Characterizations of k-normal matrices

B.K.N.Muthugobal[#1] and R.Subash [*2]

[#]Research Scholar, Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, Tamilnadu, India 612 001.

[*] Assistant Professor, Department of Mathematics, A.V.C College of Engineering, Mannampandal, Mayiladuthurai,Tamilnadu, India 609305.

Abstract: In this paper to extend and generalize lists of characterizations of k-normal and k-hermitian matrices known in the literature, by providing numerous sets of equivalent conditions referring to the notions of conjugate transpose, Moore-Penrose and group inverse.

Keywords: k-normal, k-hermitian, Moore-Penrose inverse, Group inverse and Conjugate transpose.

AMS Classification: 15A09, 15A57.

1. INTRODUCTION:

In this section concerns matrices belonging to the set $C_{n\times n}$ composed of square matrices of order n with complex entries. Referring to singular value decomposition such matrices can be written in the form

$$A = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* K \qquad (1)$$

where $U, V \in C_{n\times n}$ are k-unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r1}, \ldots, \sigma_t I_{rt})$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$, $r_1 + r_2 + \ldots + r_t = r = \mathrm{rank}(A)$ and I_{ri} in the identity matrix of order r_i, see[13, p.66], using this decomposition, Hartwig and Spindelbock [7,corollory6] derived the following representation.

Let $A \in C_{n\times n}$ be of rank r, then

$$A = U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix} U^* K \qquad (2)$$

where U and Σ are taken from the singular decomposition (1) of A, and the matrices N and M satisfy the condition $NN^* + MM^* = I_r$. Utilizing the representation (2), we see that $A = KA^* K$, that is A is k-hermitian if and only if M=0 and $\Sigma N = N^* \Sigma$, the matrix A satisfies $AA^* K = KA^* A$, that is A is k-normal if and only if M=0 and $\Sigma N = N\Sigma$. This is equivalent to the relation $AXK = KXA$, where X is the Moor-Penrose inverse of A, Uniquely defined by the four conditions $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$ and expresses the so called k-normal property of matrix A.

From (2) it follows that

$$X = KU\begin{pmatrix} N^*\Sigma^{-1} & 0 \\ M^*\Sigma^{-1} & 0 \end{pmatrix}U^* \qquad (3)$$

We say that A has the group property if $\operatorname{rank}(A) = \operatorname{rank}(A^2)$. This is equivalent to saying that the matrix N in (2) is nonsingular. In such a case, matrix A has a group inverse Y, which is characterized by the following three conditions $AYA = A$, $YAY = Y$, and $YA = AY$ ensuring that matrix Y is unique.

From (2) we obtain,

$$Y = KU\begin{pmatrix} N^{-1}\Sigma^{-1} & N^{-1}\Sigma^{-1}N^{-1}M \\ 0 & 0 \end{pmatrix}U^* \qquad (4)$$

In what follows, whenever the group inverse occurs, it is assumed to exist.

Theorem 1: Let $A \in C_{n\times n}$. Then the following conditions are equivalent.

a. A is k-normal
b. $AKAA^* = AA^*AK$
c. $KAA^*A = A^*AKA$
d. $A^*KY = YKA^*$
e. $KAA^*KY = A^*KYAK$
f. $AA^*KY = KYAA^*$
g. $KAYKA^* = YKA^*AK$
h. $A^*AYK = YKA^*A$

i. $A^{*}KA^{*}KY = A^{*}KYKA^{*}$

j. $A^{*}KXKY = YKA^{*}KX$

k. $AA^{*}KX = KXAA^{*}$

l. $A^{*}KYKA^{*} = YKA^{*}KA^{*}$

m. $A^{*}KYKX = XKA^{*}KY$

n. $A^{*}KYKY = YKA^{*}KY$

o. $XKA^{*}KY = YKXKA^{*}$

p. $XKYKA^{*}YKA^{*}KX$

q. $YKA^{*}KY = YKYKA^{*}$

Proof: By hypothesis, matrix A of the representation (2) is k-normal if and only if M=0 and $\sum N = N\sum$.

(a) $\Leftrightarrow$ (b): Assume that A is k-normal matrix.

