Disks which do not Contain any Zero of a Polynomial

M. H. Gulzar
Department of Mathematics
University of Kashmir, Srinagar 190006

Abstract: In this paper we find disks which do not contain any zero of a polynomial when the coefficients of the polynomial are restricted to certain conditions.

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1. Introduction and Statement of Results

Under certain restricted conditions on the coefficients of a polynomial, researchers have been able to find regions containing some or all or no zero of a polynomial. In this connection various published papers are available in the literature . Recently M. H. Gulzar [3] proved the following results:

Theorem A: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\rho+\alpha_n \ge \alpha_{n-1} \ge ......\alpha_{n-k+1} \ge \lambda\alpha_{n-k} \ge \alpha_{n-k-1}......\ge \alpha_1 \ge \tau\alpha_0 ,$$

for some real numbers $\lambda, \rho \ge 0, 0<\tau \le 1, 1 \le k \le n, \alpha_{n-k} \ne 0.$.

If $\alpha_{n-k-1} > \alpha_{n-k}$, then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R>0, c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \le 1$.

If $\alpha_{n-k} > \alpha_{n-k+1}$, then the number f zeros of P(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \le 1$.

Theorem B: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,2,......,n$$

and

$$|\rho + a_n| \ge |a_{n-1}| \ge ...... \ge |a_{n-k+1}| \ge \lambda|a_{n-k}| \ge |a_{n-k-1}| \ge .....|a_1| \ge \tau|a_0| \ ,$$

for some $\rho \ge 0, \lambda > 0, 1 \le k \le n, a_{n-k} \ne 0, 0 < \tau \le 1$.

If $|a_{n-k-1}| > |a_{n-k}| i.e. \lambda > 1$, then the number of zeros of P(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{M}{|a_0|} \ ,$$

where

$$M = R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|], \quad \text{for } R \ge 1$$

and

$$M = |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$

$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha\sum_{j=1, j\neq n-k}^{n-1}|a_j|], \quad \text{for } R \le 1.$$

If $|a_{n-k}| > |a_{n-k+1}| i.e. \lambda < 1$, then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{M'}{|a_0|} \quad ,$$

Where

$$M' = R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$

$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha\sum_{j=1, j\neq n-k}^{n-1}|a_j|] \quad \text{for } R \ge 1$$

and

$$M' = |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$

$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha\sum_{j=1, j\neq n-k}^{n-1}|a_j|] \quad \text{for } R \le 1.$$

The aim of this paper is to find disks which do not contain any zero of the polynomials in theorems 1 and 2. In fact, we are going to prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ......\alpha_{n-k+1} \ge \lambda\alpha_{n-k} \ge \alpha_{n-k-1}...... \ge \alpha_1 \ge \tau\alpha_0,$$

for some real numbers $\lambda, \rho \ge 0, 0 < \tau \le 1, 1 \le k \le n, \alpha_{n-k} \ne 0.$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then in $|z| \le R$, P(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \le 1$, where

$$M_1 = R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0|$$

$$+ 2\sum_{j=1}^{n}|\beta_j|]$$

and

$$M_2 = R[2\rho + |\alpha_n| + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|].$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then for $|z| \le R$, P(z) has no zero in $|z| < \frac{|a_0|}{M_3}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_4}$ for $R \le 1$, where

$$M_3 = R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|]$$

and

$$M_4 = R[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|].$$

Applying Theorem 1 to the polynomial –iP(z) , we get the following result:

Theorem 2: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and

$$\rho + \beta_n \ge \beta_{n-1} \ge ......\ge \beta_{n-k+1} \ge \lambda\beta_{n-k} \ge \beta_{n-k-1} ......\ge \beta_1 \ge \tau\beta_0,$$

for some real numbers $\lambda, \rho \ge 0, 0 < \tau \le 1, 1 \le k \le n, \beta_{n-k} \ne 0.$

If $\beta_{n-k-1} > \beta_{n-k}$, then in $|z| \le R$, P(z) has no zero in $|z| < \frac{|a_0|}{M_1^*}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2^*}$ for $R \le 1$, where

$$M_1^* = R^{n+1}[2\rho + |\beta_n| + \beta_n + (\lambda - 1)\beta_{n-k} + |\lambda - 1||\beta_{n-k}| - \tau(|\beta_0| + \beta_0) + |\beta_0| + |\alpha_0| + 2\sum_{j=1}^{n} |\alpha_j|]$$

and

$$M_2^* = R[2\rho + |\beta_n| + \beta_n + (\lambda - 1)\beta_{n-k} + |\lambda - 1||\beta_{n-k}| - \tau(|\beta_0| + \beta_0) + |\beta_0| + |\alpha_0| + 2\sum_{j=1}^{n} |\alpha_j|].$$

