_{k(r)}U)(U^{*}D_{k(r)}U) = PW$

and $B = U^{*}B_{k(1)}^{T}\bar{U} = (U^{*}L_{k(1)}U)(U^{*}V_{k(1)}\bar{U}) = LV = VL^{T}$

where $L_{k(i)}$ and $V_{k(i)}$ are k-hermit ion and k-unitary and direct sums conformable to $B_{k(1)}^{T}$ and D.

A direct check shows that $PL = LP$ and $PV = VP^T$, also $WL = U^* D_{k(u)} L_{k(1)} U = U^* L_{k(1)} D_{k(u)} U = LW$ since $D_{k(u)} B_{k(1)} B^*_{k(1)} = B_{k(1)} B^*_{k(1)} D_{k(u)}$ implies $D_{k(u)} L_{k(1)} = L_{k(1)} D_{k(u)}$

Note:

A sufficient condition for the simultaneously reduction of A and B is given by the following.

Theorem 4.4:

If A is k-normal, B is con k-normal and $AB = BA^T$ then $WAW^* = D$ and $WB^T W = F$, the k-normal form of theorem 1, where W is a k-unitary matrix, also AB is con k-normal.

Proof:

Let $UAU^* = D$, k-diagonal and $UBU^T = B_{k(2)}$ which is con k-normal. Then $AB = BA^T$ implies $D\,B_{k(2)} = UAU^* UBU^T = UBU^T \bar{U} A^T U^T = B_{k(2)} D^T = B_{k(2)} D$. Let $D = C_{k(1)} I_{k(1)} + C_{k(2)} I_{k(2)} + ... + C_{k(n)} I_{k(n)}$, where the $C_{k(i)}$ are complex and $C_{k(i)} \neq C_{k(j)}$ for $i \neq j$ and $B_{k(2)} = C_{k(1)} + ... + C_{k(n)}$. Let $V_{k(i)}$ be k-unitary such that $V_{k(i)} C_{k(i)} V^T_{k(i)} = F_{k(i)} =$ the real k-normal form of theorem1, and let $V = V_{k(1)} + V_{k(2)} + ... + V_{k(n)}$. Then $VUAU^* V^* = D$, $VUBU^T V^T = F =$ a direct sum of the $F_{k(i)}$. Also, $AB = BA^T$ implies $B^T A^T = AB^T$ and so $ABB^* A^* = AB^T \bar{B} A^* = B^T A^T \bar{A}\bar{B} = (AB)^T (\overline{AB})$.

It is also possible for the product of two k-normal matrices A and B to the con k-normal, if $U = HU = UH^T$ is con-k-normal and if $A = U$ and $B = H$ this is so or if $LV = VL^T$ is con-k-normal and if $A = UL = LU$ is k-normal with L k-hermit ion and V and U is k-unitary , for $B = V$, $AB = (UL)V = L(UV) = (UV)L^T$ con-k-normal .

But if in the first example, $U^2 H$ is not k-normal then HU is not con k-normal so that BA is not necessarily con k-normal through of theorem2 can be obtained which states the following if A is k-normal, then AB and AB^T one con k-normal if $ABB^* = B^T \bar{B} A$, $BB^* A = AB^T \bar{B}$ and $\bar{B} AA^* = A^T \bar{A}\bar{B}$. (The proof is not included here because of is similarity to that above). When B is con-k-normal, two of these conditions merge into due in theorem7.

It is possible for the product of two con-k-normal matrices to be con-k-normal, but no such single analogous necessary and sufficient conditions of exhibited above are available.

These may be seen of follows. Two non-real complex commutative matrices $M = M^T \& N = N^T$ can form a con k-normal (and non real symmetric) matrix MN (such that NM is also con k-normal) which need not be k-normal be k-normal. Then two symmetric matrices.

$$x = \begin{bmatrix} i & i+i \\ 1+l & -i \end{bmatrix}, \quad y = \begin{bmatrix} 1+2i & 3-4i \\ 3-4i & -(1+2i) \end{bmatrix}$$

Are such that Z is real, k-normal and con k-normal (and not symmetric). Finally, if U and V are two complex k-unitary matrices of the same order, they can to chosen so UV is non-real complex, k-normal and con k-normal. If $A = M + X + U$ and $B = N + Y + V$, $AB = MN + XY + UV$ where A and B are con k-normal as in AB(but not symmetric). A single impaction of these matrices shows that relations on the order of $(B^T \bar{B})A = A(BB^*) = (BB^*)$ and $(A^T \bar{A})\bar{B} = (AA^*)\bar{B} = \bar{B}(AA^*)$ do not necessarily hold, these are sufficient, however, to guarantee that AB is con k-normal (as direct verification from the definition will show).

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