To prove that $AKAA^{*} = AA^{*}AK$.

$$\text{Now,}\quad AKAA^{*} = \left[U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}U^{*}K\right]K\left[U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}U^{*}K\right]\left[U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}U^{*}K\right]^{*}$$

$$AKAA^{*} = U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Sigma N^{*} & \Sigma M^{*}\\ 0 & 0\end{pmatrix}U^{*}$$

$$AKAA^{*} = U\begin{pmatrix}\Sigma N\Sigma N & \Sigma N\Sigma M\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Sigma N^{*} & \Sigma M^{*}\\ 0 & 0\end{pmatrix}U^{*}$$

$$AKAA^{*} = U\begin{pmatrix}N\Sigma^{2}N & N\Sigma^{2}M\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Sigma N^{*} & \Sigma M^{*}\\ 0 & 0\end{pmatrix}U^{*}$$

$$AKAA^{*} = U\begin{pmatrix}N\Sigma^{2}N\Sigma N^{*} & N\Sigma^{2}N\Sigma M^{*}\\ 0 & 0\end{pmatrix}U^{*}$$

$$\text{Also,}\quad AA^{*}AK = \left[U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}U^{*}K\right]\left[U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}U^{*}K\right]^{*}\left[U\begin{pmatrix}\Sigma N & \Sigma M\\ 0 & 0\end{pmatrix}U^{*}K\right]K$$

$$AA^*AK = U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma N^* & \Sigma M^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*$$

$$AA^*AK = U\begin{pmatrix} \Sigma N\Sigma N^* & \Sigma N\Sigma M^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*$$

$$AA^*AK = U\begin{pmatrix} N\Sigma^2 N^* & N\Sigma^2 M^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*$$

$$AA^*AK = U\begin{pmatrix} N\Sigma^2 N^* \sum N & N\Sigma^2 N^*\Sigma M \\ 0 & 0 \end{pmatrix}U^*$$

If $M = 0$ implies $M^* = 0$. Thus, we obtained equivalently $M = 0$ and $N\sum^2 N^* = \sum^2$. Taking square roots, we arrive at $M = 0$ and $N\sum N^* = \sum$, that is $M = 0$ and $\sum N = N\sum$.

Therefore, $AKAA^* = AA^*AK = U\begin{pmatrix} N\Sigma^3 & 0 \\ 0 & 0 \end{pmatrix}U^*$.

Converse part we have to prove directly.

(a) $\Leftrightarrow$ (d): Assume that A is k-normal matrix.

To prove that $A^*KY = YKA^*$.

Now, $$A^*KY = \left[U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*K\right]^* K\left[KU\begin{pmatrix} N^{-1}\Sigma^{-1} & N^{-1}\Sigma^{-1}N^{-1}M \\ 0 & 0 \end{pmatrix}U^*\right]$$

$$A^*KY = KU\begin{pmatrix} \Sigma N^* & \Sigma M^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N^{-1}\Sigma^{-1} & N^{-1}\Sigma^{-1}N^{-1}M \\ 0 & 0 \end{pmatrix}U^*$$

$$A^*KY = KU\begin{pmatrix} \Sigma N^* N^{-1}\Sigma^{-1} & \Sigma N^* N^{-1}\Sigma^{-1}N^{-1}M \\ 0 & 0 \end{pmatrix}U^*$$

Also, $$YKA^* = \left[KU\begin{pmatrix} N^{-1}\Sigma^{-1} & N^{-1}\Sigma^{-1}N^{-1}M \\ 0 & 0 \end{pmatrix}U^*\right]K\left[U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*K\right]^*$$

$$YKA^* = KU\begin{pmatrix} N^{-1}\Sigma^{-1} & N^{-1}\Sigma^{-1}N^{-1}M \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma N^* & \Sigma M^* \\ 0 & 0 \end{pmatrix}U^*$$

$$YKA^* = KU\begin{pmatrix} N^{-1}\Sigma^{-1}\Sigma N^* & N^{-1}\Sigma^{-1}\Sigma M^* \\ 0 & 0 \end{pmatrix}U^*$$

It is seen that A^* and Y commute if and only if $M = 0$ implies $M^* = 0$ and $\Sigma N^* N^{-1}\Sigma^{-1} = N^{-1}\Sigma^{-1}\Sigma N^*$.

Therefore, $A^*KY = YKA^*$. Converse part is obvious.

We show the equivalence between (c) and (e) to (q) to prove the similar way.

Theorem 2: Let $A \in C_{n\times n}$. Then the following conditions are equivalent.

a. A is k-normal
b. $A^*KX = YKA^*$
c. $A^*KY = XKA^*$

Proof: (a) $\Leftrightarrow$ (b): Assume that A is k-normal matrix.

To prove that $A^*KX = YKA^*$.

Now, $$A^*KX = \left[U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*K\right]^* K\left[KU\begin{pmatrix} N^*\Sigma^{-1} & 0 \\ M^*\Sigma^{-1} & 0 \end{pmatrix}U^*\right]$$

$$A^*KX = KU\begin{pmatrix} \Sigma N^* & \Sigma M^* \\ 0 & 0 \end{pmatrix}^*\begin{pmatrix} N^*\Sigma^{-1} & 0 \\ M^*\Sigma^{-1} & 0 \end{pmatrix}U^*$$

$$A^*KX = KU\begin{pmatrix} \Sigma N^*N^*\Sigma^{-1} + \Sigma M^*M^*\Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*$$

$$A^*KX = KU\begin{pmatrix} \Sigma N^*N^*\Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*$$

Also, $YKA^* = KU\begin{pmatrix} N^{-1}\Sigma^{-1}\Sigma N^* & 0 \\ 0 & 0 \end{pmatrix}U^*$

It follows that $A^*KX = YKA^*$ implies $M^* = 0$. Moreover, by $N^* = N^{-1}$, in addition to $M^* = 0$ we get $\Sigma N^* N^* \Sigma^{-1} = N^{-1}\Sigma^{-1}\Sigma N^*$. Hence we further obtain $M = 0$ and $\sum N = N\sum$, that is, the pair of conditions corresponding to the k-normality of A. The other direction is obvious.

