Vertex- Edge Dominating Sets and Vertex-Edge Domination Polynomials of Paths

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ABSTRACT

Let G = (V, E) be a simple Graph. A set $S \subseteq V(G)$ is a vertex-edge dominating set (or simplyve-dominating set) if for all edges $e \in E(G)$, there exist a vertex $v \in S$ such that v dominates e. In this paper, we study the concept of vertex-edge domination polynomial of the path P_n. The vertex-edge domination polynomial of P_n is $D_{ve}(P_n, x) = \sum_{i=\lceil \frac{n-1}{4} \rceil}^{|V(G)|} d_{ve}(P_n, i)x^i$, where $d_{ve}(P_n, i)$ is the number of vertex edge dominating sets of P_n with cardinality i. We obtain some properties of $D_{ve}(P_n, x)$ and its co-efficients. Also, we calculate the recursive formula to derive the vertex-edge domination polynomials of paths.

Keywords: Path, vertex-edge dominating sets, vertex-edge domination polynomial, vertex-edge domination number.

1. Introduction

Let G = (V, E) be a simple graph of order |V| = n. For any vertex $v \in V$, the open neighbourhood of v is the set N(v) = $\{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is the set N[v] = $N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G, if N[S] = V, or equivalently, every vertex in V\S is adjacent to atleast one vertex in S. The domination number of a graph, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. A set of vertices in a Graph G is said to be a vertex-edge dominating set, if for all edges $e \in E(G)$ there exists a vertex $v \in S$ such that v dominates e. Otherwise for a graph G = (V, E), a vertex $u \in V(G)$ ve-dominates an edge $vw \in E(G)$ if (i) u = v or u = w (u is incident to vw) or (ii) uv or uw is an edge in G (u is incident to an edge is adjacent to vw).

The minimum cardinality of a ve-dominating set of G is called the vertex-edge domination number of G, and is denoted by $\gamma_{ve}(G)$. A path is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2 let P_n be the path with n vertices.

In the next section we construct the families of the vertex-edge dominating sets of paths by recursive method. In section 3, we use the results obtained in section 2 to study the vertex-edge domination polynominal of paths.

We use the notation $\lceil x \rceil$, for the smallest integer greater than or equal to *x*; also we denote the set {1, 2, . . . , n} by [n], throughout this paper.

2. Vertex-edge dominating sets of paths

Let $D_{ve}(P_n, i)$ be the family of vertex-edge dominating sets of P_n with cardinality i.

Lemma: 2.1 $\gamma_{ve}(P_n) = \left\lceil \frac{n-1}{4} \right\rceil$, n > 1

Lemma: 2.2 $D_{ve}(P_j, i) = \phi$, if and only if i > j or $i < \left\lceil \frac{j-1}{4} \right\rceil$.

Proof:

As Pj contains j vertices, any member of $D_{ve}(P_j, i)$ contains atmost j vertices.

Therefore, $D_{ve}(P_j, i) = \phi$ for i > j.

Also, since $\left\lceil \frac{j-1}{4} \right\rceil$ is minimum cardinality of a vertex-edge dominating set, there is no vertex-edge dominating set of cardinality less than $\left\lceil \frac{j-1}{4} \right\rceil$.

Therefore, $D_{ve}(P_j, i) = \phi$ for $i < \left\lceil \frac{j-1}{4} \right\rceil$.

Lemma: 2.3

If a Graph G contains a simple path of length 5k – 1, then every vertex-edge dominating set of G must contain atleast k vertices of the path.

Proof:

G contains a path with 5k vertices. A single vertex of P_{5k} covers 5 vertices in case of vertex-edge dominating set.

Therefore, a minimum of k vertices covers the entire 5k vertices in P_{5k}. As P_{5k} is a part of G, any vertex-edge dominating sets contain atleast k vertices.

Lemma: 2.4

If $Y \in D_{ve}(P_{n-5, i-1})$, and there exists $x \in [n]$ such that $Y \cup \{x\} \in D_{ve}(P_{n, i})$, then $Y \in D_{ve}(P_{n-4, i-1})$.

Proof:

Suppose $Y \notin D_{ve}(P_{n-4, i-1})$.

Since $Y \in D_{ve}(P_{n-5, i-1})$, Y contains atleast one vertex labeled n – 7 or n – 6 or n – 5 if $n-5 \in Y$, then $Y \in D_{ve}(P_{n-4, i-1})$ a contradiction. Hence, $n-6 \in Y$. If $n-6 \in Y$, then $Y \in D_{ve}(P_{n-4, i-1})$ a contradiction. Hence $n-7 \in Y$, n – 5 or $n-6 \notin Y$, Now we take $n-7 \in Y$. To prove $Y \in D_{ve}(P_{n-4, i-1})$ suppose, $Y \notin D_{ve}(P_{n-4, i-1})$ since $Y \notin D_{ve}(P_{n-4, i-1})$ therefore $n-7 \in Y$.

If we take any element x in [n], then it will cover atmost 5 vertices. Hence $Y \cup \{x\}$ will be in $D_{ve}(P_{m, i})$ for $m \leq n-1$ and $Y \cup \{x\} \notin D_{ve}(P_{n, i})$ a contradiction. ∴ our assumption is wrong

∴ $Y \in D_{ve}(P_{n-4, i-1})$

Lemma: 2.5

(i) If $D_{ve}(P_{n-1, i-1}) = D_{ve}(P_{n-4, i-1}) = \phi$, then $D_{ve}(P_{n-2, i-1}) = D_{ve}(P_{n-3, i-1}) = \phi$.

(ii) If $D_{ve}(P_{n-1, i-1}) \neq \phi$ and $D_{ve}(P_{n-4, i-1}) \neq \phi$, then $D_{ve}(P_{n-2, i-1}) \neq \phi$ and $D_{ve}(P_{n-3, i-1}) \neq \phi$.

(iii) If $D_{ve}(P_{n-1, i-1}) = D_{ve}(P_{n-2, i-1}) = D_{ve}(P_{n-3, i-1}) = D_{ve}(P_{n-4, i-1}) = \phi$, then $D_{ve}(P_{n, i}) = \phi$.

Proof:

(i) Since, $D_{ve}(P_{n-1, i-1}) = \phi$, by lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-2}{4} \right\rceil$.

Also since, $D_{ve}(P_{n-4, i-1}) = \phi$, by lemma 2.2, $i-1 > n-4$ or $i-1 < \left\lceil \frac{n-5}{4} \right\rceil$.

If $i-1 < \left\lceil \frac{n-5}{4} \right\rceil$, then $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$.

If $i-1 > n-1$, then $i-1 > n-3$ in this case,

$D_{ve}(P_{n-3, i-1}) = \phi$. Suppose $i-1 < \left\lceil \frac{n-5}{4} \right\rceil$.

$$\therefore i-1 < \left\lceil \frac{n-3}{4} \right\rceil .$$

If $i-1 > n-1$, then $i-1 > n-2$. In this case,

$D_{ve}(P_{n-2, i-1}) = \phi$

(ii) $D_{ve}(P_{n-1, i-1}) \neq \phi$, $D_{ve}(P_{n-4, i-1}) \neq \phi$

$$\therefore \left\lceil \frac{n-2}{4} \right\rceil \leq i-1 \leq n-1 \text{ and}$$

$$\left\lceil \frac{n-5}{4} \right\rceil \leq i-1 \leq n-4$$

$i-1 \leq n-4 \Rightarrow i-1 \leq n-3$. $i-1 \geq \left\lceil \frac{n-5}{4} \right\rceil$

$$\Rightarrow i-1 \geq \left\lceil \frac{n-4}{4} \right\rceil$$

∴ $D_{ve}(P_{n-3, i-1}) \neq \phi$. $i-1 \leq n-4 \Rightarrow i-1 \leq n-2$

$$i-1 \geq \left\lceil \frac{n-5}{4} \right\rceil \Rightarrow i-1 \geq \left\lceil \frac{n-3}{4} \right\rceil$$

∴ $D_{ve}(P_{n-2, i-1}) \neq \phi$.

(iii) Suppose that $D_{ve}(P_{n, i}) \neq \phi$, Let $Y \in D_{ve}(P_{n, i})$ Then, atleast one vertex labeled n or n – 1 or n – 2 is in Y. If $n \in Y$, then atleast one vertex labeled n – 3 or n – 4 or n – 5 is in Y. Then $Y - \{n\} \in D_{ve}(P_{n-3, i-1})$, a contradiction. If $n-1 \in Y$, then atleast one vertex labeled n – 4 or n – 5 or n – 6 is in Y.

∴ $Y - \{n-1\} \in D_{ve}(P_{n-4, i-1})$, a contradiction. Similarly if $n-2 \in Y$, then atleast one vertex labeled n – 5 or n – 6 or n – 7 is in Y. hence $Y - \{n-2\} \in D_{ve}(P_{n-4, i-1})$ a contradiction.

if $n-5 \in Y$, then $Y - \{n-2\} \in D_{ve}(P_{n-4, i-1})$ a contradiction.

if $n-6 \in Y$, then $Y - \{n-2\} \in D_{ve}(P_{n-4, i-1})$ a contradiction.

if $n-7 \in Y$, then by making some proper rearrangements (like $Y - \{n-7, n-2\} \cup \{n-6\}$) we obtain an element in $D_{ve}(P_{n-4, i-1})$ a contradiction.