If $\beta_{n-k} > \beta_{n-k+1}$, then for $|z| \le R$, P(z) has no zero in $|z| < \frac{|a_0|}{M_3^*}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_4^*}$ for $R \le 1$, where

$$M_3^* = R^{n+1}[2\rho + |\beta_n| + \beta_n + (1-\lambda)\beta_{n-k} + |1-\lambda||\beta_{n-k}| - \tau(|\beta_0| + \beta_0) + |\beta_0| + |\alpha_0| + 2\sum_{j=1}^{n}|\alpha_j|]$$

and

$$M_4^* = R[2\rho + |\beta_n| + \beta_n + (1-\lambda)\beta_{n-k} + |1-\lambda||\beta_{n-k}| - \tau(|\beta_0| + \beta_0) + |\beta_0| + |\alpha_0| + 2\sum_{j=1}^{n}|\alpha_j|].$$

Combining Theorem A and Theorem 1, we get the following result:

Corrollary 1: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ......\alpha_{n-k+1} \ge \lambda\alpha_{n-k} \ge \alpha_{n-k-1}...... \ge \alpha_1 \ge \tau\alpha_0,$$

for some real numbers $\lambda, \rho \ge 0, 0 < \tau \le 1, 1 \le k \le n, \alpha_{n-k} \ne 0.$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then the number of zeros of P(z) in $\frac{|a_0|}{M_1} \le |z| \le \frac{R}{c} (R > 0, c > 1)$, does not exceed

$$\frac{1}{\log c} \log \frac{R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (\lambda-1)\alpha_{n-k} + |\lambda-1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log c} \log \frac{|a_0| + R[2\rho + |\alpha_n| + \alpha_n + (\lambda-1)\alpha_{n-k} + |\lambda-1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \le 1$.

If $\alpha_{n-k} > \alpha_{n-k+1}$, then the number f zeros of P(z) in $\frac{|a_0|}{M_2} \le |z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \le 1$.

Here M_1 and M_2 are as in Theorem 1.

Theorem 3: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, j = 0,1,2,......,n$$

and

$$|\rho + a_n| \ge |a_{n-1}| \ge ...... \ge |a_{n-k+1}| \ge \lambda|a_{n-k}| \ge |a_{n-k-1}| \ge .....|a_1| \ge \tau|a_0| ,$$

for some $\rho \ge 0, \lambda > 0, 1 \le k \le n, a_{n-k} \ne 0, 0 < \tau \le 1$.

If $|a_{n-k-1}| > |a_{n-k}|$, then for $|z| \le R$, P(z) has no zero in $|z| < \frac{|a_0|}{M_1'}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2'}$ for $R \le 1$, where

$$M_1' = R^{n+1}[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|]$$

and

$$M_2' = R[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|].$$

If $|a_{n-k}| > |a_{n-k+1}|$, then for $|z| \le R$, P(z) has no zero in $|z| < \frac{|a_0|}{M_3'}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_4'}$ for $R \le 1$, where

$$M_3' = R^{n+1}[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha + 1 - \lambda) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]$$

and

$$M_4' = R[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha + 1 - \lambda) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|].$$

For different values of the parameters in the above results, we get many interesting results which, at the same time, generalize many known results on the subject.

2. Lemmas

For the proofs of the above results we need the following results:

Lemma 1: If f(z) is analytic in $|z| \le R$,but not identically zero, f(0)$\neq$0 and $f(a_k) = 0, k = 1,2,......,n$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta}|d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma 2: If f(z) is analytic, f(0)$\neq$0 and $|f(z)| \le M(r)$ in $|z| \le r$, then the number of zeros of f(z) in $|z| \le \frac{r}{c}, c > 1$ does not exceed

$$\frac{1}{\log c}\log\frac{M(r)}{|f(0)|}.$$

Lemma 2 is a simple deduction from Lemma 1.

Lemma 3: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, 0\le j\le n,$ and $|a_j|\ge|a_{j-1}|, 0\le j\le n,$ then

$$|a_j-a_{j-1}|\le(|a_j|-|a_{j-1}|)\cos\alpha+(|a_j|+|a_{j-1}|)\sin\alpha.$$

Lemma 3 is due to Govil and Rahman [2].

3. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z)&=(1-z)P(z)\\
&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\
&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}\\
&\quad+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......+(a_1-a_0)z+a_0\\
&=-(\alpha_n+i\beta_n)z^{n+1}+(\alpha_n-\alpha_{n-1})z^n+......+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}\\
&\quad+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......+(\alpha_1-\tau\alpha_0)z\\
&\quad+(\tau-1)\alpha_0z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+a_0\\
&=a_0+G(z),\ \text{where}
\end{aligned}$$

$$\begin{aligned}
G(z)&=-(\alpha_n+i\beta_n)z^{n+1}+(\alpha_n-\alpha_{n-1})z^n+......+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}\\
&\quad+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......+(\alpha_1-\tau\alpha_0)z\\
&\quad+(\tau-1)\alpha_0z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j
\end{aligned}$$

If $\alpha_{n-k-1}>\alpha_{n-k}$, then

$$G(z)=-(\alpha_n+i\beta_n)z^{n+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+......+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}$$

$$+(\lambda\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}-(\lambda-1)\alpha_{n-k}z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+\ldots\ldots$$
$$+(\alpha_1-\tau\alpha_0)z+(\tau-1)\alpha_0 z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j.$$

For $|z|\le R$, we have by using the hypothesis

$$\begin{aligned}|G(z)|&\le|\alpha_n|R^{n+1}+|\beta_n|R^{n+1}+|\rho|R^n+|\rho+\alpha_n-\alpha_{n-1}|R^n+\ldots\ldots+|\alpha_{n-k+1}-\alpha_{n-k}|R^{n-k+1}\\&\quad+|\lambda\alpha_{n-k}-\alpha_{n-k-1}|R^{n-k}+|\lambda-1||\alpha_{n-k}|R^{n-k}\\&\quad+|\alpha_{n-k-1}-\alpha_{n-k-2}|R^{n-k-1}+\ldots\ldots+|\alpha_1-\tau\alpha_0|R+(1-\tau)|\alpha_0|R\\&\quad+|\alpha_0|+|\beta_0|+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)R^j\\&=|\alpha_n|R^{n+1}+|\beta_n|R^{n+1}+\rho R^n+(\rho+\alpha_n-\alpha_{n-1})R^n+\ldots\ldots+(\alpha_{n-k+1}-\alpha_{n-k})R^{n-k+1}\\&\quad+(\lambda\alpha_{n-k}-\alpha_{n-k-1})R^{n-k}+|\lambda-1||\alpha_{n-k}|R^{n-k}\\&\quad+(\alpha_{n-k-1}-\alpha_{n-k-2})R^{n-k-1}+\ldots\ldots+(\alpha_1-\tau\alpha_0)R+(1-\tau)|\alpha_0|R\\&\quad+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)R^j\\&\le R^{n+1}[2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|\\&\quad+2\sum_{j=0}^{n}|\beta_j|]\\&=M_1\qquad\text{for } R\ge1\end{aligned}$$

and

$$|G(z)|\le R[2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

$$=M_2\qquad\text{for } R\le1.$$

Since G(z) is analytic for $|z|\le R$ and G(0)=0, it follows by Schwarz Lemma that

$$|G(z)|\le M_1|z| \text{ for } R\ge1$$

and

$$|G(z)|\le M_2|z| \text{ for } R\le1.$$

Hence for $|z|\le R$, $R\ge1$,

$$|F(z)|=|a_0+G(z)|$$

$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - M_1|z|$$
$$> 0$$

if

$$|z| < \frac{|a_0|}{M_1}.$$

And for $|z| \leq R$, $R \leq 1$, $|F(z)| > 0$ if $|z| < \frac{|a_0|}{M_2}$.

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \leq 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that that P(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \leq 1$.

