Stability Analysis of two Competitive Interacting Species with Optimal and Bionomic Harvesting of the Second Species

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Abstract**—This paper deals with two species competitive model with optimal harvesting of the second species under bionomic conditions. The model is characterized by a pair of first order non-linear ordinary differential equations. All the possible equilibrium points of the model are identified and the criteria for the local and global stabilities are discussed. The possibility of existence of bio economic equilibrium is being discussed and an optimal harvesting policy is given using Pontryagin's maximum principle.**

1. INTRODUCTION

There is an extensive study on several kinds of prey- predator interactions after it was initiated by Lotka [1] and Volterra [2]. Bionomics of natural resources has played a significant role in all these interactions. There is a strong impact of harvesting on the dynamic evolution of a population. In fishery, forestry, agriculture and wild life management, the exploitation of biological resources and harvesting of population species can be seen. The problems of predator-prey systems in the presence of harvesting were discussed by many authors and attention on economic policies from harvesting have also been analysed. A detailed discussion on the issues and techniques associated with the bionomic exploitation of natural resources was given by Clark [3, 4]. A study on a class of predator-prey models under constant rate of harvesting of both species simultaneously was made by Brauer and Soudack [5, 6]. Multi-species harvesting models are also studied in detail by Chaudhuri [7, 8]. Models on the combined harvesting of a two species prey predator fishery have been discussed by Ragozin and Brown [9], Chaudhuri and Saha Ray [10]. K. Shiva Reddy et.al [12] proposed the mathematical model for the three species ecosystem comprising of two predators competing for the prey. They also investigated the stability concepts using various mathematical techniques.

In this connection, a mathematical model based on the system of non-linear equations has been constructed. All the four equilibrium points are identified and their local stability is discussed. The conditions for global stability of the system are derived by using Liapunov function. Biological and Bionomical equilibria of the system are derived.

2. MATHEMATICAL MODEL

The model equations for a two species competitive system are given by the following system of non-linear ordinary differential equations

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^{\,2} - \alpha_{12} N_1 N_2 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^{\,2} - \alpha_{21} N_1 N_2 - q_2 E_2 N_2 \tag{2.2}$$

where N_1 and N_2 are the populations of the first and second species with natural growth rates (bio potentials) a_1 and a_2 respectively, α_{11} is rate of decrease of the first species due to insufficient food, α_{12} is rate of decrease of the first species due to inhibition by the second species, α_{21} is rate of decrease of the second species due to inhibition by the first species, α_{22} is rate of decrease of the second species due to insufficient food other than the first species; q_2 is the catch ability co-efficient of the second species, E_2 is the harvesting effort and $q_2 E_2 N_2$ is the catch-rate functions based on the catch-per-unit-effort hypothesis. Further both the variables N_1 and N_2 are non-negative and the model parameters $\alpha_{12}, \alpha_{21}, \alpha_{22}, q_2, E_2,\ a_1, a_2,\ \alpha_{11},\ a_2 - q_2 E_2$ are assumed to be non-negative constants.

3. EQUILIBRIUM STATES

The system has only four equilibrium states defined by $\frac{dN_1}{dt} = 0, \frac{dN_2}{dt} = 0$

E_1: The fully washed out state with the equilibrium point $\overline{N}_1 = 0; \overline{N}_2 = 0$

E_2: The state in which, only the predator survives and the prey is washed out.

The equilibrium point is $\overline{N}_1 = 0 ; \overline{N}_2 = \frac{(a_2 - q_2 E)}{\alpha_{22}}$

E_3: The state in which only the prey survives and the predator is washed out

The equilibrium point is $\overline{N}_1 = \frac{a_1}{\alpha_{11}} ; \overline{N}_2 = 0$

E_4: The co-existent state (normal steady state)

The equilibrium point is

$$\overline{N}_1 = \frac{a_1\alpha_{22} - (a_2 - q_2 E)\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} ; \overline{N}_2 = \frac{(a_2 - q_2 E)\alpha_{11} - a_1\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}$$