Lemma: 2.6

If $D_{ve}(P_{n, i}) \neq \phi$, then

(i) $D_{ve}(P_{n-1, i-1}) = D_{ve}(P_{n-2, i-1}) = D_{ve}(P_{n-3, i-1}) = \phi$ and $D_{ve}(P_{n-4, i-1}) \neq \phi$ if and only if $n = 4k+5$ and $i = k+1$, for some $k \in N$.

(ii) $D_{ve}(P_{n-2, i-1}) = D_{ve}(P_{n-3, i-1}) = D_{ve}(P_{n-4, i-1}) = \phi$ and $D_{ve}(P_{n-1, i-1}) \neq \phi$ if and only if $i = n$.

(iii) $D_{ve}(P_{n-1, i-1}) = \phi$, $D_{ve}(P_{n-2, i-1}) \neq \phi$,

$D_{ve}(P_{n-3}, {}_{i-1}) \neq \phi$ and $D_{ve}(P_{n-4}, {}_{i-1}) \neq \phi$ if and only if, $n = 4k + 3$ and $i = \left\lceil \frac{4k+3}{4} \right\rceil$ for $k \in N$

(iv) $D_{ve}(P_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(P_{n-2}, {}_{i-1}) \neq \phi$,
$D_{ve}(P_{n-3}, {}_{i-1}) \neq \phi$ and $D_{ve}(P_{n-4}, {}_{i-1}) = \phi$ if and only if $i = n-2$.

(v) $D_{ve}(P_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(P_{n-2}, {}_{i-1}) \neq \phi$,
$D_{ve}(P_{n-3}, {}_{i-1}) \neq \phi$ and $D_{ve}(P_{n-4}, {}_{i-1}) \neq \phi$, if and only if, $\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i \leq n-3$.

(vi) $D_{ve}(P_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(P_{n-2}, {}_{i-1}) \neq \phi$,
$D_{ve}(P_{n-3}, {}_{i-1}) = \phi$ and $D_{ve}(P_{n-4}, {}_{i-1}) = \phi$, if and only if, $i = n-1$.

Proof:

(i) $(\Rightarrow)$

Since, $D_{ve}(P_{n-1}, {}_{i-1}) = D_{ve}(P_{n-2}, {}_{i-1})$
$= D_{ve}(P_{n-3}, {}_{i-1}) = \phi$, by lemma 2.2,

$\left\lceil \frac{n-2}{4} \right\rceil > i-1$ or $i-1 > n-1$.

$\left\lceil \frac{n-3}{4} \right\rceil > i-1$ or $i-1 > n-2$

$\left\lceil \frac{n-4}{4} \right\rceil > i-1$ or $i-1 > n-3$

Thus, $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ or $i-1 > n-1$.

If $i-1 > n-1$, then $i > n$

$\therefore$ $D_{ve}(P_n, {}_i) = \phi$, a contradiction.

therefore, $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$.

$\therefore i < \left\lceil \frac{n-4}{4} \right\rceil + 1$. Since,

$D_{ve}(P_n, {}_i) \neq \phi$, $i \geq \left\lceil \frac{n-1}{4} \right\rceil$

$\left\lceil \frac{n-1}{4} \right\rceil \leq i$ and $i < \left\lceil \frac{n-4}{4} \right\rceil + 1$

$\therefore \left\lceil \frac{n-1}{4} \right\rceil \leq i < \left\lceil \frac{n-4}{4} \right\rceil + 1$ ----------(1)

when $n = 4k + 5$, $\left\lceil \frac{n-1}{4} \right\rceil = k + 1 = \left\lceil \frac{n-4}{4} \right\rceil$

Therefore, $k + 1 \leq i < k + 2$. Therefore $i = k + 1$. For $n \neq 4k + 5$, $\left\lceil \frac{n-1}{4} \right\rceil = \left\lceil \frac{n-4}{4} \right\rceil + 1$

Therefore (1) does not occur. Therefore, only possibility is $n = 4k + 5$ and $i = k + 1$.

Conversely, Assume $n = 4k + 5$, and $i = k + 1$, $k \in N$

$i - 1 = k = \frac{n-5}{4} < \frac{n-2}{4}$ $\therefore i - 1 < \left\lceil \frac{n-2}{4} \right\rceil$.

$D_{ve}(P_{n-1}, {}_{i-1}) = \phi$

also, $i - 1 = \frac{n-5}{4} < \frac{n-3}{4}$ $\therefore i - 1 < \left\lceil \frac{n-3}{4} \right\rceil$

$\therefore D_{ve}(P_{n-2}, {}_{i-1}) = \phi$ $\therefore i - 1 = \frac{n-5}{4} < \frac{n-4}{4}$ $\therefore$ $D_{ve}(P_{n-3}, {}_{i-1}) = \phi$

Also, since $i - 1 = \frac{n-5}{4}$, $D_{ve}(P_{n-4}, {}_{i-1}) \neq \phi$

(ii) $D_{ve}(P_{n-2}, {}_{i-1}) = D_{ve}(P_{n-3}, {}_{i-1})$
$= D_{ve}(P_{n-4}, {}_{i-1}) = \phi$ $\therefore$ by lemma, 2.2

$i - 1 > n - 2$ or $i - 1 < \left\lceil \frac{n-5}{4} \right\rceil$ if $i - 1$

$< \left\lceil \frac{n-5}{4} \right\rceil$, then $i - 1 < \left\lceil \frac{n-2}{4} \right\rceil$

$\therefore D_{ve}(P_{n-1}, {}_{i-1}) = \phi$, a contradiction

$\therefore i - 1 > n - 2 \therefore i > n - 1$. Also, since, $D_{ve}(P_{n-1}, {}_{i-1}) \neq \phi$

$\therefore$ $i - 1 \leq n - 1 \therefore i \leq n \therefore n - 1 < i$ and $i \leq n$. $n - 1 < i \leq n \therefore i = n$

Conversly, If $i = n$, then by lemma 2.2.

$D_{ve}(P_{n-2}, {}_{i-1}) = D_{ve}(P_{n-2}, {}_{n-1}) = \phi$, $D_{ve}(P_{n-3}, {}_{i-1})$
$= D_{ve}(P_{n-3}, {}_{n-1}) = \phi$

$D_{ve}(P_{n-4}, {}_{i-1}) = D_{ve}(P_{n-4}, {}_{n-1}) = \phi$, $D_{ve}(P_{n-1}, {}_{i-1})$
$= D_{ve}(P_{n-1}, {}_{n-1}) \neq \phi$

(iii) Since $D_{ve}(P_{n-1}, {}_{i-1}) = \phi$, by lemma 2.2

$i - 1 > n - 1$ or $i - 1 < \left\lceil \frac{n-2}{4} \right\rceil$

if $i-1 > n-1$, then $i-1 > n-2$ and by lemma 2.2, $D_{ve}(P_{n-2,\, i-1}) = \phi$, a contradiction

So, $i < \left\lceil \frac{n-2}{4} \right\rceil + 1$ Since, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$

$$i-1 \geq \left\lceil \frac{n-3}{4} \right\rceil \quad \therefore \left\lceil \frac{n-3}{4} \right\rceil \leq i-1$$

$$\left\lceil \frac{n-3}{4} \right\rceil + 1 \leq i \quad \text{and} \quad i < \left\lceil \frac{n-2}{4} \right\rceil + 1$$

$$\therefore \left\lceil \frac{n-3}{4} \right\rceil + 1 \leq i < \left\lceil \frac{n-2}{4} \right\rceil + 1 \qquad \text{- - - - - (1)}$$

Similarly ,

$$\left\lceil \frac{n-4}{4} \right\rceil + 1 \leq i < \left\lceil \frac{n-2}{4} \right\rceil + 1 \qquad \text{--------(2)}$$

$$\left\lceil \frac{n-5}{4} \right\rceil + 1 \leq i < \left\lceil \frac{n-2}{4} \right\rceil + 1 \qquad \text{--------(3)}$$

(1), (2) and (3) hold good when $n = 4k + 3$ and

$$i = k + 1 = \left\lceil \frac{4k+3}{4} \right\rceil.$$

When $n \neq 4k + 3$, we obtain that

$$\left\lceil \frac{n-3}{4} \right\rceil + 1 = \left\lceil \frac{n-2}{4} \right\rceil + 1$$

or $\left\lceil \frac{n-4}{4} \right\rceil + 1 = \left\lceil \frac{n-2}{4} \right\rceil + 1$ or

$$\left\lceil \frac{n-5}{4} \right\rceil + 1 = \left\lceil \frac{n-2}{4} \right\rceil + 1$$

which are not possible.by (1), (2) and (3) Therefore, the only possibility is $n = 4k + 3$ and $i = k + 1$. Conversly, assume that $n = 4k + 3$ and $i = k + 1$.

$$\left\lceil \frac{n-2}{4} \right\rceil = \left\lceil \frac{4k+3-2}{4} \right\rceil = \left\lceil \frac{4k+1}{4} \right\rceil$$

$> k = i-1$. Therefore , $i-1 < \left\lceil \frac{n-2}{4} \right\rceil$

Therefore, $D_{ve}(P_{n-1,\, i-1}) = \phi$

$$\left\lceil \frac{n-3}{4} \right\rceil = \left\lceil \frac{4k+3-3}{4} \right\rceil = k = i-1$$

Therefore , $D_{ve}(P_{n-2,\, i-1}) \neq \phi \left\lceil \frac{n-4}{4} \right\rceil$

$$= \left\lceil \frac{4k+3-4}{4} \right\rceil = \left\lceil \frac{4k-1}{4} \right\rceil \leq k = i-1$$

$i-1 > \left\lceil \frac{n-4}{4} \right\rceil$. Therefore, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$

$$\left\lceil \frac{n-5}{4} \right\rceil = \left\lceil \frac{4k+3-5}{4} \right\rceil = \left\lceil \frac{4k-2}{4} \right\rceil$$

$$\leq k = i-1, \therefore i-1 > \left\lceil \frac{n-5}{4} \right\rceil$$

Therefore, $D_{ve}(P_{n-4,\, i-1}) \neq \phi$.

