

Some Common Fixed Point Theorems for Weakly Compatible Mappings in Menger Spaces

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Abstract: In this paper, firstly we prove a common fixed point theorem for weakly compatible mappings. Secondly, we prove common fixed point theorems for weakly compatible mappings with E.A and CLR property in Menger spaces. At the end, we provide an example in support of our theorem.

Key Words: Menger space, weakly compatible mappings, t-norm of Hadzic- type, E.A. and (CLR) property.

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1. INTRODUCTION

The theory of probabilistic metric spaces is an important part of stochastic Analysis, and so it is of interest to develop the fixed point theory in such spaces. The first result from the fixed point theory in probabilistic metric spaces is obtained by Sehgal and Bharucha- Reid [16]. Since then many fixed points theorems for single valued and multi valued mappings in probabilistic metric spaces have been proved in [2]-[5].

A probabilistic metric space is an ordered pair (X, F) , where X is an arbitrary set and F is a mapping from X^2 into the set of distribution functions. The distribution function $F_{x,y}(t)$ will denote the value of $F_{x,y}$ at the real number t . The function $F_{x,y}$ are assumed to satisfy the following conditions:

- (i) $F_{x,y}(0) = 0$ for all $x, y \in X$
- (ii) $F_{x,y}(t) = 1$ for all $t > 0$ iff $x = y$
- (iii) For distinct points $x, y \in X$, $F_{x,y}(t) \neq 1$ for $t > 0$
- (iv) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$
- (v) If $F_{x,y}(t_1) = 1, F_{y,z}(t_2) = 1$, then

$F_{x,z}(t_1 + t_2) = 1$ for all $x, y, z \in X$ and $t_1, t_2 > 0$.

In 2003, Ren and Wang [18] gave the notion of n-th order t-norm as follows:

Definition 1.1. A mappings $\Delta : \prod_{i=1}^n [0,1] \rightarrow [0,1]$ is called a n-th order t-norm if following conditions are satisfied:

- (i) $\Delta(0,0,\dots,0) = 0, \Delta(a, 1,1,\dots,1) = a$ for

all $a \in [0,1]$

$$(ii) \Delta(a_1, a_2, a_3, \dots, a_n) = \Delta(a_2, a_1, a_3, \dots, a_n) \\ = \Delta(a_2, a_3, a_1, \dots, a_n) \\ = \Delta(a_2, a_3, a_4, \dots, a_n, a_1)$$

(iii) $a_i \geq b_i, i=1,2,3,\dots,n$ implies

$$\Delta(a_1, a_2, a_3, \dots, a_n) \geq \Delta(b_1, b_2, b_3, \dots, b_n)$$

$$(iv) \Delta(\Delta(a_1, a_2, a_3, \dots, a_n), b_2, b_3, \dots, b_n) \\ = \Delta(a_1, \Delta(a_2, a_3, \dots, a_n, b_2), b_3, \dots, b_n) \\ = \Delta(a_1, a_2, \Delta(a_3, a_4, \dots, a_n, b_2, b_3), b_4, \dots, b_n) \\ = \dots \\ = \Delta(a_1, a_2, a_3, \dots, a_{n-1}, \Delta(a_n, b_2, b_3, \dots, b_n)).$$

For $n=2$, we have a binary t- norm, which is commonly known as t- norm.

Basics examples of t- norm are the Lukasiewicz

t- norm $\Delta_L, \Delta_L(a, b) = \max(a+b-1, 0)$, t- norm $\Delta_P, \Delta_P(a, b) = ab$ and t- norm $\Delta_M, \Delta_M(a, b) = \min\{a, b\}$.

Definition 1.2.[3] Let Δ be a t- norm and let

$\Delta_n: [0,1] \rightarrow [0,1] (n \in \mathbb{N})$ be defined by

$$\Delta_1(x) = \Delta(x, x), \Delta_{n+1}(x) = \Delta(\Delta_n(x), x)$$

($n \in \mathbb{N}, x \in [0,1]$).