This state would exit only when $a_1\alpha_{22} > (a_2 - q_2 E)\alpha_{12}$, $(a_2 - q_2 E)\alpha_{11} > a_1\alpha_{21}$, $\alpha_{11}\alpha_{22} > \alpha_{12}\alpha_{21}$

4. STABILITY OF THE EQUILIBRIUMSTATES

4.1 Stability of the equilibrium state E_1

To discuss the stability of equilibrium point $\overline{N}_1 = 0$; $\overline{N}_2 = 0$, we consider slight deviations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{4.1.1}$$

$$N_2 = \overline{N}_2 + u_2(t). \tag{4.1.2}$$

Substituting (4.1.1) and (4.1.2) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = a_1 u_1 - \alpha_{11} u_1^2 - \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = (a_2 - q_2 E) u_2 - \alpha_{22} u_2^2 - \alpha_{21} u_1 u_2$$

On neglecting products and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = a_1 u_1 \tag{4.1.3}$$

and

$$\frac{du_2}{dt} = (a_2 - q_2 E) u_2 \tag{4.1.4}$$

The characteristic equation is

$[\lambda - a_1][\lambda - (a_2 - q_2 E)] = 0$,

whose roots $a_1, (a_2 - q_2 E)$ are both positive. Hence the equilibrium state is unstable.

The solutions for equations (4.1.3) and (4.1.4) are

$$u_1 = u_{10}\, e^{a_1 t}$$

$$u_2 = u_{20}\, e^{(a_2 - q_2 E)t}$$

where u_{10}, u_{20} are the initial values of u_1 and u_2.

4.2 Stability of the equilibrium state E_2

Substituting (4.1.1) and (4.1.2) in (2.1) and (2.2), we get,

$$\frac{du_1}{dt} = a_1u_1 - \alpha_{11}u_1^2 - \alpha_{12}u_1u_2 - \left(a_2 - q_2E\right)\frac{\alpha_{12}}{\alpha_{22}}u_1$$

$$\frac{du_2}{dt} = -\left(a_2 - q_2E\right)u_2 - \alpha_{22}u_2^2 - \alpha_{21}u_1u_2 - \left(a_2 - q_2E\right)\frac{\alpha_{21}}{\alpha_{22}}u_1$$

On neglecting products, and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = \left[a_1 - \left(a_2 - q_2E\right)\frac{\alpha_{12}}{\alpha_{22}}\right]u_1 \tag{4.2.1}$$

and

$$\frac{du_2}{dt} = -\left(a_2 - q_2E\right)\frac{\alpha_{21}}{\alpha_{22}}u_1 - \left(a_2 - q_2E\right)u_2 \tag{4.2.2}$$

The characteristic equation is

$$\left[\lambda + \left(a_2 - q_2E\right)\right]\left\{\lambda - \left[a_1 - \left(a_2 - q_2E\right)\frac{\alpha_{12}}{\alpha_{22}}\right]\right\} = 0 \tag{4.2.3}$$

Case (i): When $a_1 > \frac{\left(a_2 - q_2E\right)\alpha_{12}}{\alpha_{22}}$, one root of equation (4.2.3) is $\lambda_1 = -\left(a_2 - q_2E\right)$ and the other root,

$\lambda_2 = \left[a_1 - \left(a_2 - q_2E\right)\frac{\alpha_{12}}{\alpha_{22}}\right]$ is positive. Hence the equilibrium state is unstable.

Case (ii): When $a_1 < \frac{\left(a_2 - q_2E\right)\alpha_{12}}{\alpha_{22}}$, one root of equation (4.2.3) is $\lambda_1 = -\left(a_2 - q_2E\right)$ and other root,

$\lambda_2 = \left[a_1 - \left(a_2 - q_2E\right)\frac{\alpha_{12}}{\alpha_{22}}\right]$ is negative.