(iv) Since $D_{ve}(P_{n-4,\, i-1}) = \phi$, by lemma 2.2.

$$i-1 > n-4 \text{ or } i-1 < \left\lceil \frac{n-5}{4} \right\rceil$$

Since, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$, by lemma 2.2.

$$\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 \leq n-3$$

Therefore , $\left\lceil \frac{n-4}{4} \right\rceil + 1 \leq i \leq n-2$

Therefore , $i-1 < \left\lceil \frac{n-5}{4} \right\rceil$ is not possible.

$\therefore i-1 > n-4$. $\therefore i > n-3$. Therefore, $i \geq n-2$. But $i \leq n-2$

Therefore, $i = n-2$. Conversly, if $i = n-2$, then by lemma 2.2,

$$D_{ve}(P_{n-1,\, i-1}) = D_{ve}(P_{n-1,\, n-2-1})$$
$$= D_{ve}(P_{n-1,\, n-3}) \neq \phi$$
$$D_{ve}(P_{n-2,\, i-1}) = D_{ve}(P_{n-2,\, n-2-1})$$
$$= D_{ve}(P_{n-2,\, n-3}) \neq \phi$$
$$D_{ve}(P_{n-3,\, i-1}) = D_{ve}(P_{n-3,\, n-2-1})$$
$$= D_{ve}(P_{n-3,\, n-3}) \neq \phi$$
$$D_{ve}(P_{n-4,\, i-1}) = D_{ve}(P_{n-4,\, n-2-1})$$
$$= D_{ve}(P_{n-4,\, n-3}) = \phi$$

(v) We have, $D_{ve}(P_{n-1,\, i-1}) \neq \phi$, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$ and $D_{ve}(P_{n-4,\, i-1}) \neq \phi$.

Since, $D_{ve}(P_{n-1,\,i-1}) \neq \phi$. $\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 \leq n-1$

Therefore , $\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i \leq n$ ------------(i)

Similarly, we have

$\left\lceil \frac{n-3}{4} \right\rceil + 1 \leq i \leq n-1$ ------------(ii)

$\left\lceil \frac{n-4}{4} \right\rceil + 1 \leq i \leq n-2$ -----------(iii)

$\left\lceil \frac{n-5}{4} \right\rceil + 1 \leq i \leq n-3$ ------------(iv)

combining (i), (ii), (iii) and (iv), we have,

$$\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i \leq n-3$$

Conversly, Assume $\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i \leq n-3$

From this, we obtain, $\left\lceil \frac{n-5}{4} \right\rceil \leq i-1 \leq n-4$,

$\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 \leq n-3$,

$\left\lceil \frac{n-3}{4} \right\rceil \leq i-1 \leq n-2$,

$\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 \leq n-1$.

Hence ,

$D_{ve}(P_{n-1,\,i-1}) \neq \phi$, $D_{ve}(P_{n-2,\,i-1}) \neq \phi$, $D_{ve}(P_{n-3,\,i-1}) \neq \phi$, and $D_{ve}(P_{n-4,\,i-1}) \neq \phi$.

(vi)

$D_{ve}(P_{n-1,\,i-1}) \neq \phi$, then by lemma 2.2

$i-1 \leq n-1$ and $i-1 \geq \left\lceil \frac{n-2}{4} \right\rceil$ ------- (1)

$D_{ve}(P_{n-2,\,i-1}) \neq \phi$, then by lemma 2.2

$i-1 \leq n-2$ and $i-1 \geq \left\lceil \frac{n-3}{4} \right\rceil$ ------- (2)

$D_{ve}(P_{n-3,\,i-1}) = \phi$, then by lemma 2.2

$i-1 > n-3$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ ------- (3)

$D_{ve}(P_{n-4,\,i-1}) = \phi$, then by lemma 2.2

$i-1 > n-4$ or $i-1 < \left\lceil \frac{n-5}{4} \right\rceil$ ------- (4)

from (1),

$\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 \leq n-1$ if $i-1 = \left\lceil \frac{n-2}{4} \right\rceil$

$\therefore D_{ve}(P_{n-3,i-1}) = D_{ve}(P_{n-3,\left\lceil \frac{n-2}{4} \right\rceil}) \neq \phi$, contradiction.

$\therefore\ i-1 \neq \left\lceil \frac{n-2}{4} \right\rceil \therefore i-1 \leq n-1 \therefore i \leq n \therefore i = n,$ $n-1, \ldots$

if $i = n$, $D_{ve}(P_{n-1,\,i-1}) = D_{ve}(P_{n-1,\,n-1}) \neq \phi$,

$D_{ve}(P_{n-2,\,i-1}) = D_{ve}(P_{n-2,\,n-1}) = \phi$ a contradiction.

$\therefore i \neq n$,

if $i = n-1$,

$D_{ve}(P_{n-1,\,i-1}) = D_{ve}(P_{n-1,\,n-2}) \neq \phi$, $D_{ve}(P_{n-2,\,i-1}) = D_{ve}(P_{n-2,\,n-2}) \neq \phi$

$D_{ve}(P_{n-3,\,i-1}) = D_{ve}(P_{n-3,\,n-2}) = \phi$, $D_{ve}(P_{n-4,\,i-1}) = D_{ve}(P_{n-4,\,n-2}) = \phi$

if $i = n-2$,

$D_{ve}(P_{n-1,\,i-1}) = D_{ve}(P_{n-1,\,n-3}) \neq \phi$, $D_{ve}(P_{n-2,\,i-1}) = D_{ve}(P_{n-2,\,n-3}) \neq \phi$

$D_{ve}(P_{n-3,i-1}) = D_{ve}(P_{n-3,\,n-3}) \neq \phi$,

a contradiction $\therefore i \neq n-2, n-3, \ldots\ldots$

$\therefore i = n-1$ . . . . . (i)

From (3),

if $i-1 > n-3$, $i > n-2 \therefore i \geq n-1$,

$\therefore i = n-1, n, n+1, \ldots, i \neq n, n+1, \ldots$

$\therefore i = n-1$ . . . . (ii)

from (2),

$i-1 \leq n-2$, $i \leq n-1$, $i = n-1, n-2, \ldots,$

$\therefore i = n-1$ . . . . (iii) if $i = n-1$,

$\therefore$ (4) $\Rightarrow$ $n-1 > n-4$ is true. Conversly, if $i = n-1$

$\therefore$ $D_{ve}(P_{n-1,\,i-1}) = D_{ve}(P_{n-1,\,n-2}) \neq \phi$,

$D_{ve}(P_{n-2,\,i-1}) = D_{ve}(P_{n-2,\,n-2}) \neq \phi$,

$D_{ve}(P_{n-3,\,i-1}) = D_{ve}(P_{n-3,\,n-2}) = \phi$,

$D_{ve}(P_{n-4,\,i-1}) = D_{ve}(P_{n-4,\,n-2}) = \phi$

Hence the result.