If $\alpha_{n-k} > \alpha_{n-k+1}$, then

$$G(z) = -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \dots\dots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1}$$
$$+ (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k}z^{n-k+1} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots\dots$$
$$+ (\alpha_1 - \alpha_0)z + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j.$$

For $|z| \leq R$, we have by using the hypothesis

$$|G(z)| \leq |\alpha_n|R^{n+1} + |\beta_n|R^{n+1} + |\rho|R^n + |\rho + \alpha_n - \alpha_{n-1}|R^n + \dots\dots + |\alpha_{n-k+1} - \lambda\alpha_{n-k}|R^{n-k+1}$$
$$+ |\alpha_{n-k} - \alpha_{n-k-1}|R^{n-k} + |1-\lambda||\alpha_{n-k}|R^{n-k+}$$
$$+ |\alpha_{n-k-1} - \alpha_{n-k-2}|R^{n-k-1} + \dots\dots + |\alpha_1 - \tau\alpha_0|R + (1-\tau)|\alpha_0|R$$
$$+ \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)R^j$$

$$\leq R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0|$$
$$+ 2\sum_{j=1}^{n}|\beta_j|]$$
$$= M_3 \quad \text{for } R \geq 1$$

and

$$|G(z)| \leq R[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|$$
$$= M_4 \quad \text{for } R \leq 1.$$

Since G(z) is analytic for $|z| \le R$ and G(0)=0, it follows by Schwarz Lemma that

$$|G(z)| \le M_3 |z| \text{ for } R \ge 1$$

and

$$|G(z)| \le M_4 |z| \text{ for } R \le 1.$$

Hence for $|z| \le R$, $R \ge 1$,

$$\begin{aligned} |F(z)| &= |a_0 + G(z)| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - M_3 |z| \\ &> 0 \end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_3}.$$

And for $|z| \le R$, $R \le 1$, $|F(z)| > 0$ if $|z| < \dfrac{|a_0|}{M_4}$.

This shows that F(z) has no zero in $|z| < \dfrac{|a_0|}{M_3}$ for $R \ge 1$ and no zero in $|z| < \dfrac{|a_0|}{M_4}$ for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that P(z) has no zero in $|z| < \dfrac{|a_0|}{M_3}$ for $R \ge 1$ and no zero in $|z| < \dfrac{|a_0|}{M_4}$ for $R \le 1$.

That proves Theorem 1.

Proof of Theorem 3: Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots\dots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \dots\dots + (a_{n-k+1} - a_{n-k}) z^{n-k+1} + (a_{n-k} - a_{n-k-1}) z^{n-k} \\ &\quad + (a_{n-k-1} - a_{n-k-2}) z^{n-k-1} + \dots\dots + (a_1 - a_0) z + a_0 \\ &= a_0 + G(z), \text{ where} \end{aligned}$$

$$\begin{aligned} G(z) == -a_n z^{n+1} &+ (a_n - a_{n-1}) z^n + \dots\dots + (a_{n-k+1} - a_{n-k}) z^{n-k+1} + (a_{n-k} - a_{n-k-1}) z^{n-k} \\ &+ (a_{n-k-1} - a_{n-k-2}) z^{n-k-1} + \dots\dots + (a_1 - a_0) z. \end{aligned}$$

If $|a_{n-k-1}| > |a_{n-k}|$, i.e. $\lambda > 1$, then

$$\begin{aligned} G(z) = -a_n z^{n+1} - \rho z^n &+ (\rho + a_n - a_{n-1}) z^n + \dots\dots + (a_{n-k+1} - a_{n-k}) z^{n-k+1} \\ &+ (\lambda a_{n-k} - a_{n-k-1}) z^{n-k} - (\lambda - 1) a_{n-k} z^{n-k} + (a_{n-k-1} - a_{n-k-2}) z^{n-k-1} + \dots\dots \\ &+ (a_1 - \tau a_0) z + (\tau - 1) a_0 z \end{aligned}$$

so that for $|z| \le R$, we have by using the hypothesis and Lemma 3,

$$\begin{aligned}|G(z)| &\le |a_n|R^{n+1} + \rho R^n + |\rho + a_n - a_{n-1}|R^n + ...+|a_{n-k+1} - a_{n-k}|R^{n-k+1} + |\lambda a_{n-k} - a_{n-k-1}|R^{n-k} \\ &\quad + |\lambda - 1||a_{n-k}|R^{n-k} + |a_{n-k-1} - a_{n-k-2}|R^{n-k-1} + ...+|a_1 - \tau a_0|R + (1-\tau)|a_0|R \\ &\le R^{n+1}[|a_n| + \rho + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + ... \\ &\quad + (|a_{n-k+1}| - |a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + |a_{n-k}|)\sin\alpha + (\lambda - 1)|a_{n-k}| \\ &\quad + (\lambda|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (\lambda|a_{n-k}| + |a_{n-k-1}|)\sin\alpha \\ &\quad + (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha + ... \\ &\quad + (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0|] \\ &\le R^{n+1}[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha \\ &\qquad - \lambda\sin\alpha - \lambda + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|\,] \\ &= M_1{}' \quad \text{for } R \ge 1\end{aligned}$$

and

$$\begin{aligned}&\le R[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha \\ &\qquad - \lambda\sin\alpha - \lambda + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|\,] \\ &= M_2{}' \quad \text{for } R \le 1.\end{aligned}$$