As the roots of the equation (4.2.3) are both negative, the equilibrium state is stable.
The solutions for equations (4.2.1) and (4.2.2) are

$$u_1 = u_{10}\, e^{\lambda_2 t}$$

and $u_2 = \rho_1\, e^{\lambda_2 t} + (u_{20} - \rho_1)\, e^{-(a_2 - q_2E)t}$

where $\rho_1 = \dfrac{-(a_2 - q_2E)u_{10}\alpha_{21}}{a_1\alpha_{22} + (a_2 - q_2E)\left[\alpha_{22} - \alpha_{12}\right]}$

Case (iii): When $a_1 = \frac{\left(a_2 - q_2E\right)\alpha_{12}}{\alpha_{22}}$, one root of the equation (4.2.3) is $\lambda_1 = -\left(a_2 - q_2E\right)$ and the other root,

$\lambda_2 = 0$. Hence the equilibrium state is "neutrally" stable.

The solutions for (4.2.1) and (4.2.2) are

$$u_1 = u_{10}$$

and

$$u_2 = \frac{a_1\alpha_{21}}{(a_2 - q_2E)\alpha_{12}}u_{10} + \left[u_{20} + \frac{a_1\alpha_{21}}{\left(a_2 - q_2E\right)\alpha_{12}}u_{10}\right]e^{-\left(a_2 - q_2E\right)t}$$

4.3. *Stability of the equilibrium state E_3*

Substituting (4.1.1) and (4.1.2) in (2.1) and (2.2), we get,

$$\frac{du_1}{dt} = -a_1u_1 - \alpha_{11}u_1^2 - \alpha_{12}u_1u_2 - \frac{a_1\alpha_{12}u_2}{\alpha_{11}}$$

$$\frac{du_2}{dt} = (a_2 - q_2E)u_2 - \alpha_{22}u_2^2 - \alpha_{21}u_1u_2 - \frac{a_1\alpha_{21}u_2}{\alpha_{11}}$$

By neglecting products, and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = -a_1u_1 - \frac{a_1\alpha_{12}u_2}{\alpha_{11}} \tag{4.3.1}$$

and

$$\frac{du_2}{dt} = \left[(a_2 - q_2E) - \frac{a_1\alpha_{21}}{\alpha_{11}}\right]u_2 \tag{4.3.2}$$

and the characteristic equation is

$$[\lambda + a_1]\left\{\lambda - \left[(a_2 - q_2E) - \frac{a_1\alpha_{21}}{\alpha_{11}}\right]\right\} = 0 \tag{4.3.3}$$

Case (i): When $(a_2 - q_2E) > \frac{a_1\alpha_{21}}{\alpha_{11}}$, one root of equation (4.3.3) is $\lambda_1 = -a_1$ while the other root,

$\lambda_2 = \left[(a_2 - q_2E) - \frac{a_1\alpha_{21}}{\alpha_{11}}\right]$ is positive. Hence the equilibrium state is unstable.

Case (ii): When $(a_2 - q_2E) < \frac{a_1\alpha_{21}}{\alpha_{11}}$, one root of equation (4.3.3) is $\lambda_1 = -a_1$ and other root,

$\lambda_2 = \left[(a_2 - q_2E) - \frac{a_1\alpha_{21}}{\alpha_{11}}\right]$ is negative.

As the roots of the equation (4.3.3) are both negative, the equilibrium state is stable.
The solutions for equations (4.3.1) and (4.3.2) are

$$u_1 = \rho_2 u_{20} e^{dt} + (u_{10} + \rho_2)\, e^{-a_1 t}$$

$$u_2 = u_{20} e^{dt}$$

where $d = \left[(a_2 - q_2E) - \frac{a_1\alpha_{21}}{\alpha_{11}}\right]$; $\rho_2 = \frac{a_1\alpha_{12}}{\left[(a_2 - q_2E)\alpha_{11} + a_1(\alpha_{11} - \alpha_{21})\right]}$