Theorem 2.7

For every $n \geq 5$ and $i \geq \left\lceil \frac{n-1}{4} \right\rceil$

(i) If $D_{ve}(P_{n-1,\, i-1}) = D_{ve}(P_{n-2,\, i-1}) = D_{ve}(P_{n-3,\, i-1}) = \phi$ and $D_{ve}(P_{n-4,\, i-1}) \neq \phi$, then, $D_{ve}(P_{n,\, i}) = \{\{3, 7, 11, \ldots, n-6, n-2\}\}$

(ii) If $D_{ve}(P_{n-2,\, i-1}) = D_{ve}(P_{n-3,\, i-1}) = D_{ve}(P_{n-4,\, i-1}) = \phi$, and $D_{ve}(P_{n-1,\, i-1}) \neq \phi$, then $D_{ve}(P_{n,\, i}) = \{[n]\}$

(iii) If $D_{ve}(P_{n-1,\, i-1}) = \phi$, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$ and $D_{ve}(P_{n-4,\, i-1}) \neq \phi$, then $D_{ve}(P_{n,\, i}) = \{\{3, 7, 11, \ldots, n-4, n\}\} \cup \{\{n-2\} \cup X \mid X \in D_{ve}(P_{n-4,\, i-1})\}$
$\cup \{\{n-1\} \cup X \mid X \in D_{ve}(P_{n-3,\, i-1})\}$
$\cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-2,\, i-1})\}$

(iv) If $D_{ve}(P_{n-4,\, i-1}) = \phi$, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$ and $D_{ve}(P_{n-1,\, i-1}) \neq \phi$, then $D_{ve}(P_{n,\, i}) = \{[n] - \{x, y\} \mid x, y \in [n]\}$

(v) If $D_{ve}(P_{n-1,\, i-1}) \neq \phi$, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$ and $D_{ve}(P_{n-4,\, i-1}) \neq \phi$, then $D_{ve}(P_{n,\, i}) = \{\{n\} \cup X_1, \{n-1\} \cup X_2, \{n-2\} \cup X_3 \mid X_1 \in D_{ve}(P_{n-1,\, i-1}),$
$X_2 \in D_{ve}(P_{n-2,\, i-1}),\ X_3 \in D_{ve}(P_{n-3,\, i-1})\}$
$\cup \{\{n-2\} \cup X \mid X \in D_{ve}(P_{n-4,\, i-1}) \backslash D_{ve}(P_{n-3,\, i-1})\}$
$\cup \{\{n-1\} \cup X \mid X \in D_{ve}(P_{n-3,\, i-1}) \backslash D_{ve}(P_{n-2,\, i-1})\}$
$\cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-3,\, i-1}) \cap D_{ve}(P_{n-2,\, i-1})\}$

(vi) If $D_{ve}(P_{n-1,\, i-1}) \neq \phi$, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$, $D_{ve}(P_{n-3,\, i-1}) = \phi$, $D_{ve}(P_{n-4,\, i-1}) = \phi$ and then $D_{ve}(P_{n,\, i}) = \{[n] - \{x\} \mid x \in [n]\}$

Proof:

(i) If $D_{ve}(P_{n-1,\, i-1}) = D_{ve}(P_{n-2,\, i-1}) = D_{ve}(P_{n-3,\, i-1}) = \phi$, and $D_{ve}(P_{n-4,\, i-1}) \neq \phi$, then by lemma 2.6 (i), $n = 4k + 5$, $i = k + 1$ for some $k \in N$ $\therefore 4k = n - 5$

$$k = \frac{n-5}{4},\ i = k + 1 = \frac{n-5}{4} + 1$$

$$= \frac{n-5+4}{4} = \frac{n-1}{4}$$

$$\therefore D_{ve}(P_{n,\, i}) = D_{ve}\left(P_n, \frac{n-1}{4}\right)$$

Clearly, $\{3, 7, 11, \ldots, 4k-5, 4k-1, 4k+3\}$ is a vertex-edge dominating set with $\frac{4k+3+1}{4} = \frac{4k+4}{4} = k + 1 = i = \frac{n-1}{4}$ elements

Also, no other set of cardinality $\frac{n-1}{4}$ is a vertex-edge dominating set.

Therefore, $D_{ve}(P_{n,\, i}) = D_{ve}\left(P_n, \frac{n-1}{4}\right)$
$= \{\{3, 7, 11, \ldots, 4k-5, 4k-1, 4k+3\}\}$
$= \{\{3, 7, 11, \ldots, n-6, n-2\}\}$

$$\therefore D_{ve}(P_{n,\, i}) = D_{ve}\left(P_n, \frac{n-1}{4}\right)$$

$= \{\{3, 7, 11, \ldots, n-10, n-6, n-2\}\}$

(ii) We have, $D_{ve}(P_{n-2,\, i-1}) = D_{ve}(P_{n-3,\, i-1}) = D_{ve}(P_{n-4,\, i-1}) = \phi$, and $D_{ve}(P_{n-1,\, i-1}) \neq \phi$, by lemma 2.6 (ii) we have $i = n$. So, $D_{ve}(P_{n,\, i}) = D_{ve}(P_{n,\, n}) = \{\{1, 2, 3, \ldots, n\}\} = \{[n]\}$

(iii) We have $D_{ve}(P_{n-1,\, i-1}) = \phi$, $D_{ve}(P_{n-2,\, i-1}) \neq \phi$, $D_{ve}(P_{n-3,\, i-1}) \neq \phi$ and $D_{ve}(P_{n-4,\, i-1}) \neq \phi$, by lemma2.6 (iii), $n = 4k + 3$ and

$$i = \left\lceil \frac{4k+3}{4} \right\rceil = k + 1 \text{ for some } k \in N.$$

Since $X = \{3, 7, 11, \ldots, 4k-1\} \in D_{ve}(P_{4k+1}, k)$,
$X \cup \{4k+3\} \in D_{ve}(P_{4k+3}, k+1)$
If $X \in D_{ve}(P_{4k}, k)$, then $X \cup \{4k+2\} \in D_{ve}(P_{4k+3}, k+1)$
If $X \in D_{ve}(P_{4k-1}, k)$, then $X \cup \{4k+1\} \in D_{ve}(P_{4k+3}, k+1)$
If $X \in D_{ve}(P_{4k+3}, k)$, then $X \cup \{4k\} \in D_{ve}(P_{4k+3}, k+1)$
$\therefore$ we have, $\{\{3, 7, 11, \ldots, 4k-1, 4k+3\}\}$
$\cup \{X \cup \{4k+2\} \mid X \in D_{ve}(P_{4k}, k)\}$
$\cup \{X \cup \{4k+1\} \mid X \in D_{ve}(P_{4k-1}, k)\}$
$\cup \{X \cup \{4k\} \mid X \in D_{ve}(P_{4k+3}, k)\}$
$\subseteq D_{ve}(P_{4k+4}, k+1)$ ....(1)

ie, $\{\{3, 7, 11, \ldots, n-6, n-2\}\} \cup \{X \cup \{n-2\} \mid X \in D_{ve}(P_{n-3,\, i-1})\} \cup \{X \cup \{n-1\} \mid X \in D_{ve}(P_{n-4,\, i-1})\} \cup \{X \cup \{n\} \mid X \in D_{ve}(P_{n,\, i-1})\}$
$\subseteq D_{ve}(P_{n,\, i})$ ....1(a)

Now, let $Y \in D_{ve}(P_{4k+3}, k+1)$. Then, $4k+3$ or $4k+2$ or $4k+1$ is in Y.

If $4k + 3 \in Y$, then by lemma 2.3 atleast one vertex labeled $4k + 2$ or $4k + 1$ or $4k$ or $4k - 1$ is in Y.

If $4k + 2$ or $4k + 1$ or $4k$ is in Y, then

$Y - \{4k + 3\} \in D_{ve}(P_{4k+2}, k)$, a contradiction.

Since $D_{ve}(P_{4k+2}, k) = \phi$. Hence, $4k - 1 \in Y$ and $4k \notin Y$ and $4k + 1 \notin Y$, $4k + 2 \notin Y$.

Therefore, in this case $Y = X \cup \{4k + 3\}$ for some $X \in D_{ve}(P_{4k+1}, k)$.

ie $Y = \{3, 7, 11, \ldots, 4k - 1, 4k + 3\}$

Now, Suppose that $4k + 2 \in Y$ (and $4k + 3 \notin Y$) then by lemma 2.3 atleast one vertex labeled $4k + 1$ or $4k$ or $4k - 1$ or $4k - 2$ is in Y. If $4k + 1 \in Y$,

then $Y - \{4k + 2\} \in D_{ve}(P_{4k+1}, k) = \{\{3, 7, 11, \ldots, 4k - 1\}\}$. $\{4k + 1\} \in D_{ve}(P_{4k+1}, k)$. Therefore contradiction. Since $4k + 1 \notin X$ for any $X \in D_{ve}(P_{4k+1}, k)$. Therefore, $4k$, $4k - 1$, or $4k - 2$ is in Y, but $4k + 1\} \notin Y$. Thus $Y = X \cup \{4k + 2\}$ for some $X \in D_{ve}(P_{4k}, k)$. Now, suppose that $4k + 1 \in Y$. And $4k + 2 \notin Y$, $4k + 3 \notin Y$. by lemma 2.3, atleast one vertex labeled $4k$, $4k - 1$, $4k - 2$, or $4k - 3$ is in Y. If $4k \in Y$, then $Y - \{4k + 1\} \in D_{ve}(P_{4k}, k)$, a contradiction since $4k \notin X$.