Then we say that the t- norm Δ is of Hadzic- type if the family $\{\Delta_n(x), n \in \mathbb{N}\}$ is equicontinuous at $x=1$. The family $\{\Delta_n(x), n \in \mathbb{N}\}$ is equicontinuous at $x=1$ if for every $\lambda \in (0,1)$, there $\delta(\lambda) \in (0,1)$ such that

$$x > 1 - \delta(\lambda) \text{ implies } \Delta_n(x) > 1 - \lambda (n \in \mathbb{N}).$$

A trivial example of t- norm of Hadzic- type is $\Delta = \Delta_M$.

Remark 1.3.[4] (i) If there exists a strictly increasing sequence $\{b_n\}_{n \in \mathbb{N}}$ in $[0,1]$ such that $\lim_{n \rightarrow \infty} b_n = 1$ and $\Delta(b_n, b_n) = b_n$ for all $n \in \mathbb{N}$, then Δ is of Hadzic – type.

(ii) If Δ is continuous and Δ is of Hadzic- type, then there exists a sequence $\{b_n\}_{n \in \mathbb{N}}$ as in (i).

Definition 1.4[4]. If Δ is a t- norm and $(x_1, x_2, x_3, \dots, x_n) \in [0,1]^n (n \in \mathbb{N})$, then $\Delta_{i=1}^n x_i$ is defined recurrently by 1, if $n=1$ and $\Delta_{i=1}^n x_i = \Delta(\Delta_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $[0,1]$, then $\Delta_{i=1}^\infty x_i$ is defined as $\lim_{n \rightarrow \infty} \Delta_{i=1}^n x_i$.

Definition 1.5. Let X be any non-empty set and D the set of all left-continuous distribution functions. A triplet (X, F, Δ) is said to be a Menger space if the probabilistic metric space (X, F) satisfies the following condition:

$$(vi) F_{x,z}(t) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)),$$

where $t_1, t_2 > 0, t_1 + t_2 = t$ and $x, y, z, \in X$ and Δ is the t- norm.

Definition 1.6 A sequence $\{x_n\}$ in a Menger space (X, F, Δ) is said to be

(i) convergent with limit x if $\lim_{n \rightarrow \infty} F_{x_n, x}(t) = 1$ for all $t > 0$.

(ii) Cauchy sequence in X if given

$\epsilon > 0, \lambda > 0$, there exists a positive

integer $N_{\epsilon, \lambda}$ such that

$$F_{x_n, x_m}(\epsilon) > 1 - \lambda \text{ for all } m, n > N_{\epsilon, \lambda}.$$

(iii) Complete if every Cauchy sequence in X is convergent in X .

Definition 1.7[12]. Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Definition 1.8 Two self-mapping f and g of a Menger space (X, F, Δ) are said to be weakly commuting if $F(fgx, gfx, t) \geq F(fx, gx, t)$, for each $x \in X$ and for each $t > 0$.

Definition 1.9[14]. Let f and g mapping from a Menger space (X, F, Δ) into itself. A pair of map $\{f, g\}$ is said to be compatible if $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

Definition 1.10. Let f and g self-mapping from on Menger space (X, F, Δ) . The mappings f and g are said to be non-compatible if $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t) \neq 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

In 2007, Kohli et.al [13] introduced the notion of variants of R-weak commutative maps as follows:

Definition 1.11. A pair of self-mappings (f, g) of a Menger space (X, F, Δ) is said to be

- (i) Weakly commuting if $F(fgx, gfx, t) \geq F(fx, gx, t)$
- (ii) R- Weakly commuting if there exists some $R > 0$ such that $F(fgx, gfx, t) \geq F(fx, gx, t/R)$
- (iii) R- Weakly commuting mappings of the type (i) if there exists some $R > 0$ such that $F(fgx, ffx, t) \geq F(fx, gx, t/R)$
- (iv) R- Weakly commuting mappings of the type (ii) if there exists some $R > 0$ such that $F(fgx, ggx, t) \geq F(fx, gx, t/R)$
- (v) R- Weakly commuting mappings of the type (iii) if there exists some $R > 0$ such that $F(ffx, ggx, t) \geq F(fx, gx, t/R)$, for all $x \in X$ and $t > 0$.