4.4. *Stability of the equilibrium state E_4*

Substituting (4.1.1) and (4.1.2) in (2.1) and (2.2), we get,

$\frac{du_1}{dt} = -\alpha_{11}u_1^2 - \alpha_{12}u_1u_2 - \alpha_{11}\overline{N}_1u_1 - \alpha_{12}\overline{N}_1u_2$ $\frac{du_2}{dt} = -\alpha_{22}u_2^2 - \alpha_{21}u_1u_2 - \alpha_{22}\overline{N}_2u_2 - \alpha_{21}\overline{N}_2u_1$ By neglecting products, and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N}_1u_1 - \alpha_{12}\overline{N}_1u_2 \tag{4.4.1}$$

and

$$\frac{du_2}{dt} = -\alpha_{21}\overline{N}_2u_1 - \alpha_{22}\overline{N}_2u_2 \tag{4.4.2}$$

The characteristic equation is

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)\lambda + [\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}]\ \overline{N}_1\overline{N}_2 = 0 \tag{4.4.3}$$

Since the sum of the roots of (4.4.3) is negative and product of the roots is positive, the roots of which can be noted to be negative. Hence the co-existent equilibrium state is stable.
The solutions for (4.4.1) & (4.4.2) are given by

$$u_1 = \left[\frac{u_{10}(\lambda_1+\alpha_{22}\overline{N}_2)-u_{20}\alpha_{12}\overline{N}_1}{\lambda_1-\lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}(\lambda_2+\alpha_{22}\overline{N}_2)-u_{20}\alpha_{12}\overline{N}_1}{\lambda_2-\lambda_1}\right] e^{\lambda_2 t}$$

$$u_2 = \left[\frac{u_{20}(\lambda_1+\alpha_{11}\overline{N}_1)-u_{10}\alpha_{21}\overline{N}_2}{\lambda_1-\lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2+\alpha_{11}\overline{N}_1)-u_{10}\alpha_{21}\overline{N}_2}{\lambda_2-\lambda_1}\right] e^{\lambda_2 t}$$

where λ_1, λ_2 are the roots of the equation (4.4.3)

5. GLOBAL STABILITY

Theorem: The Equilibrium point $(\overline{N}_1, \overline{N}_2)$ is globally asymptotically stable.

Proof: Let us consider the following Liapunov function

$$V(N_1,N_2) = N_1 - \overline{N}_1 - \overline{N}_1 \ln\left[\frac{N_1}{\overline{N}_1}\right] + l\left\{N_2 - \overline{N}_2 - \overline{N}_2 \ln\left[\frac{N_2}{\overline{N}_2}\right]\right\}$$

where 'l' is a positive constant

Differentiating V w.r.to 't' we get

$$\frac{dV}{dt} = \left(\frac{N_1-\overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + l\left(\frac{N_2-\overline{N}_2}{N_2}\right)\frac{dN_2}{dt}$$

$$\frac{dV}{dt} = \left(\frac{N_1-\overline{N}_1}{N_1}\right)\left\{N_1\left[a_1-\alpha_{11}N_1-\alpha_{12}N_2\right]\right\} + l\left(\frac{N_2-\overline{N}_2}{N_2}\right)\left\{N_2\left[(a_2-q_2E)-\alpha_{22}N_2-\alpha_{21}N_1\right]\right\}$$

$$\frac{dV}{dt} = \left(N_1-\overline{N}_1\right)\left\{a_1-\alpha_{11}N_1-\alpha_{12}N_2\right\} + l\left(N_2-\overline{N}_2\right)\left\{(a_2-q_2E)-\alpha_{22}N_2-\alpha_{21}N_1\right\}$$

$$= \left(N_1-\overline{N}_1\right)\left\{\alpha_{11}\overline{N}_1+\alpha_{12}\overline{N}_2-\alpha_{11}N_1-\alpha_{12}N_2\right\} + l\left(N_2-\overline{N}_2\right)\left\{\alpha_{22}\overline{N}_2-\alpha_{21}\overline{N}_1-\alpha_{22}N_2-\alpha_{21}N_1\right\}$$

$$= \left(N_1-\overline{N}_1\right)\left\{-\alpha_{11}\left(N_1-\overline{N}_1\right)-\alpha_{12}\left(N_2-\overline{N}_2\right)\right\} + l\left(N_2-\overline{N}_2\right)\left\{-\alpha_{21}\left(N_1-\overline{N}_1\right)-\alpha_{22}\left(N_2-\overline{N}_2\right)\right\}$$