Next, equivalent conditions we have to prove the similar way.

Theorem 3: Let $A \in C_{n\times n}$. Then the following conditions are equivalent.

a. A is k-normal
b. $KAA^*KX = A^*$
c. $KAA^*KY = A^*$
d. $XKA^*AK = A^*$
e. $YKA^*AK = A^*$

Proof: We again show the equivalence (a) $\Leftrightarrow$ (b) only.

Assume that A is k-normal matrix.

To prove that $KAA^*KX = A^*$.

Now, $$KAA^*KX = K\left[U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*K\right]\left[U\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}U^*K\right]^* K\left[KU\begin{pmatrix} N^*\Sigma^{-1} & 0 \\ M^*\Sigma^{-1} & 0 \end{pmatrix}U^*\right]$$

$$KAA^*KX = KU\begin{pmatrix} \Sigma N & \Sigma M \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma N^* & \Sigma M^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N^*\Sigma^{-1} & 0 \\ M^*\Sigma^{-1} & 0 \end{pmatrix}U^*$$

$$KAA^*KX = KU\begin{pmatrix} \Sigma N\Sigma N^* & \Sigma N\Sigma M^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N^*\Sigma^{-1} & 0 \\ M^*\Sigma^{-1} & 0 \end{pmatrix}U^*$$

$$KAA^*KX = KU\begin{pmatrix} N\Sigma^2 N^* N^*\Sigma^{-1} + \Sigma N\Sigma M^* M^*\Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*$$

$$KAA^*KX = KU\begin{pmatrix} \Sigma^2 N^* \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*$$

and $$A^* = KU\begin{pmatrix} \Sigma N^* & \Sigma M^* \\ 0 & 0 \end{pmatrix} U^*$$

Consequently, the condition $KAA^*KX = A^*$ is equivalent to $M = 0$ and $\Sigma^2 N^* \Sigma^{-1} = \Sigma N^*$, that is $M = 0$ and $\sum^2 N = N \sum^2$, which according to the derivations from the above is equivalent to the k-normality of A. the converse part is obvious.

Similarly, we have to prove the other equivalent conditions.

Remark: In this section is concluded with some observations concerning k-hermitian matrices, for, by exploiting representation (2), one can obtain also several characterizations of such matrices. For, instance, the following list of conditions is equivalent to $A = KA^*K$.

a. $AKA = KA^*A$
b. $KAX = A^*KX$
c. $KAY = A^*KX$
d. $KAY = A^*KY$
e. $KAY = XKA^*$
f. $XAK = YKA^*$

Similarly, the following list provides selected conditions of the form $A = A^*XY$, which are satisfied if and only if A is k-hermitian.

a. $KAKAX = A^*$
b. $AA^*KXK = A$
c. $KA^*AYK = A$
d. $A^*KA^*KY = A^*$
e. $A^*KXKX = Y$
f. $A^*KXKY = X$
g. $A^*KXKY = Y$

h. $A^*KYKY = Y$

i. $YKA^*KY = X$

The final observation is that analogous results to the ones referring to k-hermitian matrices can be obtained for skew k-hermitian matrices. This observation follows directly from the fact that a matrix A is skew k-hermitian whenever iA is k-hermitian.

Bibliography:

[1]. S.Krishnamoorthy and R.Subash, "On k-normal matrices", International J. of Math. Sci. & Engg. Appls. 5(II) (2011), 119-130.

[2]. R.Penrose, "A Generalized inverse for matrices" Proc.Cambridge Philos.Soc., 51(1955), 406-413.

[3]. G.Wang , Y.Wei and S.Qiao, "Generalized Inverses" Theory and Computations, Science Press (Beijing) 2004.

[4]. R.D.Hill and R.S.Water, "On k-real and k-hermitian matrices," Linear Alg. Appl. Vol.169(1992), pp.17-29.

[5]. L.Elsner and Kh.D.Ikramov, Normal matrices: an update, Linear Algebra Appl. 285(1998), 291-303.

[6]. Oskar Maria Baksalary and Gotz Trenkler, "Characterizations of EP, Norma and Hermitian matrices" Linear and Multilinear Alg. Vol.56, No.3 (2008), 299-304.