In our further discussion, we adopt the terminology from the paper of Imdad et.al.[8] .

We rename R- Weakly commuting mappings of the type (i), R- Weakly commuting mappings of the type (ii) and R- Weakly commuting mappings of the type (iii) by R Weakly commuting mappings of the type (A_g) , R- Weakly commuting mappings of the type (A_f) and R- Weakly commuting mappings of the type (P) , respectively. One can notice that definition 1.11.(iii) and 1.11.(iv) was inspired by Imdad et.al. [8] from the paper of Pathak et. al. [15], whereas definition 1.11.(v) was introduced by Imdad et.al. [8].

In 2002, Aamri and Moutawakil [1] generalized the notion of non compatible mapping to E.A. property. It was pointed out in [1], that property E.A. buys containment of ranges without any continuity requirements besides minimizes the commutativity conditions of the maps at their points of coincidence. Moreover, E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Recently, common fixed point theorems in probabilistic metric spaces/fuzzy metric spaces using E.A. property along with weak compatibility have been recently obtained in ([8],[10]).

Definition 1.12[1]. Let f and g be two self-maps of a metric (X, d) then they are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u, \text{ for some } u \in X.$$

Now in a similar mode, we can state E.A. property in Menger space as follows:

Definition 1.13. A pair of self-mapping (f, g) of Menger spaces (X, F, Δ) is said to hold E.A. property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} F_{f_{x_n}, g_{x_n}}(t) = 1 \text{ for all } t > 0.$$

Example.1.14. Let $X = [0, \infty)$ be the usual metric space. Define $f, g: X \rightarrow X$ by $fx = \frac{x}{4}$ and $gx = \frac{3x}{4}$ for all $x \in X$. Consider the sequence $\{x_n\} = \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 0$, then f and g satisfy the E.A. property.

Although E.A property is generalization of the concept of non compatible maps, yet it requires either completeness of the whole space or any of the range space or continuity of maps. Recently, the new notion of CLR property (common limit range property) was given by Sintunavarat and Kuman [19] that does not impose such conditions. Their importance of CLR property ensures that one does not require the closeness of range subspaces.

Definition 1.15 [19]. Two maps f and g on Menger spaces X are satisfy the common limit in the range of g (CLR g) property if $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = gx$, for some $x \in X$.

Example.1.16. Let $X = [0, \infty)$ be the usual metric space. Define $f, g: X \rightarrow X$ by $fx = x + 1$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = 1 + \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 2 = g1$, therefore f and g satisfy the (CLR g) property.

Now we state a Lemma which is useful in our study:

LEMMA 1.17[14]. Let (X, F, Δ) be a Menger space. If there exists $q \in (0, 1)$ such that $F(x, y, qt) \geq F(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

2. Main Theorems

Now, we prove fixed point theorems for weakly compatible mappings.

THEOREM 2.1. Let (X, F, Δ) be a complete Menger space with continuous t-norm of Hadzic type . Let A, B, S and T be self mappings on X satisfying the following conditions:

- (2.1) $A(X) \subset T(X), B(X) \subset S(X)$
- (2.2) the pairs (A, S) and (B, T) are weakly compatible
- (2.3) there exists $q \in (0, 1)$
 $F(Ax, By, qt) \geq \min \{ F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t), F(Ax, Ty, t) \}$
 for all $x, y \in X$ and $t > 0$,
- (2.4) One of the subspaces $A(X), B(X), S(X)$ or $T(X)$ is a closed subspace of X .