$$= \left\{-\alpha_{11}\left(N_1-\overline{N}_1\right)^2-\alpha_{12}\left(N_1-\overline{N}_1\right)\left(N_2-\overline{N}_2\right)\right\} + l\left\{-\alpha_{21}\left(N_1-\overline{N}_1\right)\left(N_2-\overline{N}_2\right)-\alpha_{22}\left(N_2-\overline{N}_2\right)^2\right\}$$

$$= -\alpha_{11}\left(N_1-\overline{N}_1\right)^2-\alpha_{12}\left(N_1-\overline{N}_1\right)\left(N_2-\overline{N}_2\right)-l\alpha_{21}\left(N_1-\overline{N}_1\right)\left(N_2-\overline{N}_2\right)-l\alpha_{22}\left(N_2-\overline{N}_2\right)^2$$

$$= -\alpha_{11}\left(N_1-\overline{N}_1\right)^2-\frac{\alpha_{12}}{2}\left[\left(N_1-\overline{N}_1\right)^2+\left(N_2-\overline{N}_2\right)^2\right]-\frac{l\alpha_{21}}{2}\left[\left(N_1-\overline{N}_1\right)^2+\left(N_2-\overline{N}_2\right)^2\right]$$
$$-l\alpha_{22}\left(N_2-\overline{N}_2\right)^2$$

$$\frac{dV}{dt} = -\left(\frac{\alpha_{12}+l\alpha_{21}}{2}+\alpha_{11}\right)\left(N_1-\overline{N}_1\right)^2-\left(\frac{\alpha_{12}+l\alpha_{21}}{2}+l\alpha_{22}\right)\left(N_2-\overline{N}_2\right)^2$$
$$< 0$$

Therefore , the equilibrium point $(\overline{N}_1, \overline{N}_2)$ is globally asymptotically stable.

6. BIONOMIC EQUILIBRIUM

The term bionomic equilibrium is an amalgamation of the concepts of biological equilibrium as well as economic equilibrium. The economic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost for the effort devoted to harvesting.

Let $c_2 =$ fishing cost per unit effort of the predator, $p_2 =$ price per unit biomass of the predator. The net economic revenue of the predator at any time *t* is given by $R_2 = \left(p_2 q_2 N_2 - c_2\right)E$ (6.1)

The biological equilibrium is $\left((N_1)_\infty, (N_2)_\infty, (E)_\infty\right)$,

where $(N_1)_\infty, (N_2)_\infty, (E)_\infty$ are the positive solutions of

$$a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 = 0 \tag{6.2}$$

$$\left(a_2 - q_2 E\right) N_2 - \alpha_{22} N_2^2 - \alpha_{21} N_1 N_2 = 0 \tag{6.3}$$

and

$$\left(p_2 q_2 N_2 - c_2\right) E = 0 \tag{6.4}$$

From (6.4), we have

$$\left\{p_2 q_2 (N_2)_\infty - c_2\right\}(E)_\infty = 0$$

$$\Rightarrow \left\{p_2 q_2 (N_2)_\infty - c_2\right\} = 0$$

$$\Rightarrow (N_2)_\infty = \frac{c_2}{p_2 q_2} \tag{6.5}$$

From (6.2), we have

$$(N_1)_\infty \left\{a_1 - \alpha_{11}(N_1)_\infty - \alpha_{12}(N_2)_\infty\right\} = 0$$

$$\Rightarrow \left\{a_1 - \alpha_{11}(N_1)_\infty - \alpha_{12}(N_2)_\infty\right\} = 0$$

$$\Rightarrow \left\{a_1 - \alpha_{11}(N_1)_\infty - \alpha_{12}\frac{c_2}{p_2 q_2}\right\} = 0$$

$$\Rightarrow \left\{\left(a_1 - \alpha_{12}\frac{c_2}{p_2 q_2}\right) - \alpha_{11}(N_1)_\infty\right\} = 0 \Rightarrow (N_1)_\infty = \frac{1}{\alpha_{11}}\left(a_1 - \alpha_{12}\frac{c_2}{p_2 q_2}\right) \tag{6.6}$$