$\therefore$ $4k - 1$ or $4k - 2$ or $4k - 3$ is in Y but $4k \notin Y$, Then $Y = X \cup \{4k + 1\} \in D_{ve}(P_{4k-1}, k)$.

$$\therefore D_{ve}(P_{4k+3}, k+1) \subseteq \{\{3, 7, 11, \ldots, 4k - 1, 4k + 3\}\} \cup \{X \cup \{4k + 1\} \mid X \in D_{ve}(P_{4k-1}, k)\} \cup \{X \cup \{4k + 2\} \mid X \in D_{ve}(P_{4k}, k)\} \cup \{X \cup \{4k + 3\} \mid X \in D_{ve}(P_{4k+1}, k)\}$$

$$\therefore D_{ve}(P_n, i) \subseteq \{\{3, 7, 11, \ldots, n - 4, n\}\} \cup \{X \cup \{n - 2\} \mid X \in D_{ve}(P_{n-4}, i - 1)\} \cup \{X \cup \{n - 1\} \mid X \in D_{ve}(P_{n-3}, i - 1)\} \cup \{X \cup \{n\} \mid X \in D_{ve}(P_{n-2}, i - 1)\} \text{---------(2)}$$

$\therefore$ from 1(a) and (2)

$$\therefore D_{ve}(P_n, i) = \{\{3, 7, 11, \ldots, n - 4, n\}\} \cup \{X \cup \{n - 2\} \mid X \in D_{ve}(P_{n-4}, i - 1)\} \cup \{X \cup \{n - 1\} \mid X \in D_{ve}(P_{n-3}, i - 1)\} \cup \{X \cup \{n\} \mid X \in D_{ve}(P_{n-2}, i - 1)\}$$

(iv) $D_{ve}(P_{n-4}, i - 1) = \phi$, $D_{ve}(P_{n-3}, i - 1) \neq \phi$
$D_{ve}(P_{n-2}, i - 1) \neq \phi$ and $D_{ve}(P_{n-1}, i - 1) \neq \phi$.
By lemma 2.6 (iv), $i = n - 2$

$\therefore D_{ve}(P_n, i) = D_{ve}(P_n, n - 2)$

If we have n vertices, we remove two vertices that will cover all the vertices.

$\therefore D_{ve}(P_n, i) = D_{ve}(P_n, n - 2) = \{[n] - \{x, y\} \mid x, y \in [n]\}$.

(v) $D_{ve}(P_{n-1}, i-1) \neq \phi$, $D_{ve}(P_{n-2}, i-1) \neq \phi$, $D_{ve}(P_{n-3}, i-1) \neq \phi$, and $D_{ve}(P_{n-4}, i-1) \neq \phi$.

Let $X_1 \in D_{ve}(P_{n-1}, i-1)$, so atleast one vertex labeled $n - 1$ or $n - 2$ or $n - 3$ is in X_1.

If $n - 1$ or $n - 2 \in X_1$ or $n - 3 \in X_1$ then,

$X_1 \cup \{n\} \in D_{ve}(P_{n,i})$. Let $X_2 \in D_{ve}(P_{n-2}, i-1)$, then atleast one vertex labeled $n - 2$ or $n - 3$ or $n - 4$ is in X_2. If $n - 2$ or $n - 3$ or $n - 4 \in X_2$, then

$X_2 \cup \{n - 1\} \in D_{ve}(P_{n,i})$.

Let $X_3 \in D_{ve}(P_{n-3}, i-1)$, then atleast one vertex labeled $n - 3$ or $n - 4$ or $n - 5$ is in X_3. If $n - 3$ or $n - 4$ or $n - 5 \in X_3$ then $X_3 \cup \{n - 2\} \in D_{ve}(P_{n,i})$.

Let $X_4 \in D_{ve}(P_{n-4}, i-1)$, then atleast one vertex labeled $n - 4$ or $n - 5$ or $n - 6$ is in X_4. If $n - 4 \in X_4$, then $X_4 \cup \{x\} \in D_{ve}(P_{n,i})$ for $x \in \{n, n - 1\}$. if $n - 5 \in X_4$, then $X_4 \cup \{x\} \in D_{ve}(P_{n,i})$ for $x \in \{n, n - 1, n - 2\}$. if $n - 6 \in X_4$, then

$X_4 \cup \{n - 2\} \in D_{ve}(P_{n,i})$.

Therefore, we have, $\{\{n\} \cup X_1, \{n - 1\} \cup X_2, \{n - 2\} \cup X_3 \mid X_1 \in D_{ve}(P_{n-1}, i - 1), X_2 \in D_{ve}(P_{n-2}, i - 1), X_3 \in D_{ve}(P_{n-3}, i - 1)\}$
$\cup \{\{n - 2\} \cup X \mid X \in D_{ve}(P_{n-4}, i - 1) \setminus D_{ve}(P_{n-3}, i - 1)\} \cup \{\{n - 1\} \cup X \mid X \in D_{ve}(P_{n-3}, i - 1) \setminus D_{ve}(P_{n-2}, i - 1)\}$
$\cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-3}, i - 1) \cap D_{ve}(P_{n-2}, i - 1)\} \subseteq D_{ve}(P_{n,i})$ ……(i)

Let $Y \in D_{ve}(P_{n,i})$, then $n \in Y$, $n - 1 \in Y$ or $n - 2 \in Y$. If $n \in Y$ then, by lemma 2.3, atleast one vertex labeled $n - 1$, $n - 2$, $n - 3$ or $n - 4$ is in Y. If $n - 1 \in Y$ or $n - 2 \in Y$ or $n - 3 \in Y$ then $Y = X \cup \{n\}$ for some $X \in D_{ve}(P_{n-1}, i - 1)$.

If $n - 4 \in Y$, $n - 3 \notin Y$, $n - 2 \notin Y$, $n - 1 \notin Y$, then $Y = X \cup \{n\}$ for some

$X \in D_{ve}(P_{n-3}, i - 1) \cap D_{ve}(P_{n-2}, i - 1)$.

Now, suppose that $n - 1 \in Y$ and $n \notin Y$, then by lemma 2.3 atleast one vertex labeled $n - 2$, $n - 3$, $n - 4$ or $n - 5$ is in Y.

If $n - 2 \in Y$, $n - 3 \in Y$, $n - 4 \in Y$ then $Y = X \cup \{n - 1\}$ for some $X \in D_{ve}(P_{n-2}, i - 1)$. If $n - 5 \in Y$, $n - 4 \notin Y$, $n - 3 \notin Y$ then $Y = X \cup \{n - 2\}$ for some

$X \in D_{ve}(P_{n-4}, i-1) \setminus D_{ve}(P_{n-3}, i-1)$.

So, $D_{ve}(P_{n,i}) \subseteq \{\{n\} \cup X_1, \{n-1\} \cup X_2, \{n-2\} \cup X_3 \mid X_1 \in D_{ve}(P_{n-1}, i-1), X_2 \in D_{ve}(P_{n-2}, i-1), X_3 \in D_{ve}(P_{n-3}, i-1) \cup \{\{n-2\} \cup X \mid X \in D_{ve}(P_{n-4}, i-1) \setminus D_{ve}(P_{n-3}, i-1)\} \cup \{\{n-1\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1) \setminus D_{ve}(P_{n-2}, i-1)\} \cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1) \cap D_{ve}(P_{n-2}, i-1)\}$ ……(ii)

$\therefore$ From (i) and (ii) $D_{ve}(P_{n,i}) = \{\{n\} \cup X_1, \{n-1\} \cup X_2, \{n-2\} \cup X_3 \mid X_1 \in D_{ve}(P_{n-1}, i-1), X_2 \in D_{ve}(P_{n-2}, i-1), X_3 \in D_{ve}(P_{n-3}, i-1)$
$\cup \{\{n-2\} \cup X \mid X \in D_{ve}(P_{n-4}, i-1) \setminus D_{ve}(P_{n-3}, i-1)\}$
$\cup \{\{n-1\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1) \setminus D_{ve}(P_{n-2}, i-1)\}$
$\cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1) \cap D_{ve}(P_{n-2}, i-1)\}$

(vi) $D_{ve}(P_{n-1}, i-1) \neq \phi$, $D_{ve}(P_{n-2}, i-1) \neq \phi$, $D_{ve}(P_{n-3}, i-1) = \phi$, $D_{ve}(P_{n-4}, i-1) = \phi$,

by lemma 2.6 (vi), $i = n-1$.

Therefore, $D_{ve}(P_{n,i}) = D_{ve}(P_n, n-1) = \{[n] - \{x\} \mid x \in [n]\}$

Example:2.8

Consider P_9 with $V(P_9) = [9]$. we use theorem 2.7 to construct $D_{ve}(P_{9,i})$ for i = 2, 3, 8, 9

Solution:

$D_{ve}(P_{9,2})$

$D_{ve}(P_{8,1}) = \phi$, $(P_{7,1}) = \phi$, $D_{ve}(P_{6,1}) = \phi$ and,

$D_{ve}(P_{5,1}) \neq \phi$, $D_{ve}(P_{5,1}) = \{3\}$,

$\therefore$ by theorem 2.7, $D_{ve}(P_{9,2}) = \{\{3, 7\}\}$ $D_{ve}(P_{9,3})$

$D_{ve}(P_{8,2}) = \{\{3, 6\}, \{2, 6\}, \{3, 7\}\}$

$D_{ve}(P_{7,2}) = \{\{1,5\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{3, 7\}\}$

$D_{ve}(P_{6,2}) = \{\{1,4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}$

$D_{ve}(P_{5,2}) = \{\{1,3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\}$

$D_{ve}(P_{5,2}) \setminus D_{ve}(P_{6,2}) = \{\{1, 3\}, \{2, 3\}\}$

$D_{ve}(P_{6,2}) \setminus D_{ve}(P_{7,2}) = \{\{1, 4\}, \{2, 4\}, \{3, 4\}\}$

$D_{ve}(P_{6,2}) \cap D_{ve}(P_{7,2}) = \{\{1,5\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}\}$