Assume that there exists $x_0, x_1 \in X$ such that for

$$y_1 = Ax_0 = Tx_1, y_2 = Bx_1 = Sx_2 \text{ and}$$

$$\mu \in (q, 1)$$

$$\lim_{n \rightarrow \infty} \Delta_{i=n}^{\infty} F(y_1, y_2, \frac{1}{\mu^i}) = 1.$$

Then $A, B, S,$ and T have a unique common fixed point in X .

Proof. Since $B(X) \subset S(X)$ there exists $x_1, x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we construct two sequences $\{x_n\}$ and $\{y_n\}$ of X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and}$$

$$y_{2n} = Sx_{2n} = Bx_{2n-1} \text{ for } n = 1, 2, \dots$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (2.3), we have that for all $t > 0$,

$$\begin{aligned} F(y_{2n+1}, y_{2n+2}, qt) &= F(Ax_{2n}, Bx_{2n+1}, qt) \\ &\geq \min \{ F(Sx_{2n}, Tx_{2n+1}, t), F(Ax_{2n}, Sx_{2n}, t), \\ &\quad F(Bx_{2n+1}, Tx_{2n+1}, t), F(Ax_{2n}, Tx_{2n+1}, t) \} \\ &\geq \min \{ F(y_{2n}, y_{2n+1}, t), F(y_{2n+1}, y_{2n}, t), \\ &\quad F(y_{2n+2}, y_{2n+1}, t), F(y_{2n+1}, y_{2n+2}, t) \}, \end{aligned}$$

If we take $F(y_{2n+1}, y_{2n+2}, qt) \geq F(y_{2n+2}, y_{2n+1}, t)$, which is a contradiction by Lemma 1.17 since $q \in (0, 1)$. Therefore we have

$$F(y_{2n+1}, y_{2n+2}, qt) \geq F(y_{2n}, y_{2n+1}, t).$$

Also, letting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (2.3), we have that for all $t > 0$

$$F(y_{2n+2}, y_{2n+3}, qt) \geq F(y_{2n+1}, y_{2n+2}, t).$$

In general, for any $n \in \mathbb{N}$, we have

$$F(y_n, y_{n+1}, qt) \geq F(y_{n-1}, y_n, t).$$

It follows that

$$\begin{aligned} F(y_n, y_{n+1}, qt) &\geq F(y_{n-1}, y_n, \frac{t}{q}) \\ &\geq F(y_{n-1}, y_{n-2}, \frac{t}{q^2}) \\ &\quad \dots \\ &\geq F(y_1, y_2, \frac{t}{q^{n-1}}) \end{aligned}$$

Thus for all $t > 0$ and $n=1, 2, 3, \dots$

$$F(y_n, y_{n+1}, qt) \geq F(y_1, y_2, \frac{t}{q^{n-1}}).$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in X .

Let $\sigma = \frac{q}{\mu}$. Since $0 < \sigma < 1$ the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $m_0 \in \mathbb{N}$ such that $\sum_{i=m_0}^{\infty} \sigma^i < 1$.

