Fourier transform and Plancherel Theorem for Nilpotent Lie Group

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Abstract

As will known the connected and simply connected nilpotent Lie group N has an important role in quantum mechanics. In this paper we show how the Fourier transform on the $n-$dimensional vector Lie group $\mathbb{R}^n$ can be generalized to N in order to obtain the Plancherel theorem. In addition we define the Fourier transform for the subgroup $NA = A \propto N$ of the real semi-simple Lie group $SL(n,\mathbb{R})$ to get also the Plancherel formula for NA

Keywords: Nilpotent Lie Group, Semi-simple Lie Group, Fourier Transform and Plancherel Theorem

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1 Notations and Results.

1.1. The fine structure of the nilpotent Lie groups will help us to do the Fourier transform on a nilpotent Lie groups N. As well known any group connected and simply connected N has the following form

$$N = \begin{pmatrix}
1 & x_1^1 & x_1^2 & x_1^3 & . & . & . & . & . & x_1^{n-2} & x_1^{n-1} & x_1^n \\
0 & 1 & x_2^2 & x_2^3 & . & . & . & . & . & x_2^{n-2} & x_2^{n-1} & x_2^n \\
0 & 0 & 1 & x_3^3 & . & . & . & . & . & x_3^{n-2} & x_3^{n-1} & x_3^n \\
0 & 0 & 0 & 1 & . & . & . & . & . & x_4^{n-2} & x_4^{n-1} & x_4^n \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & x_{n-2}^{n-2}. & .x_{n-2}^{n-1} & x_{n-2}^n \\
0 & 0 & 0 & 0 & . & . & . & . & . & 1 & x_{n-1}^{n-2} & x_{n=1}^n \\
0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 1 & x_n^n \\
0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 1
\end{pmatrix} \qquad (1)$$

As shown, the matrix (1) is formed by the subgroup $\mathbb{R}$, $\mathbb{R}^2$,...., $\mathbb{R}^{n-1}$, and $\mathbb{R}^n$

$$\left(\mathbb{R} = \begin{bmatrix} x_1^1 \\ 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbb{R}^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ 1 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \end{bmatrix}, ..., \mathbb{R}^{n-1} = \begin{bmatrix} x_1^{n-1} \\ x_2^{n-1} \\ x_3^{n-1} \\ x_4^{n-1} \\ . \\ . \\ . \\ . \\ .x_{n-2}^{n-1} \\ x_{n-1}^{n-2} \\ 1 \\ 0 \end{bmatrix}, \mathbb{R}^n = \begin{bmatrix} x_1^n \\ x_2^n \\ x_3^n \\ x_4^n \\ . \\ . \\ . \\ . \\ x_{n-2}^n \\ x_{n=1}^n \\ x_n^n \\ 1 \end{bmatrix} \right) \tag{2}$$

Each $\mathbb{R}^i$ is a subgroup of N of dimension i , $1 \le i \le n$, put $d = n + (n-1) + ... + 2 + 1$, which is the dimension of N . According to [6,7], the group N is isomorphic onto the following group

$$(((((\mathbb{R}^n \propto_{\rho_n})\mathbb{R}^{n-1}) \propto_{\rho_{n-1}})\mathbb{R}^{n-2} \propto_{\rho_{n-2}} .....) \propto_{\rho_2} \mathbb{R}^2) \propto_{\rho_1} \mathbb{R} \tag{3}$$

That means

$$N;(((((\mathbb{R}^n \propto_{\rho_n})\mathbb{R}^{n-1}) \propto_{\rho_{n-1}})\mathbb{R}^{n-2} \propto_{\rho_{n-2}} .....) \propto_{\rho_3} \mathbb{R}^3 \propto_{\rho_2} \mathbb{R}^2) \propto_{\rho_1} \mathbb{R} \tag{4}$$