$\therefore$ by theorem 2.7,

$D_{ve}(P_{9,3}) = \{\{9\} \cup X_1, \{8\} \cup X_2, \{7\} \cup X_3 \mid X_1 \in D_{ve}(P_{8,2}), X_2 \in D_{ve}(P_{7,2}), X_3 \in D_{ve}(P_{6,2})\} \cup \{\{7\} \cup X \mid X \in D_{ve}(P_{5,2}) \setminus D_{ve}(P_{6,2})\} \cup \{\{8\} \cup X \mid X \in D_{ve}(P_{6,2}) \setminus D_{ve}(P_{7,2})\} \cup \{\{9\} \cup X \mid X \in D_{ve}(P_{6,2}) \cap D_{ve}(P_{7,2})\}$

$D_{ve}(P_{9,3}) = \{\{3, 6, 9\}, \{2, 6, 9\}, \{3, 7, 9\}, \{1, 5, 8\}, \{2, 5, 8\}, \{2, 6, 8\}, \{3, 5, 8\}, \{3, 6, 8\}, \{3, 7, 8\}, \{1, 4, 7\}, \{1, 5, 7\}, \{2, 4, 7\}, \{2, 5, 7\}, \{2, 6, 7\}, \{3, 4, 7\}, \{3, 5, 7\}, \{3, 6, 7\}, \{1, 3, 7\}, \{2, 3, 7\},\{1, 4, 8\}, \{2, 4, 8\}, \{3, 4, 8\}, \{1,5, 9\}, \{2, 5, 9\}, \{3, 5, 9\}\}$

$D_{ve}(P_{9,8})$

$D_{ve}(P_{8,7}) = \{\{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6, 8\}, \{1, 2, 3, 4, 6, 7, 8\}, \{1, 2, 3, 5, 6, 7, 8\}, \{1, 2, 4, 5, 6, 7, 8\}, \{1, 3, 4, 5, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8\}, \{1, 2, 3, 4, 5, 7, 8\}\}$.

$D_{ve}(P_{7,7}) = \{\{1, 2, 3, 4, 5, 6, 7\}\}$

$D_{ve}(P_{6,7}) = \phi$, $D_{ve\,5}{}^7 = \phi$.

Thus $D_{ve}(P_{n-1,i-1}) \neq \phi$, $D_{ve}(P_{n-2,i-1}) \neq \phi$, $D_{ve}(P_{n-3,i-1}) = \phi$ and $D_{ve}(P_{n-4,i-1}) = \phi$.

$\therefore$ by part (iv) of theorem 2.7,

$D_{ve}(P_{n,i}) = \{[n] - \{x\} \mid x \in [n]\}$

$D_{ve}(P_{9,8}) = \{\{1, 2, 3, 4, 5, 6, 7, 9\}, \{1, 2, 3, 4, 5, 6, 8, 9\}, \{1, 2, 3, 4, 6, 7, 8, 9\}, \{1, 2, 3, 5, 6, 7, 8, 9\}, \{1, 2, 4, 5, 6, 7, 8, 9\}, \{1, 3, 4, 5, 6, 7, 8, 9\}, \{2, 3, 4, 5, 6, 7, 8, 9\}, \{1, 2, 3, 4, 5, 6, 7, 8, \}, \{1, 2, 3, 4, 5, 7, 8, 9\}\}$.

$D_{ve}(P_{9,9})$

$D_{ve}(P_{8,8}) = \{\{1, 2, 3, 4, 5, 6, 7, 8\}\}$

$D_{ve}(P_{7,8}) = \phi$, $D_{ve}(P_{6,8}) = \phi$, $D_{ve}(P_{5,8}) = \phi$,

$\therefore D_{ve}(P_{n-1,i-1}) \neq \phi$, $D_{ve}(P_{n-2,i-1}) = \phi$, $D_{ve}(P_{n-3,i-1}) = \phi$ and $D_{ve}(P_{n-4,i-1}) = \phi$.

then by part (ii) of theorem 2.7, $D_{ve}(P_{n,i}) = \{[n]\}$, $\therefore$ $D_{ve}(P_{9,9}) = \{\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\} = \{[9]\}$.

Table I: $d_{ve}(P_{n,j})$, The number of vertex- edge dominating sets of Pn with cardinality j.

n \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2	2	1													
3	3	3	1												
4	2	6	4	1											
5	1	8	10	5	1										
6	0	8	18	15	6	1									
7	0	6	25	33	21	7	1								
8	0	3	28	57	54	28	8	1							
9	0	1	25	81	110	82	36	9	1						
10	0	0	18	96	186	191	118	45	10	1					
11	0	0	10	96	267	371	308	163	55	11	1				
12	0	0	4	81	330	617	672	470	218	66	12	1			
13	0	0	1	57	354	893	1261	1134	687	284	78	13	1		
14	0	0	0	33	330	1137	2072	2359	1812	970	362	91	14	1	
15	0	0	0	15	267	1281	3018	4313	4126	2772	1331	453	105	15	1

3. Vertex-Edge Domination Polynomial of a path

Let $D_{ve}(P_n, x) = \sum_{i=\lceil \frac{n-1}{4} \rceil}^{|V(G)|} d_{ve}(P_n, i)\, x^i$ be the vertex-edge domination polynomial of a path P_n.

Theorem:3.1

1. If $D_{ve}(P_n,i)$ is the family of the vertex-edge dominating sets of P_n with cardinality i, where $i \geq \left\lceil \frac{n-1}{4} \right\rceil$. $| D_{ve}(P_n, i) | = | D_{ve}(P_{n-1}, i-1) | + | D_{ve}(P_{n-2}, i-1) | + | D_{ve}(P_{n-3}, i-1) | + | D_{ve}(P_{n-4}, i-1)|$.
2. For every $n \geq 5$, $D_{ve}(P_n, x) = x[D_{ve}(P_{n-1}, x) + D_{ve}(P_{n-2}, x) + D_{ve}(P_{n-3}, x) + D_{ve}(P_{n-4}, x)]$ with initial values
   $D_{ve}(P_1, x) = x$, $D_{ve}(P_2, x) = x^2 + 2x$, $D_{ve}(P_3, x) = x^3 + 3x^2 + 3x$, $D_{ve}(P_4, x) = x^4 + 4x^3 + 6x^2 + 2x$

Proof:

1. From Theorem 2.7, we consider the cases given below, where $i \geq \left\lceil \frac{n-1}{4} \right\rceil$ and $n \geq 5$

(i) If $D_{ve}(P_{n-1}, i-1) = D_{ve}(P_{n-2}, i-1) = D_{ve}(P_{n-3}, i-1) = \phi$ and $D_{ve}(P_{n-4}, i-1) \neq \phi$, then $D_{ve}(P_n, i) = \{\{3, 7, 11, \ldots, n-6, n-2\}\}$

(i) If $D_{ve}(P_{n-2}, i-1) = D_{ve}(P_{n-3}, i-1) = D_{ve}(P_{n-4}, i-1) = \phi$ and $D_{ve}(P_{n-1}, i-1) \neq \phi$, then $D_{ve}(P_n, i) = \{[n]\}$

(iii) If $D_{ve}(P_{n-1}, i-1) = \phi$, $D_{ve}(P_{n-2}, i-1) \neq \phi$, $D_{ve}(P_{n-3}, i-1) \neq \phi$ and $D_{ve}(P_{n-4}, i-1) \neq \phi$, then $D_{ve}(P_n, i) = \{\{3, 7, 11, \ldots, n-4, n\}\}$
$\cup \{\{n-2\} \cup X \mid X \in D_{ve}(P_{n-4}, i-1)\} \cup \{\{n-1\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1)\}$
$\cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-2}, i-1)\}$

(iv) If $D_{ve}(P_{n-4}, i-1) = \phi$, $D_{ve}(P_{n-3}, i-1) \neq \phi$, $D_{ve}(P_{n-2}, i-1) \neq \phi$ and $D_{ve}(P_{n-1}, i-1) \neq \phi$ then
$D_{ve}(P_n, i) = \{[n] - \{x, y\} \mid x, y \in [n]\}$

(v) If $D_{ve}(P_{n-1}, i-1) \neq \phi$, $D_{ve}(P_{n-2}, i-1) \neq \phi$, $D_{ve}(P_{n-3}, i-1) = \phi$ and $D_{ve}(P_{n-4}, i-1) \neq \phi$, then $D_{ve}(P_n, i) = \{\{n\} \cup X_1, \{n-1\} \cup X_2, \{n-2\} \cup X_3|$
$X_1 \in D_{ve}(P_{n-1}, i-1)$, $X_2 \in D_{ve}(P_{n-2}, i-1)$, $X_3 \in D_{ve}(P_{n-3}, i-1)$, $\cup \{\{n-2\} \cup X \mid X \in D_{ve}(P_{n-4}, i-1) \setminus D_{ve}(P_{n-3}, i-1)\}$
$\cup \{\{n-1\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1) \setminus D_{ve}(P_{n-2}, i-1)\}$
$\cup \{\{n\} \cup X \mid X \in D_{ve}(P_{n-3}, i-1) \cap D_{ve}(P_{n-2}, i-1)\}$