Hence for every $m > m_0 + 1$ and every $s \in \mathbb{N}$

$$t > t \sum_{i=m_0}^{\infty} \sigma^i > t \sum_{i=m-1}^{m+s-1} \sigma^i.$$

Now

$$\begin{aligned} &F(y_{m+s+1}, y_m, t) \\ &\geq F(y_{m+s+1}, y_m, t \sum_{i=m-1}^{m+s-1} \sigma^i) \\ &\geq F(y_{m+s+1}, y_m, t\sigma^{m-1} + t\sigma^{m-1+1} + \dots + t\sigma^{m-1+s}) \\ &\geq F(y_{m+s+1}, y_m, t\sigma^{m-1} + t\sigma^{m-1+1} + t\sigma^{m-1+2} + \dots + \\ &\quad t\sigma^{m-1+s}) \\ &\geq \Delta_1(F(y_{m+s+1}, y_{m+1}, t\sigma^{m-1+1} + t\sigma^{m-1+2} + \dots + t\sigma^{m-1+s}), F(y_{m+1}, y_m, t\sigma^{m-1})) \\ &\geq \Delta_2(\Delta(F(y_{m+s+1}, y_{m+2}, t\sigma^{m-1+2} + \dots + t\sigma^{m-1+s}), \\ &\quad F(y_{m+2}, y_{m+1}, t\sigma^m), F(y_{m+1}, y_m, t\sigma^{m-1}))) \\ &\geq \Delta_3(\Delta(\Delta(F(y_{m+s+1}, y_{m+3}, t\sigma^{m-1+3} + \dots + t\sigma^{m-1+s}), \\ &\quad F(y_{m+3}, y_{m+2}, t\sigma^{m+1}), F(y_{m+2}, y_{m+1}, t\sigma^m), \\ &\quad F(y_{m+1}, y_m, t\sigma^{m-1})))) \end{aligned}$$

$$\begin{aligned}
 &\geq \dots \\
 &\geq \underbrace{\Delta(\Delta \dots \Delta)}_{s\text{-times}}(F(y_{m+s+1}, y_{m+s}, t\sigma^{m-1+s}), \\
 &\quad F(y_{m+s}, y_{m+s-1}, t\sigma^{m+s-2}), \\
 &\quad \dots, F(y_{m+1}, y_m, t\sigma^{m-1}))) \dots) \\
 &\geq \underbrace{\Delta(\Delta \dots \Delta)}_{s\text{-times}}(F(y_1, y_2, t\frac{\sigma^{m-1+s}}{q^{m-1+s}}), F(y_1, y_2, t\frac{\sigma^{m-2+s}}{q^{m-2+s}}), \\
 &\quad \dots, F(y_1, y_2, t\frac{\sigma^{m-1}}{q^{m-1}}))) \dots) \\
 &\geq \underbrace{\Delta(\Delta \dots \Delta)}_{s\text{-times}}(F(y_1, y_2, \frac{t}{\mu^{m-1+s}}), F(y_1, y_2, \frac{t}{\mu^{m-2+s}}), \\
 &\quad \dots, F(y_1, y_2, \frac{t}{\mu^{m-1}}))) \dots) \\
 &\geq \Delta_{i=m-1}^{m+s-1} F(y_1, y_2, \frac{t}{\mu^i}) \\
 &\geq \Delta_{i=m-1}^{\infty} F(y_1, y_2, \frac{t}{\mu^i}).
 \end{aligned}$$

It is obvious that

$$\lim_{n \rightarrow \infty} \Delta_{i=n}^{\infty} F(y_1, y_2, \frac{1}{\mu^i}) = 1, \text{ implies } \lim_{n \rightarrow \infty} \Delta_{i=n}^{\infty} F(y_1, y_2, \frac{t}{\mu^i}) = 1$$

for every $t > 0$. Now for every $t > 0$ and every $\lambda \in (0, 1)$, there exists $m_1(t, \lambda)$ such that

$F(y_{m+s+1}, y_m, t) > 1 - \lambda$ for every $m \geq m_1(t, \lambda)$ and every $s \in \mathbb{N}$. Hence the sequence

$\{y_n\}$ is a Cauchy sequence in X . Since X is complete, therefore, there exists a point z in X such that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} y_n &= z \text{ and this gives } \lim_{n \rightarrow \infty} Sx_{2n} = \lim_{n \rightarrow \infty} Tx_{2n-1} = \lim_{n \rightarrow \infty} Ax_{2n-2} \\
 &= \lim_{n \rightarrow \infty} Bx_{2n-1} = z
 \end{aligned}$$

for all $n \in \mathbb{N}$. Without loss of generality, we assume that $S(X)$ is a complete subspace of X . Then $z = Su$ for some $u \in X$. Subsequently, we have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} Sx_n &= \lim_{n \rightarrow \infty} Tx_n \\
 &= \lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z = Su.
 \end{aligned}$$