From (6.3), (6.5) & (6.6), we get

$$\left(a_2 \frac{c_2}{p_2 q_2} - \alpha_{22}\frac{c_2^{\,2}}{p_2^{\,2} q_2^{\,2}} - \alpha_{21}\frac{c_2}{p_2 q_2}(N_1)_\infty - q_2 (E)_\infty \frac{c_2}{p_2 q_2}\right) = 0$$

$$\Rightarrow q_2 (E)_\infty \frac{c_2}{p_2 q_2} = \left(a_2 \frac{c_2}{p_2 q_2} - \alpha_{22}\frac{c_2^{\,2}}{p_2^{\,2} q_2^{\,2}} - \alpha_{21}\frac{c_2}{p_2 q_2}(N_1)_\infty\right)$$

$$\Rightarrow (E)_\infty = \frac{1}{q_2}\left[\left(a_2 - \alpha_{22}\frac{c_2}{p_2 q_2}\right) - \alpha_{21}(N_1)_\infty\right] \tag{6.7}$$

It is clear that $(E)_\infty > 0$ if $\left(a_2 - \alpha_{22}\frac{c_2}{p_2 q_2}\right) > \alpha_{21}(N_1)_\infty$ (6.8)

Thus the bionomic equilibrium $\left((N_1)_\infty, (N_2)_\infty, (E)_\infty\right)$ exists, if inequality (6.8) holds.

7. OPTIMAL HARVESTING POLICY

The present value *J* of a continuous time-stream of revenues is given by

$$J = \int_0^\infty e^{-\delta t}\left(p_2 q_2 N_2 - c_2\right) E\,dt \tag{7.1}$$

where δ denotes the instantaneous annual rate of discount. Our problem is to maximize *J* subject to the state equations (2.1) & (2.2) and to the control constraints $0 \le E \le (E)_{\max}$ by invoking Pontryagin's maximum principle [11].

The Hamiltonian for the problem is given by

$$H = e^{-\delta t}\left(p_2 q_2 N_2 - c_2\right)E + \lambda_1\left(a_1 N_1 - \alpha_{11} N_1^{\,2} - \alpha_{12} N_1 N_2\right)$$
$$+\lambda_2\left(a_2 N_2 - \alpha_{22} N_2^{\,2} - \alpha_{21} N_1 N_2 - q_2 E N_2\right) \tag{7.2}$$

where λ_1, λ_2 are the adjoint variables.

Let us assume that the control constraints are not binding i.e. the optimal solution does not occur at $(E)_{\max}$. At $(E)_{\max}$ we have a singular control

By Pontryagin's maximal principle,

$$\frac{\partial H}{\partial E} = 0 \;\; ; \; \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} \;\; ; \; \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2}$$

$$\frac{\partial H}{\partial E} = 0 \Rightarrow e^{-\delta t}\left(p_2 q_2 N_2 - c_2\right) - \lambda_2 q_2 N_2 = 0$$

$$\Rightarrow \lambda_2 = e^{-\delta t}\left(p_2 - \frac{c_2}{q_2 N_2}\right) \tag{7.3}$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} = -\left\{\lambda_1\left(a_1 - 2\alpha_{11}N_1 - \alpha_{12}N_2\right) + \lambda_2\left(-\alpha_{21}N_2\right)\right\}$$

$$\Rightarrow \frac{d\lambda_1}{dt} = \left(\lambda_1 \alpha_{11} N_1 + \lambda_2 \alpha_{21} N_2 E\right) \tag{7.4}$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_2} = -\left\{e^{-\delta t} p_2 q_2 E + \lambda_1\left(-\alpha_{12}N_1\right) + \lambda_2\left[\left(a_2 - q_2 E\right) - \alpha_{21}N_1 - 2\alpha_{22}N_2\right]\right\}$$

$$\Rightarrow \frac{d\lambda_2}{dt} = \left(\lambda_2 \alpha_{22} N_2 + \lambda_1 \alpha_{12} N_1 - e^{-\delta t} p_2 q_2 E\right) \tag{7.5}$$