(vi) If $D_{ve}(P_{n-1}, i-1) \neq \phi$, $D_{ve}(P_{n-2}, i-1) \neq \phi$, $D_{ve}(P_{n-3}, i-1) = \phi$ and $D_{ve}(P_{n-4}, i-1) = \phi$, then $D_{ve}(P_n, i) = \{[n] - \{x\} \mid x \in [n]\}$ from the above construction, in each case we obtain that, $|D_{ve}(P_n, i)| = | D_{ve}(P_{n-1}, i-1) | + | D_{ve}(P_{n-2}, i-1) | + | D_{ve}(P_{n-3}, i-1) | + | D_{ve}(P_{n-4}, i-1)|$

2. By definition,

$$D_{ve}(P_n, x) = \sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n,i})\, x^i$$

$$= x \sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n,i})\, x^{i-1} \text{ by using part (i)}$$

$$= x \left(\sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} \left(\begin{array}{l} (d_{ve}(P_{n-1}, i-1) + d_{ve}(P_{n-2}, i-1) \\ + d_{ve}(P_{n-3}, i-1) + d_{ve}(P_{n-4}, i-1) \end{array} \right) x^{i-1} \right)$$

$$= x \left(\sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-1, i-1})\, x^{i-1} + \sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-2, i-1})\, x^{i-1} + \sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-3, i-1})\, x^{i-1} + \sum_{i=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-4}, i-1)\, x^{i-1} \right)$$

$= x[D_{ve}(P_{n-1}, x) + D_{ve}(P_{n-2}, x) + D_{ve}(P_{n-3}, x)$

$+ D_{ve}(P_{n-4}, x)]$

The initial values are $D_{ve}(P_1, x) = \sum_{i=\left\lceil \frac{n-1}{4} \right\rceil}^{1} d_{ve}(P_{1,i})\, x^i$

$= \sum_{i=0,1} d_{ve}(P_{1,i})\, x^i = d_{ve}(P_{1,0})\, x^0 + d_{ve}(P_{1,1})\, x^1$

$= 0.1 + 1.x = x$

$D_{ve}(P_2, x) = \sum_{i=\left\lceil \frac{n-1}{4} \right\rceil}^{2} d_{ve}(P_{2,i})\, x^i = \sum_{i=1}^{2} d_{ve}(P_{2,i})\, x^i$

$= d_{ve}(P_{2,1})\, x^1 + d_{ve}(P_{2,2})\, x^2$

$= 2.x + 1.x^2 = x^2 + 2x$

$D_{ve}(P_3, x) = \sum_{i=\left\lceil \frac{n-1}{4} \right\rceil}^{3} d_{ve}(P_{3,i})\, x^i = \sum_{i=1}^{3} d_{ve}(P_{3,i})\, x^i$

$= d_{ve}(P_{3,1})\, x^1 + d_{ve}(P_{3,2})\, x^2 + d_{ve}(P_{3,3})\, x^3$

$= 3.x^1 + 3.x^2 + 1.x^3 = x^3 + 3x^2 + 3x$

$D_{ve}(P_4, x) = \sum_{i=\left\lceil \frac{4-1}{4} \right\rceil}^{4} d_{ve}(P_{4,i})\, x^i = \sum_{i=1}^{4} d_{ve}(P_{4,i})\, x^i$

$= d_{ve}(P_{4,1})\, x^1 + d_{ve}(P_{4,2})\, x^2 + d_{ve}(P_{4,3})\, x^3 + d_{ve}(P_{4,4})\, x^4$

$= 2.x^1 + 6.x^2 + 4.x^3 + 1.x^3 = x^4 + 4x^3 + 6x^2 + 2x$

Theorem:3.2

The following properties hold for the co-efficients of $D_{ve}(P_n, x)$:

(i) $d_{ve}(P_{n,} n) = 1$

(ii) $d_{ve}(P_{n,} n-1) = n$ for $n \geq 2$

(iii) $d_{ve}(P_{n,} n-2) = \frac{n(n-1)}{2}$, $n \geq 3$

(iv) $d_{ve}(P_{n,} n-3) = \binom{n}{3} - 2$, $n > 3$

(v) $d_{ve}(P_{4k+1}, k) = 1$

(vi) $d_{ve}(P_{4k}, k) = k + 1$

(vii) If $S_n = \sum_{j=\left\lceil \frac{n-1}{4} \right\rceil}^{n} d_{ve}(P_n, j)$ for $n \geq 5$, then

$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$ with initial values $S_1 = 1$, $S_2 = 3$, $S_3 = 7$, $S_4 = 13$.

(viii) $d_{ve}(P_{4n+2}, n) = 0$, $n \in N$

(ix) $\sum_{n=1}^{\infty} d_{ve}(P_{n,i}) = 4 \sum_{n=1}^{\infty} d_{ve}(P_{n,i-1})$

Proof:

(i) Proof is obvious.

(ii) We prove this by induction on n.

Obviously the result is true for n = 2.

Now, suppose that the result is true for all numbers less than n. Now, we prove it for n. We have,

$d_{ve}(P_n, n-1) = d_{ve}(P_{n-1}, n-2) + d_{ve}(P_{n-2}, n-2) + d_{ve}(P_{n-3}, n-2) + d_{ve}(P_{n-4}, n-2)$

$= (n-1) + 1 + 0 + 0 = n$ (by induction hypothesis, and part (i)).

∴ The result is true for all n, Hence by principle of induction, The result is true for all $n \geq 2$.

(iii) We prove this by induction on n.

Obviously, the result is true for n = 3.

Now, suppose that the result is true for all numbers less than n.Now, we prove it for n. We have

$d_{ve}(P_n, n-2) = d_{ve}(P_{n-1}, n-3) + d_{ve}(P_{n-2}, n-3) + d_{ve}(P_{n-3}, n-3) + d_{ve}(P_{n-4}, n-3)$

$= \frac{(n-1)(n-2)}{2} + n - 2 + 1 + 0$ (by induction hypothesis and part (i) & part (ii)).

$= \frac{(n-1)(n-2)}{2} + (n-1) = (n-1)\left[\frac{n-2}{2} + 1\right]$

$= \frac{n(n-1)}{2}$

Therefore, the result is true for n. Hence by principle of induction, the result is true for all $n \geq 3$.

(iv) We prove this by induction on n. Obviously, the result is true for n = 4. Now, suppose that the result is true for all numbers less than n.Now, we prove it for n.

we have $d_{ve}(P_{n}, {}_{n-3}) = d_{ve}(P_{n-1}, {}_{n-4}) + d_{ve}(P_{n-2}, {}_{n-4}) + d_{ve}(P_{n-3}, {}_{n-4}) + d_{ve}(P_{n-4}, {}_{n-4})$

$= \frac{(n-1)(n-2)(n-3)}{6} - 2 + \frac{(n-2)(n-3)}{2} + n - 3 + 1$ by induction hypothesis and part (i),(ii), (iii)

$= \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-2)(n-3)}{2} + n - 4$

$= \binom{n}{3} - 2$

Hence, by principle of induction, the result is true for all $n \geq 4$, $n \in N$

(v) We prove this by induction on k.

Obviously,the result is true for k = 1. now, suppose that the result is true for all numbers less than k. Now, we prove it for k.We have,

$d_{ve}(P_{4k+1,k}) = d_{ve}(P_{4k,k-1}) + d_{ve}(P_{4k-1,k-1}) + d_{ve}(P_{4k-2,k-1}) + d_{ve}(P_{4k-3,k-1})$

$= 0 + 0 + 0 + 1 = 1$

Therefore, the result is true for k. Hence, by principle of induction, the result is true for all k.

(vi) We prove this by induction on k

Obviously, the result is true for k = 1.

Now, suppose that the result is true for all numbers less than k. Now, we prove it for k. We have,

$d_{ve}(P_{4k,k}) = d_{ve}(P_{4k-1,k-1}) + d_{ve}(P_{4k-2,k-1}) + d_{ve}(P_{4k-3,k-1}) + d_{ve}(P_{4k-4,k-1}) = 0 + 0 + 1 + k = k + 1$

Therefore, the result is true for k. Hence, by Principle of induction,

The result is true for all k, $k \in N$.

(vii) We have

$$S_n = \sum_{j=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n,j})$$

$$= \sum_{j=\lceil \frac{n-1}{4} \rceil}^{n} (d_{ve}(P_{n-1,j-1}) + d_{ve}(P_{n-2,j-1}) + d_{ve}(P_{n-3,j-1}) + d_{ve}(P_{n-4,j-1}))$$

$$= \sum_{j=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-1,j-1}) + \sum_{j=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-2,j-1}) + \sum_{j=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-3,j-1}) + \sum_{j=\lceil \frac{n-1}{4} \rceil}^{n} d_{ve}(P_{n-4,j-1})$$

$$= \sum_{j=\lceil \frac{n-1}{4} \rceil-1}^{n-1} d_{ve}(P_{n-1,j}) + \sum_{j=\lceil \frac{n-1}{4} \rceil-1}^{n-2} d_{ve}(P_{n-2,j}) + \sum_{j=\lceil \frac{n-1}{4} \rceil-1}^{n-3} d_{ve}(P_{n-3,j}) + \sum_{j=\lceil \frac{n-1}{4} \rceil-1}^{n-4} d_{ve}(P_{n-4,j})$$

$= S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$, $n \geq 5$.