Next, we claim that $Au = Su$. For this purpose, we put $x = u$ and $y = x_n$ in (2.3), then this gives

$$\begin{aligned}
 &F(Au, Bx_n, qt) \\
 &\geq \min \{ F(Su, Tx_n, t), F(Au, Su, t), \\
 &\quad F(Bx_n, Tx_n, t), F(Au, Tx_n, t) \}.
 \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we have

$$\begin{aligned}
 F(Au, z, qt) &\geq \min \{ F(z, z, t), F(Au, z, t), \\
 &F(z, z, t), F(Au, z, t) \} \\
 &= F(Au, z, t).
 \end{aligned}$$

By Lemma 1.17, we have $Au = z$, hence $Au = Su = z$. Since $A(X) \subset T(X)$, therefore there exists a point $v \in X$ such that $Au = z = Tv$.

Next, we claim that $Tv = Bv$. Putting $x = u$ and $y = v$ in (2.3), we have

$$\begin{aligned}
 &F(Au, Bv, qt) \\
 &\geq \min \{ F(Su, Tv, t), F(Au, Su, t), \\
 &\quad F(Bv, Tv, t), F(Au, Tv, t) \}, \\
 &= \min \{ F(z, z, t), F(z, z, t),
 \end{aligned}$$

$$F(Bv, Tv, t), F(z, z, t)\}, \\ = F(Bv, Tv, t).$$

Therefore we have $Tv = Bv$. Thus $Au = Su = Tv = Bv = z$. Since the pairs (A, S) and (B, T) are weakly compatible and u and v are their coincidence points respectively, we obtain $Az = A(Su) = S(Au) = Sz$ and $Bz = B(Tv) = T(Bv) = Tz$.

Now, we prove that z is a common fixed point of A, B, S and T . For this purpose, put $x = z, y = v$ in (2.3), we get

$$F(Az, Bv, qt) \\ \geq \min \{ F(Sz, Tv, t), F(Az, Sz, t), \\ F(Bv, Tv, t), F(Az, Tv, t)\}, \\ = \min\{F(Az, Bv, t), F(Az, Az, t), \\ F(Bv, Bv, t), F(Az, Bv, t)\}, \\ = F(Az, Bv, t).$$

Which implies that $Az = Bv$. Hence $z = Az = Sz$ and z is a common fixed point of A and S . One can prove that $Bv = z$ is also a common fixed point of B and T .

Finally, in order to prove the uniqueness, suppose $w (z \neq w)$ be another fixed point of A, B, S and T . Then, for all $t > 0$, we have

$$F(z, w, qt) \\ = F(Az, Bw, qt) \\ \geq \min \{ F(Sz, Tw, t), F(Az, Sz, t), \\ F(Bw, Tw, t), F(Az, Tw, t)\}, \\ = \min \{ F(z, w, t), F(z, z, t), F(w, w, t),$$

$$F(z, w, t)\},$$

Thus we have $z = w$. Hence z is a unique common fixed point of A, B, S and T . This completes the proof.

Next, we prove fixed point theorems for weakly compatible maps with E.A. property.

Theorem 2.5. Let (X, F, Δ) be a Menger space with continuous t-norm of Hadzic type. Let A, B, S and T be self mapping on X satisfying (2.1), (2.2), (2.4) and the following conditions:

(2.6) pairs (A, S) or (B, T) satisfy E.A. property

(2.7) there exists $q \in (0, 1)$ such that

$$F(Ax, By, qt) \geq \min\{F(Sx, Ty, t), F(Ax, Sx, t), \\ F(By, Ty, t), F(Ax, Ty, t), F(By, Sx, t)\}.$$

for all $x, y \in X$ and $t > 0$,

Then $A, B, S,$ and T have a unique common fixed point in X .