1.2. Denote by $L^1(N)$ the Banach algebra that consists of all complex valued functions on the group N , which are integrable with respect to the Haar measure of N and multiplication is defined by convolution on N as follows:

$$g * f(X) = \int_N f(Y^{-1}X)g(Y)dY \tag{5}$$

for any $f \in L^1(N)$ and $g \in L^1(N)$, where $X = (X^1,\ X^2,\ X^3, ..., X^{n-2}, X^{n-1},\ X^n)$, $X^1 = x_1^1$, $X^2 = (x_1^2, x_2^2)$, $X^3 = (x_1^3, x_2^3, x_3^3)$,...., $X^{n-2} = (x_1^{n-2}, x_2^{n-2}, x_3^{n-2}, x_4^{n-2}, ..., x_{n-2}^{n-2})$, $X^{n-1} = (x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, x_4^{n-1}, ..., x_{n-2}^{n-1}, x_{n-1}^{n-1})$, $X^n = (x_1^n, x_2^n, x_3^n, x_4^n, ..., x_{n-2}^n, x_{n-1}^n, x_n^n)$,$Y = (Y^1,\ Y^2,\ Y^3, ..., \ Y^{n-2},\ Y^{n-1},\ Y^n)$, $Y^1 = y_1^1$, $Y^2 = (y_1^2,\ y_2^2)$,....,

$Y^{n-2} = (y_1^{n-2},\ y_2^{n-2},\ y_3^{n-2}, y_4^{n-2}, ..., y_{n-2}^{n-2}), Y^{n-1} = (y_1^{n-2}, y_2^{n-2}, y_3^{n-2}, y_4^{n-2}, ..., y_{n-2}^{n-2}, y_{n-1}^{n-1})$,

$Y^n = (y_1^{n-2}, y_2^{n-2}, y_3^{n-2}, y_4^{n-2}, ..., y_{n-2}^{n-2}, y_{n-1}^{n-1}, y_n^n)$, and $dY = dY^1 dY^2 dY^3 ....dY^{n-2} dY^{n-1} dY^n$ is the Haar measure on N and $*$ denotes the convolution product on N. We denote by $L^2(N)$ its Hilbert space

Let $M = \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2} \times ..... \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^d$ be the Lie group, which is the direct product of $\mathbb{R}^n, \mathbb{R}^{n-1}, \mathbb{R}^{n-2}, ..., \mathbb{R}^3, \mathbb{R}^2$ and $\mathbb{R}$. Denote by $L^1(M)$ the Banach algebra consists of all complex valued functions on the group M , which are integrable with respect to the Lebesgue measure on M and multiplication is defined by convolution on M as:

$$g *_c f(X) = \int_M f(X - Y)g(Y)dY \tag{6}$$

for any $f \in L^1(M)$, $g \in L^1(M)$, where $*_c$ signifies the convolution product on the abelian group M. In this paper, we use the methods in [3,4,6,7] to show the powerful of the Fourier transform on $\mathbb{R}^d$ which can be generalized on N in order to obtain the Plancherel theorem.

2 Fourier Transform and Plancherel Formula for N.

Definition 2.1. *For $1 \le i \le n$, let $\Im^i$ be the classical Fourier transform on $\mathbb{R}^i$, we can define the Fourier transform on N as*

$$\Im f(\lambda) = \int_N f(X) e^{-i(\langle \lambda, X \rangle)} dX \tag{7}$$

for any $f \in L^1(N)$, where $X = (X^1, X^2, X^3, ..., X^{n-2}, X^{n-1}, X^n)$, $dX = dX^1 dX^2 dX^3 ... dX^{n-2} dX^{n-1} dX^n$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3, ..., \lambda^{n-2}, \lambda^{n-1}, \lambda^n)$, where $\mathcal{F}$ is the commutative Fourier transform on $\mathbb{R}^d$