From (7.3) & (7.4), we get

$$\frac{d\lambda_1}{dt} - \lambda_1 \alpha_{11} N_1 = -B_1 e^{-\delta t}$$

where

$$B_1 = \alpha_{21}\overline{N_2}\left(p_2 - \frac{c_2}{q_2 \overline{N_2}}\right)$$

whose solution is given by $\lambda_1 = \dfrac{B_1}{\left(\alpha_{11}\overline{N_1} + \delta\right)} e^{-\delta t}$ (7.6)

From (7.5) & (7.6), we get

$$\frac{d\lambda_2}{dt} - \lambda_2 \alpha_{22} N_2 = -B_2 e^{-\delta t}$$

where

$$B_2 = \left[p_2 q_2 E - \frac{B_1 \alpha_{12} \overline{N_1}}{\left(\alpha_{11} \overline{N_1} + \delta\right)} \right]$$

whose solution is given by $\lambda_2 = \frac{B_2}{\left(\alpha_{22} \overline{N_2} + \delta\right)} e^{-\delta t}$ (7.7)

From (7.3) & (7.7), we get a singular path

$$\left(p_2 - \frac{c_2}{q_2 \overline{N_2}} \right) = \frac{B_2}{\left(\alpha_{22} \overline{N_2} + \delta\right)} \quad (7.8)$$

Thus (7.8) can be written as

$$F(\overline{N_2}) = \left(p_2 - \frac{c_2}{q_2 \overline{N_2}} \right) - \frac{B_2}{\left(\alpha_{22} \overline{N_2} + \delta\right)}$$

There exists a unique positive root

$\overline{N_2} = \left(N_2\right)_\delta$ of $F(\overline{N_2}) = 0$ in the interval $0 < \overline{N_2} < k_2$, if the following hold $F(0) < 0$,

$F(k_2) > 0$, $F'(\overline{N_2}) > 0$ for $\overline{N_2} > 0$.

For $\overline{N_2} = \left(N_2\right)_\delta$, we get

$$\left(N_1\right)_\delta = \frac{1}{\alpha_{11}} \left(a_1 - \alpha_{12} \frac{c_2}{p_2 q_2} \right) \text{ and}$$

$$\left(E\right)_\delta = \frac{1}{q_2} \left[\left(a_2 - \alpha_{22} \frac{c_2}{p_2 q_2} \right) - \alpha_{21} \left(N_1\right)_\infty \right]$$

Hence once the optimal equilibrium $\left(\left(N_1\right)_\delta, \left(N_2\right)_\delta\right)$ is determined, the optimal harvesting effort $\left(E\right)_\delta$ can be determined.

From (7.3), (7.6) and (7.7), we found that λ_1, λ_2 do not vary with time in optimal equilibrium. Hence they remain bounded as $t \to \infty$.

From (7.8), we also note that

$$\left(p_2 - \frac{c_2}{q_2 \overline{N_2}} \right) = \frac{B_2}{\left(\alpha_{22} \overline{N_2} + \delta\right)} \to 0 \text{ as } \delta \to \infty$$

Thus, the net economic revenue of the predator $R_2 = 0$.

This implies that if the discount rate increases, then the net economic revenue decreases and even may tend to zero if the discount rate tend to infinity. Thus it has been concluded that high interest rate will cause high inflation rate.

8. CONCLUSIONS

In this paper, the consequences of two species competitive model with optimal harvesting of the second species under bionomic conditions have been studied. The existence of the possible steady states along with their local stability is discussed and also conditions for global stability of the system are derived by using Liapunov function. The conditions for the existence of Biological and Bionomical equilibria of the system are derived. Further, the optimal harvesting policy has been discussed by using Pontryagin's Maximum Principle [11]. It has been found that the total user cost of harvest per unit of effort equals the discounted value of the future profit at the steady-state effort level. It has also been noted that if the discount rate increases, then the economic rent decreases and even may tend to zero if the discount rate tend to infinity. Thus it has been concluded that high interest rate will cause high inflation rate.

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