Proof. Without loss of generality, we assume that the pair (B, T) satisfies the property E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = z$, for some $z \in X$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n \rightarrow \infty} Sy_n = z$. Since $A(X) \subset T(X)$, there exists a sequence $\{y'_n\}$ in X such that $Ay'_n = Tx_n$. Hence $\lim_{n \rightarrow \infty} Ay'_n = z$. Suppose that $S(X)$ is a closed subspace of X . Then $z = Su$ for some $u \in X$. Subsequently, we have

$$\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Ay'_n = \lim_{n \rightarrow \infty} Sy_n = z = Su, \text{ for some } u \in X.$$

Next, we claim that $Au = Su$. For this purpose, we put $x = u$ and $y = x_n$ in (2.7), then this gives

$$F(Au, Bx_n, qt) \geq \min \{F(Su, Tx_n, t), F(Au, Su, t), \\ F(Bx_n, Tx_n, t), F(Au, Tx_n, t), F(Bx_n, Su, t)\}.$$

Taking limit as $n \rightarrow \infty$, we have

$$F(Au, z, qt) \geq \min \{F(z, z, t), F(Au, z, t), \\ F(z, z, t), F(Au, z, t), F(z, z, t)\} \\ = F(Au, z, t).$$

Thus by Lemma 1.17, we have $Au = z$, hence $Au = Su = z$. Since $A(X) \subset T(X)$, therefore there exists a point $v \in X$ such that $Au = z = Tv$.

Next, we claim that $Tv = Bv$. Putting $x = u$ and $y = v$ in (2.7), we have

$$F(Au, Bv, qt) \geq \min \{F(Su, Tv, t), F(Au, Su, t), \\ F(Bv, Tv, t), F(Au, Tv, t), F(Bv, Su, t)\}, \\ = \min \{F(z, z, t), F(z, z, t), \\ F(Bv, Tv, t), F(z, z, t), F(Bv, Tv, t)\}, \\ = F(Bv, Tv, t).$$

Hence, we have $Tv = Bv$. Thus $Au = Su = Tv = Bv = z$. Since the pairs (A, S) and (B, T) are weakly compatible and u and v are their coincidence points respectively, then $Az = A(Su) = S(Au) = Sz$ and $Bz = B(Tv) = T(Bv) = Tz$.

Now, we prove that z is a common fixed point of A, B, S and T . For this purpose, put $x = z, y = v$ in (2.7), we get

$$F(Az, Bv, qt) \geq \min \{F(Sz, Tv, t), F(Az, Sz, t), \\ F(Bv, Tv, t), F(Az, Tv, t), F(Bv, Sz, t)\}, \\ = \min \{F(Az, Bv, t), F(Az, Az, t), \\ F(Bv, Bv, t), F(Az, Bv, t), F(Az, Bv, t)\} \\ = F(Az, Bv, t).$$

Thus, we have $Az = Bv = z$. Hence $z = Az = Sz$ and z is a common fixed point of A and S . One can prove that $Bv = z$ is also a common fixed point of B and T .

Finally, in order to prove the uniqueness, suppose $w (z \neq w)$ be another fixed point of A, B, S and T . Then, for all $t > 0$, we have

$$F(z, w, qt) \\ = F(Az, Bw, qt) \\ \geq \min \{F(Sz, Tw, t), F(Az, Sz, t), F(Bw, Tw, t), \\ F(Az, Tw, t), F(Bw, Sz, t)\} \\ = \min \{F(z, w, t), F(z, z, t), F(w, w, t), F(z, w, t), \\ F(z, w, t)\} \\ = F(z, w, t),$$

Thus we have we have $z = w$. Hence $z = w$ is a unique common fixed point of A, B, S and T .

Finally, we prove a fixed point theorem for weakly compatible maps with (CLR_S) property.