Plancherel formula (Theorem 2.1). *For every function $f \in L^1(N)$, we have*

$$\int_{\mathbb{R}^d} \left| \Im f(\lambda) \right|^2 d\lambda = \int_N \left| f(X) \right|^2 dX \tag{8}$$

where $d\lambda$ is the lebesgue measure on $\mathbb{R}^d$

Proof: For each $1 \le j \le n$, let $\Im^j$ be the Fourier transform on $\mathbb{R}^j$. If we denote $T_j^n = \Im^n \Im^{n-1} \Im^{n-1} \Im^{n-2} ... \Im^j$, $\mathbb{R}_j^n = \mathbb{R}^n \times \mathbb{R}^{n=1} \times \mathbb{R}^{n=2} \times ... \mathbb{R}^j$, $X_j^n = X^n X^{n-1} X^{n-2} ... X^j$, $dX_j^n = dX^n dX^{n-1} dX^{n-2} ... dX^j$, $\lambda_j^n = \lambda^n \lambda^{n-1} \lambda^{n-2} ... \lambda^3 \lambda^2 \lambda^j$, and $d\lambda_j^n = d\lambda^n d\lambda^{n-1} d\lambda^{n-2} ... d\lambda^j$, then we get $\Im = T_1^n$, $\mathbb{R}^d = \mathbb{R}_1^n, X = X_1^n, dX = dX_1^n$, $\lambda = \lambda_1^n$, $d\lambda = d\lambda_1^n$. Now if we refer to [6,3], then we get

$$\int_{\mathbb{R}_2^n} \int_{\mathbb{R}} \left| T_2^n \Im^1 f(\lambda_2^n, \lambda^1) \right|^2 d\lambda_2^n d\lambda^1 = \int_{\mathbb{R}_2^n} \int_{\mathbb{R}} \left| f(X_2^n, X^1) \right|^2 dX_2^n dX^1 \tag{9}$$

and

$$\int_{\mathbb{R}_3^n} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| T_3^n \Im^2 \Im^1 f(\lambda_3^n, \lambda^2, \lambda^1) \right|^2 d\lambda_2^n d\lambda^2 d\lambda^1 = \int_{\mathbb{R}_3^n} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| f(X_3^n, X^2, X^1) \right|^2 dX_3^n dX^2 dX^1 \tag{10}$$

By indiction we get

$$\begin{aligned} &\int_{\mathbb{R}_n^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} ... \int_{\mathbb{R}3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| f(X^n, X^{n-1}, X^{n-2}, ..., X^3, X^2, X^1) \right|^2 \\ &dX^n dX^{n-1} dX^{n-2} ... dX^3 dX^2 dX^1 \\ &= \int_{\mathrm{N}} \left| f(X^n, X^{n-1}, X^{n-2}, ..., X^3, X^2, X^1) \right|^2 dX^n dX^{n-1} dX^{n-2} ... dX^3 dX^2 dX^1 \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} ... \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| \Im^n \Im^{n-1} \Im^{n-2} ... \Im^3 \Im^2 \Im^1 f(\lambda^n, \lambda^{n-1}, \lambda^{n-2}, ... \lambda^3, \lambda^2, \lambda^1) \right|^2 \\ &d\lambda^n d\lambda^{n-1} d\lambda^{n-2} ... d\lambda^3 d\lambda^2 d\lambda^1 \\ &= \int_{\mathbb{R}^d} \left| \Im^n \Im^{n-1} \Im^{n-2} ... \Im^3 \Im^2 \Im^1 f(\lambda^n, \lambda^{n-1}, \lambda^{n-2}, ... \lambda^3, \lambda^2, \lambda^1) \right|^2 \\ &d\lambda^n d\lambda^{n-1} d\lambda^{n-2} ... d\lambda^3 d\lambda^2 d\lambda^1 \end{aligned} \tag{11}$$

Hence the proof of our theorem.**2.1.**

3 Plancherel Formula for the Solvable Lie Group AN

3.1. Let $G = SL(n,\mathbb{R})$ be the real semi-simple Lie group and let $G = KAN$ be the Iwasawa decomposition of G, where $K = SO(n,\mathbb{R})$,

$$N = \begin{pmatrix} 1 & * & . & . & * \\ 0 & 1 & * & . & * \\ . & . & . & . & * \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 1 \end{pmatrix}, \tag{12}$$

and

$$A = \begin{pmatrix} a_1 & 0 & 0 & . & 0 \\ 0 & a_2 & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & 0 & a_n \end{pmatrix} \tag{13}$$