Since G(z) is analytic for $|z| \le R$ and G(0)=0, it follows by Schwarz Lemma that

$$|G(z)| \le M_1{}'|z| \ \text{ for } R \ge 1$$

and

$$|G(z)| \le M_2{}'|z| \ \text{ for } R \le 1.$$

Hence for $|z| \le R$, $R \ge 1$,

$$\begin{aligned}|F(z)| &= |a_0 + G(z)| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - M_1{}'|z| \\ &> 0\end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_1{}'}.$$

And for $|z| \le R$, $R \le 1$, $|F(z)| > 0$ if $|z| < \dfrac{|a_0|}{M_2{}'}$.

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{M_1'}$ for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_2'}$ for $R \leq 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that that P(z) has no zero in $|z| < \frac{|a_0|}{M_1'}$ for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_2'}$ for $R \leq 1$.

If $|a_{n-k}| > |a_{n-k+1}|$, i.e. $\lambda < 1$, then

$$\begin{aligned} G(z) &= -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1}) z^n + ...+ (a_{n-k+1} - \lambda a_{n-k}) z^{n-k+1} \\ &\quad + (a_{n-k} - a_{n-k-1}) z^{n-k} - (1-\lambda) a_{n-k} z^{n-k+1} + (a_{n-k-1} - a_{n-k-2}) z^{n-k-1} + ... \\ &\quad + (a_1 - a_0) z \end{aligned}$$

so that for $|z| \leq R$, we have by hypothesis and Lemma 3,

$$\begin{aligned} |G(z)| &\leq |a_n| R^{n+1} + \rho R^n + |\rho + a_n - a_{n-1}| R^n + ...+ |a_{n-k+1} - \lambda a_{n-k}| R^{n-k+1} + |a_{n-k} - a_{n-k-1}| R^{n-k} \\ &\quad + |1-\lambda||a_{n-k}| R^{n-k+1} + |a_{n-k-1} - a_{n-k-2}| R^{n-k-1} + ... \\ &\quad + |a_1 - \tau a_0| R + (1-\tau)|a_0| R \\ &\leq R^{n+1}[|a_n| + \rho + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + ... \\ &\quad + (|a_{n-k+1}| - \lambda |a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + \lambda |a_{n-k}|)\sin\alpha + |1-\lambda||a_{n-k}| \\ &\quad + (|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (|a_{n-k}| + |a_{n-k-1}|)\sin\alpha \\ &\quad + (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha + ... \\ &\quad + (|a_1| - \tau |a_0|)\cos\alpha + (|a_1| + \tau |a_0|)\sin\alpha + (1-\tau)|a_0|] \\ &\leq R^{n+1}[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha \\ &\quad + \lambda\sin\alpha + 1 - \lambda) - \tau |a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|] \\ &= M_3' \quad \text{for } R \geq 1 \end{aligned}$$

and

$$\begin{aligned} |G(z)| &\leq R[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha \\ &\quad + \lambda\sin\alpha + 1 - \lambda) - \tau |a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|] \\ &= M_4' \quad \text{for } R \leq 1. \end{aligned}$$

Since G(z) is analytic for $|z| \leq R$ and G(0)=0, it follows by Schwarz Lemma that

$$|G(z)| \le M_3{}' |z| \text{ for } R \ge 1$$

and

$$|G(z)| \le M_4{}' |z| \text{ for } R \le 1.$$

Hence for $|z| \le R$, $R \ge 1$,

$$\begin{aligned} |F(z)| &= |a_0 + G(z)| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - M_3{}' |z| \\ &> 0 \end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_3{}'}.$$

And for $|z| \le R$, $R \le 1$, $|F(z)| > 0$ if $|z| < \frac{|a_0|}{M_4{}'}$.

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{M_3{}'}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_4{}'}$ for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that that P(z) has no zero in $|z| < \frac{|a_0|}{M_3{}'}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_4{}'}$ for $R \le 1$.

That proves Theorem 3.

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