Theorem 2.8. Let (X, F, Δ) be Menger space with continuous t -norm of Hadzic type. Let A, B, S and T be self mapping on X satisfying (2.1), (2.2), (2.7) and the following conditions:

(2.9) pairs (A, S) or (B, T) satisfy (CLR_S) property,

(2.10) One of the $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a closed subspace of X .

Then A , B , S , and T have a unique common fixed point in X .

Proof. If the pair (A,S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, where $z \in S(X)$, therefore, there exists a point $u \in X$ such that $Su = z$. Since $T(X)$ is a closed subspace of X and $A(X) \subset T(X)$, so for each $\{x_n\}$ in X , there corresponds a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, $\lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} Ax_n = z$, where $z \in S(X)$. Thus, we have

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = z.$$

Now, we show that $\lim_{n \rightarrow \infty} By_n = z$.

Putting $x = x_n, y = y_n$ in (2.7), we get

$$F(Ax_n, By_n, qt) \geq \min \{F(Sx_n, Ty_n, t), F(Ax_n, Sx_n, t), F(By_n, Ty_n, t), F(Ax_n, Ty_n, t), F(By_n, Sx_n, t)\}.$$

Let $\lim_{n \rightarrow \infty} By_n = l \neq z$ for $t > 0$. Then taking limit as $n \rightarrow \infty$, we have

$$F(z, l, qt) \geq \min \{F(z, z, t), F(z, z, t), F(l, z, t), F(z, z, t), F(l, z, t)\} = F(l, z, t).$$

By Lemma 1.17, we have $z = l$, then $\lim_{n \rightarrow \infty} By_n = z$. Therefore

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} By_n = z = Su,$$

for some $u \in X$. Using Theorem 2.5 and Lemma 1.17, we can easily prove that z is a unique common fixed point of A , B , S and T . This completes the proof.

Example.2.11. Let $X = [0,2]$ equipped with the Euclidian distance and the Menger spaces induced by (X, d) i.e., $F(x, y, t) = \frac{t}{t+d(x,y)}$. Clearly (X,F,Δ) be Menger space with continuous t-norm of Hadzic type $\Delta = \min \{a,b\}$.

Define the self maps A, B, S and $T: X \rightarrow X$ by

$$Ax = \begin{cases} 0, & \text{if } x = 0 \\ 0.25 & \text{if } x > 0. \end{cases} \quad Bx = \begin{cases} 0, & \text{if } x = 0 \\ 0.45 & \text{if } x > 0. \end{cases}$$

$$Sx = \begin{cases} 0, & \text{if } x = 0 \\ 0.40 & \text{if } 0 < x \leq 0.6 \\ x - 0.45 & \text{if } x > 0.6 \end{cases} \quad Tx = \begin{cases} 0, & \text{if } x = 0 \\ 0.25 & \text{if } 0 < x \leq 0.6 \\ x - 0.25 & \text{if } x > 0.6. \end{cases}$$

$$AX = 0 \cup 0.25, BX = 0 \cup 0.45, SX = 0 \cup (0.15, 1.55), TX = 0 \cup 0.25 \cup (0.35, 1.75).$$

Consider the sequence $\{x_n\} = \left\{0.60 + \frac{1}{n}\right\}$, then $Ax_n \rightarrow 0.25, Bx_n \rightarrow 0.45, Sx_n \rightarrow 0.15, Tx_n \rightarrow 0.35, ASx_n \rightarrow 0.25, ASx_n \rightarrow 0.25, SAX_n \rightarrow 0.45, BTx_n \rightarrow 0.45, TBx_n \rightarrow 0.25$. The pairs (A,S) and (B, T) are compatible at coincidence point. If we take $q=0.6$, and $t=1$, then A,B,S and T satisfy the conditions of the Theorem 2.1 and zero is the unique common fixed point of A, B, S and T . Moreover, A, B, S and T are discontinuous at the fixed point zero.

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