(viii)

We prove this by induction on n.

Obviously, the result is true for n = 1.

Now, suppose that the result is true for all numbers less than n.Now, we prove it for n.We have,

$d_{ve}(P_{4n+2,n}) = d_{ve}(P_{4n+1,n-1}) + d_{ve}(P_{4n,n-1}) + d_{ve}(P_{4n-1,n-1}) + d_{ve}(P_{4n-2,n-1})$

$= 0 + 0 + 0 + 0$ (by induction hypothesis)

$= 0$

Therefore, the result is true for n

Hence, by Principle of induction, the result is true for all n, $n \in N$.

(ix) $\sum_{n=5}^{\infty} d_{ve}(P_{n,i}) = \sum_{n=5}^{\infty} (d_{ve}(P_{n-1,i-1}) + d_{ve}(P_{n-2,i-1}) + d_{ve}(P_{n-3,i-1}) + d_{ve}(P_{n-4,i-1}))$

$= d_{ve}(P_{4,i-1}) + d_{ve}(P_{5,i-1}) + d_{ve}(P_{6,i-1}) + \ldots + d_{ve}(P_{3,i-1}) + d_{ve}(P_{4,i-1}) + d_{ve}(P_{5,i-1}) + \ldots + d_{ve}(P_{2,i-1}) + d_{ve}(P_{3,i-1}) + d_{ve}(P_{4,i-1}) + d_{ve}(P_{5,i-1}) + \ldots + d_{ve}(P_{1,i-1}) + d_{ve}(P_{2,i-1}) + d_{ve}(P_{3,i-1}) + d_{ve}(P_{4,i-1}) + d_{ve}(P_{5,i-1}) + \ldots$

$= d_{ve}(P_{1,i}) + d_{ve}(P_{2,i}) + d_{ve}(P_{3,i}) + d_{ve}(P_{4,i}) + d_{ve}(P_{1,i-1}) + 2d_{ve}(P_{2,i-1}) + 3d_{ve}(P_{3,i-1}) + 4(\sum_{n=4}^{\infty} d_{ve}(P_{n,i-1}))$

$= 4\, d_{ve}(P_{1,i-1}) + 4\, d_{ve}(P_{2,i-1}) + 4\, d_{ve}(P_{3,i-1}) + 4\, d_{ve}(P_{4,i-1}) + 4\, d_{ve}(P_{5,i-1}) + \ldots$ ($\square$ $d_{ve}(P_{0,i-1}) = 0$)

$= 4 \sum_{n=1}^{\infty} d_{ve}(P_{n,i-1})$

$\therefore \sum_{n=1}^{\infty} d_{ve}(P_{n,i}) = 4 \sum_{n=1}^{\infty} d_{ve}(P_{n,i-1})$

Theorem: 3.3

For every $n \in N$, and $\lceil \frac{n-1}{4} \rceil \leq i \leq n$, $|D_{ve}(P_{n,i})|$ is the co-efficient of $u^n v^i$ in the Expansion of the function.

$$f(u, v) = \frac{6u^4v^2+4u^4v^3+u^4v^4+6u^5v^2+4u^5v^3+u^5v^4+5u^6v^2 + 4u^6v^3+u^6v^4+3u^7v^2+3u^7v^3+u^7v^4}{1-uv-u^2v-u^3v-u^4v}$$

Proof:

Set $f(u,v) = \sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n,i})| u^n v^i$ by recursive formula for $|D_{ve}(P_{n,i})|$ in theorem 3.1 we can write f(u, v) in the following form.

$$f(u,v) = \sum_{n=4}^{\infty}\sum_{i=1}^{\infty} (|D_{ve}(P_{n-1,i-1})| + |D_{ve}(P_{n-2,i-1})| + |D_{ve}(P_{n-3,i-1})| + |D_{ve}(P_{n-4,i-1})|) u^n v^i$$

$$= uv\left(\sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-1,i-1})| u^{n-1}v^{i-1}\right)$$

$$+ u^2v\left(\sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-2,i-1})| u^{n-2}v^{i}\right)$$

$$+ u^3v\left(\sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-3,i-1})| u^{n-3}v^{i-1}\right)$$

$$+ u^4v\left(\sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-4,i-1})| u^{n-4}v^{i-1}\right)$$

$= uv(|D_{ve}(P_{3,0})| u^3 + |D_{ve}(P_{3,1})| u^3v + |D_{ve}(P_{3,2})| u^3v^2 + |D_{ve}(P_{3,3})| u^3v^3 + \sum_{i=5}^{\infty}\sum_{i=1}^{\infty} D_{ve}(P_{n-1,i-1})| u^{n-1}v^{i-1}) + u^2v(|D_{ve}(P_{2,0})| u^2 + |D_{ve}(P_{2,1})| u^2v$

$+ |D_{ve}(P_{2,2})| u^2v^2 + |D_{ve}(P_{3,0})| u^3 + |D_{ve}(P_{3,1})| u^3v + |D_{ve}(P_{3,2})| u^3v^2$

$+ |D_{ve}(P_{3,3})| u^3v^3 + \sum_{i=6}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-2,i-1})| u^{n-2}v^{i-1}$

$+ u^3v(|D_{ve}(P_{1,0})| u + |D_{ve}(P_{1,1})| uv$

$+ |D_{ve}(P_{2,0})| u^2 + |D_{ve}(P_{2,1})| u^2v + |D_{ve}(P_{2,2})| u^2v^2$

$+ |D_{ve}(P_{3,0})| u^3 + |D_{ve}(P_{3,1})| u^3v + |D_{ve}(P_{3,2})| u^3v^2 + |D_{ve}(P_{3,3})| u^3v^3 + \sum_{n=7}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-3,i-1})| u^{n-3}v^{i-1}$

$+ u^4v(|D_{ve}(P_{0,0})| + |D_{ve}(P_{1,0})| u + |D_{ve}(P_{1,1})| uv + |D_{ve}(P_{2,0})| u^2 + |D_{ve}(P_{2,1})| u^2v + |D_{ve}(P_{2,2})| u^2v^2$

$+ |D_{ve}(P_{3,0})| u^3 + |D_{ve}(P_{3,1})| u^3v + |D_{ve}(P_{3,2})| u^3v^2 + |D_{ve}(P_{3,3})| u^3v^3 + \sum_{n=8}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n-4,i-1})| u^{n-4}v^{i-1}$

$D_{ve}(P_{n,i})$ is the family of vertex-edge dominating set with cardinality i of P_n.

$\therefore |D_{ve}(P_{n,0})| = 0$, $n \in N$ and $|D_{ve}(P_{0,0})| = 0$

$D_{ve}(P_{1,i})$ is the family of vertex-edge dominating set with cardinality 1 of P_1 $\therefore |D_{ve}(P_{1,1})| = 1$

From table I,

$|D_{ve}(P_{2,1})| = 2$, $|D_{ve}(P_{2,2})| = 1$, $|D_{ve}(P_{3,1})| = 3$,

$|D_{ve}(P_{3,2})| = 3$, $|D_{ve}(P_{3,3})| = 1$

$|D_{ve}(P_{4,1})| = 2$, $|D_{ve}(P_{4,2})| = 6$, $|D_{ve}(P_{4,3})| = 4$,

$|D_{ve}(P_{4,4})| = 1$

$$= uv(3u^3v+3u^3v^2 + u^3v^3 + \sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n,i})| u^n v^i)$$

$$+ u^2v(2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3$$

$$+ \sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n,i})| u^n v^i)$$

$$+ u^3v(uv + 2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3$$

$$+ \sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n,i})| u^n v^i)$$

$$+ u^4v(uv + 2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3$$

$$+ \sum_{n=4}^{\infty}\sum_{i=1}^{\infty} |D_{ve}(P_{n,i})| u^n v^i)(|D_{ve}(P_{n,0})| = 0$$

$f(u,v) = uv(3u^3v + 3u^3v^2 + u^3v^3 + f(u,v))$

$+ u^2v(2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3 + f(u,v))$

$+ u^3v(uv + 2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3$

$+ f(u,v)) + u^4v(uv + 2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3 + f(u,v)) = 3u^4v^2 + 3u^4v^3 + u^4v^4 + uv\, f(u,v) + 2u^4v^2$

$+ u^4v^3 + 3u^5v^2 + 3u^5v^3 + u^5v^4 + u^2v,\ f(u,v) + u^4v^2$

$+2u^5v^2 + u^5v^3 + 3u^6v^2 + 3u^6v^3 + u^6v^4 + u^3v\, f(u,v) + u^5v^2$

$+2u^6v^2 + u^6v^3 + 3u^7v^2 + 3u^7v^3 + u^7v^4 + u^4v\, f(u,v)$

$$f(u,v) = \frac{6u^4v^2 + 4u^4v^3 + u^4v^4 + 6u^5v^2 + 4u^5v^3 + u^5v^4 + 5u^6v^2 + 4u^6v^3 + u^6v^4 + 3u^7v^2 + 3u^7v^3 + u^7v^4}{1 - uv - u^2v - u^3v - u^4v}$$

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