where $a_1.a_2....a_n = 1$ and $a_i \in \mathbb{R}_+^*$

The product AN is a closed subgroup of G and is isomorphic (algebraically and topologically) to the semi-direct product of A and N with N normal in AN. Then the group AN is nothing but the group $S = N \propto_\rho A$, where $\rho : A \to Aut(N)$ the group homomorphism from A into $Aut(N)$ of all automorphisms of N , which is defined by

$$\rho(a)(m) = ama^{-1} \tag{14}$$

So the product of two elements X and Y by

$$(x,a)(m,b) = (x.\rho(a)m, a.b) = (ama^{-1}, a.b) \tag{15}$$

for any $X = (x, a) \in S$ and $Y = (m, b) \in S$. Let $dnda$ be the right haar measure on S and let $L^2(S)$ be the Hilbert space of the group S. Let $L^1(S)$ be the Banach algebra that consists of all complex valued functions on the group S , which are integrable with respect to the Haar measure of S and multiplication is defined by convolution on S as

$$g * f = \int_S f((m,b)^{-1}(n,a))g(m,b)dmdb \tag{16}$$

Where

$$m = (m_1, m_2, ..., m_{r=1}, m_r), b = (b_1, b_2, ..., b_{n-1}, b_n) \tag{17}$$

and

$$dmdb = dm_1 dm_2 .... dm_{r=1} dm_r dbdb_1 db_2 .... db_{n=1}) \tag{18}$$

In the following we prove the Plancherel theorem for NA. Therefore let $T = N \times A$ be the Lie group which is the direct product of the two Lie groups N and A, and let $H = N \times A \times A$ be the Lie group, with multiplication

$$(n,t,r)(m,s,q)=(n\rho(r)m,ts,rq)$$

for all $(n,t,r)\in H$ and $(m,s,q)\in H$. In this case the group S can be identified with the closed subgroup $N\times\{0\}\times_{\rho}A$ of H and T with the subgroup $N\times A\times\{0\}$ of H

Definition 3.1. *For every functon* f defined on S, *one can define a function* on L *as follows:*

$$\tilde{f}(n,a,b)=\tilde{f}(\rho(a)n,ab) \tag{19}$$

for all $(n,a,b)\in H$. *So every function* $\psi(n,a)$ *on* S *extends uniquely as an invariant function* $\tilde{\psi}$ $(n,\ b,\ a)$ *on* L.

Remark 3.1. *The function* $\tilde{f}$ *is invariant in the following sense:*

$$\tilde{f}(\rho(s)n,as^{-1},sb)=\tilde{f}(n,a,b) \tag{20}$$

for any $(n,a,b)\in H$ *and* $s\in A$.

Lemma 3.1. *For every function* $f\in L^1(S)$ *and for every* $g\in L^1(S)$, *we have*

$$g*\tilde{f}(n,a,b)=g*_c\tilde{f}(n,a,b) \tag{21}$$

for every $(n,a,b)\in H$, *where* $*$ *signifies the convolution product on* S *with respect the variables* (n,b) *and* $*_c$ *signifies the commutative convolution product on* B *with respect the variables* (n,a).

Proof: In fact we have

$$g*\tilde{f}(n,a,b)=\int_G\tilde{f}((m,c)^{-1}(n,a,b))g(m,s)dmdc$$

$$=\int_S\tilde{f}\big[(\rho(c^{-1})(m^{-1}),c^{-1})(n,a,b)\big]g(m,s)dmdc$$

$$=\int_S\tilde{f}\big[\rho(c^{-1})(m^{-1}n),a,c^{-1}b\big]g(m,c)dmdc$$

$$=\int_S\tilde{f}\big[m^{-1}n,c^{-1}a,b\big]g(m,c)dmdc=g*_c\tilde{f}(n,a,b) \tag{22}$$

Definition 3.2. *If* $f\in L^1(S)$, *one can define its Fourier transform* $\Im f$ *by* :

$$\Im f(\xi,\lambda)=\int_S f(n,a)e^{-i(\langle\xi,n\rangle)}a^{-i\lambda}dnda \tag{23}$$

for any $\xi=(\xi_1,\xi_2,...,\xi_r)\in\mathbb{R}^r, n=(x_1,x_2,...,x_r)\in\mathbb{R}^r$, $\lambda=(\lambda_1,\lambda_2,...,\lambda_{n-1})\in\mathbb{R}^{n-1}$ *and* $a=(a_1,a_2,...,a_{n-1},a_n)$, $a_i\in\mathbb{R}_+^*$, $1\le i\le n$, $a_1a_2...a_{n-1}a_n=1$, *where* $\langle\xi\ x\rangle=\xi_1x_1+\xi_2x_2+...+\xi_rx_r$ $dn=dx_1dx_2...dx_r$, $da=\frac{da_1}{a_1}.\frac{da_2}{a_2}...\frac{da_{n-1}}{a_{n-1}}$, *and* $a^{-i\lambda}=a_1^{-i\lambda_1}a_2^{-i\lambda_2}...a_{n-1}^{-i\lambda_{n-1}}$. *Denote by* $\mathcal{S}(S)$ *the Schwartz space of the group* $S=N\propto_{\rho}A$, *it is clear that if* $f\in\mathcal{S}(S)$, then $\Im f\in\mathcal{S}(S)$ *and the mapping* $f\to\Im f$ *is topological isomorphism of the topological vector space* $\mathcal{S}(S)$ *onto* $\mathcal{S}(\mathbb{R}^{r+n-1})$.

Definition 3.3. *If* $f\in L^1(S)$, *we define the Fourier transform of its invariant* $\tilde{f}$ *as follows*

$$\Im(\tilde{f})(\xi,\lambda,0)=\int_{H\times\mathbb{R}^{n-1}}\tilde{f}(n,a,b)e^{-i(\langle\xi,n\rangle)}a^{-i\lambda}b^{-i\lambda}dndadbd\mu \tag{24}$$

where $\mu=(\mu_1,\mu_2,...,\mu_{n-1})\in\mathbb{R}^{n-1}$ *and* $b=(b_1,b_2,...,b_{n-1})\in\mathbb{R}^{n-1}$

Theorem 3.1. *For every* $g \in L^1(S)$, *and* $f \in L^1(S)$, *we have*

$$\int_{\mathbb{R}^{n-1}} (g * \widetilde{f})(\xi, \lambda, \mu) d\mu = \int_{\mathbb{R}^{n-1}} \Im(\widetilde{f})(\xi, \lambda, \mu) \mathsf{F}(g)(\xi, \lambda) d\mu = \Im(\widetilde{f})(\xi, \lambda, 0) \mathsf{F}(g)(\xi, \lambda) \quad (25)$$

Proof: By equation (21) we get immediately

$$\int_{\mathbb{R}^{n-1}} \Im(g * \widetilde{f})(\xi, \lambda, \mu) d\mu = \int_{\mathbb{R}^{n-1}} \Im(g *_c \widetilde{f})(\xi, \lambda, \mu) d\mu = \Im(\widetilde{f})(\xi, \lambda, 0) \Im(g)(\xi, \lambda) \quad (26)$$

Plancherel's theorem 3.2. *For any* $f \in L^1(S) \cap L^2(S)$, *we have*

$$\int_S |f(n,a)|^2 dnda = \int_{\mathbb{R}^r \times \mathbb{R}^{n-1}} |\Im f(\xi, \lambda)|^2 d\xi d\lambda \quad (27)$$

Proof: First, let $\widetilde{\widetilde{f}}$ be the function defined by

$$\widetilde{\widetilde{f}}(n,a,b) = \overline{f((\rho(a)n, ab)^{-1})} \quad (28)$$

then first we have

$$f * \widetilde{\widetilde{f}}(I_N, I_A, I_A) = \int_S \widetilde{\widetilde{f}}\left[(n,a)^{-1}(I_N, I_A, I_A)\right] f(n,a) dnda$$

$$= \int_S \widetilde{\widetilde{f}}\left[(\rho(a^{-1})(n^{-1}, a^{-1}))(I_N, I_A, I_A\right] f(n,a) dnda$$

$$\int_S \widetilde{\widetilde{f}}\left[(\rho(a^{-1})(n^{-1}))(I_N)), I_A, I_A a^{-1}\right] f(n,a) dnda$$

$$= \int_S \widetilde{\widetilde{f}}\left[\rho(a^{-1})(n^{-1}), I_A, a^{-1}\right] f(n,a) dnda$$

$$= \int_S \check{f}\left[\rho(a^{-1})(n^{-1}), a^{-1}\right] f(n,a) dxdt = \int_S \overline{f(n,a)} f(n,a) dnda$$

$$= \int_S |f(n,a|^2 dnda \quad (29)$$

Secondly by (27), we obtain

$$f * \widetilde{\widetilde{f}}(I_N, I_A, I_A)$$

$$= \int_{\mathbb{R}^{r+2(n-1)}} \Im(f * \widetilde{\widetilde{f}})(\xi, \lambda, \mu) d\xi d\lambda d\mu = \int_{\mathbb{R}^{r+2n-2}} \Im(f *_c \widetilde{\widetilde{f}})(\xi, \lambda, \mu) d\xi d\lambda d\mu$$

$$= \int_{\mathbb{R}^{r+n-1}} \Im(f *_c \widetilde{\widetilde{f}})(\xi, \lambda, 0) d\xi d\lambda = \int_{\mathbb{R}^{r+n-1}} \Im(\widetilde{\widetilde{f}})(\xi, \lambda, 0) \Im(f)(\xi, \lambda) d\xi d\lambda$$

$$= \int_{\mathbb{R}^{r+n-1}} \overline{\Im(f)(\xi, \lambda)} \Im(f)(\xi, \lambda) d\xi d\lambda = \int_{\mathbb{R}^{r+n-1}} |\Im(f)(\xi, \lambda)|^2 d\xi d\lambda = \int_S |f(x,t)|^2 dxdt$$

which is the Plancherel's formula on S. *So the Fourier transform can be extended to an isometry of* $L^2(S)$ onto $L^2(\mathbb{R}^{r+n-1})$.

Corollary 2.2. *In equation* (29), *replace the first* f by g, *we obtain*

$$\int_S \overline{f(x,t)}g(x,t)dxdt = \int_{\mathbb{R}^{r+n-1}} \overline{\Im(f)(\xi,\lambda}\Im g(\xi,\lambda)d\xi d\lambda \quad (30)$$

which is the Parseval formula on S.

4 References

[1] K. El- Hussein, A Fundamental Solution of an Invariant Differential Operator on the Heisenberg Group, Mathematical Forum, 4, no. 12, 601 - 612. 2009
[2] K. El- Hussein, Eigendistributions for the Invariant Differential operators on the Affine Group. Int. Journal of Math. Analysis, Vol. 3, no. 9, 419-429. 2009
[3] K. El- Hussein, Fourier transform and invariant differential operators on the solvable Lie group G4, in Int. J. Contemp. Maths Sci. 5. No. 5-8, 403-417. 2010
[4] K. El- Hussein, On the left ideals of group algebra on the affine group, in Int. Math Forum, Int, Math. Forum 6, No. 1-4, 193-202. 2011
[5] K. El- Hussein, Note on the Solvability of the Mizohata Operator, International Mathematical Forum, 5, no. 37, 1833 - 1838. 2010
[6] K. El- Hussein, Non Commutative Fourier Transform on Some Lie Groups and Its Application to Harmonic Analysis, International Journal of Engineering Research & Technology (IJERT) Vol. 2 Issue 10, 2429- 2442. 2013.
[7] K. El- Hussein, Abstract Harmonic Analysis and Ideals of Banach Algebra on 3-Step Nilpotent Lie Groups, International Journal of Engineering Research & Technology (IJERT), Vol. 2 Issue 11, November - 2013
[8] S. Helgason, The Abel, Fourier and Radon Transforms on Symmetric Spaces. Indagationes Mathematicae. 16, 531-551. 2005
[9] W.Rudin, *Fourier Analysis on Groups*, Interscience Publishers, New York, NY. 1962.