Equivalence Of Super Magic Labelings

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ABSTRACT

Let G (V, E) be a graph with p vertices and q edges. A bijection $f : V \cup E \rightarrow \{1,2,\ldots, p+q\}$ is called a super magic labeling of a graph G, if $f(V) = \{1, 2,\ldots, p\}$ and for any edge $xy \in E$, $f(x)+f(y)+f(xy) = c(f)$, a constant. The super magic strength of a graph G, sm(G) is defined as the minimum of all c(f) where the minimum runs over all super magic labelings f of G. Recently a new version of super magic labeling has been introduced. For the graph G(V, E), a bijection $f : V \cup E \rightarrow \{1, 2,\ldots, p+q\}$ is called a super magic (special) labeling of G, if $f(E) = \{1, 2,\ldots, q\}$ for any edge $xy \in E$ such that $f(x)+f(y)+f(xy) = c'(f)$, a constant. The super magic (special) strength of a graph G, sms(G) is defined as the minimum of all c'(f) where the minimum is taken over all super magic special labelings f of G. In this paper, we prove that *sms(G) = 2q-p+sm(G).*

Key Words: Graph labeling, Magic labeling, Magic strength, Super magic labeling, Super magic strength.

AMS Subject Classification Code: 05C (Primary)

INTRODUCTION

In this paper, we consider only finite simple undirected graphs. For notations and terminology, we follow [4]. Throughout this paper, we denote a path on n vertices by P_n and a cycle on n vertices by C_n. The n - bistar $B_{n,n}$ is the graph obtained from two copies of the star $K_{1,n}$ by joining the centers of two copies of $K_{1,n}$ by means of an edge e. In $B_{n,n}$, if the edge e is subdivided once, then the resultant graph is denoted by $<K_{1,n} : 2>$. The graph nP_2 is the disjoint union of n copies of P_2. The square graph of P_n is denoted by P_n^2 and is obtained from P_n by adding the edges between two vertices in P_n if the distance between them is two in P_n. The wheel W_n on n+1 vertices is the join of C_n and K_1.

In 1970, Kotzig and Rosa [6] defined a new labeling called the magic labeling of a graph G (V, E). A bijection $f : V \cup E \rightarrow \{1, 2,\ldots, p+q\}$ is called a *magic labeling* if for all edge s $xy \in E$, $f(x)+f(y)+f(xy)$ is a constant, c(f), say.

For example, a magic labeling f of $K_{2,3}$ with c(f) = 16 is shown in Figure 1.

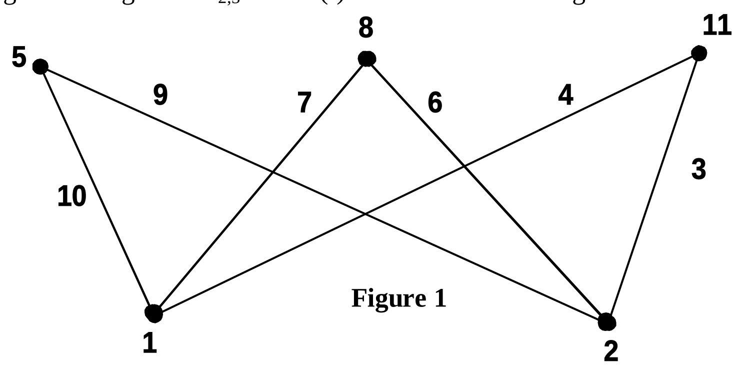


Figure 1

A graph is said to be *magic* if it has a magic labeling. Ringel and Llado [7] called this graph as *edge magic*. Many graphs like $K_{m,n}$, C_n $(n \geq 3)$, nP_2 (n is odd) etc. have been proved to be magic in [6].

In [7], it was proved that if G is a graph in which both p and q are even such that $p+q \equiv 2 \pmod 4$ and in which each vertex has odd degree, then G is not magic.

In 2000, Selvam Avadayappan et al. [2] introduced the concept of magic strength of a graph. We know that for any magic labeling f of G, there is a constant c(f) such that $f(x) + f(y) + f(xy) = c(f)$ for all edges $xy \in E$. The *magic strength* of G, m(G) is defined as the minimum of all c(f) where the minimum is taken over all magic labelings f of G. That is,

$$m(G) = \min\{\ c(f) : f \text{ is a magic labeling of } G\ \}.$$

For example, the magic strength of some graphs are shown in Figure 2.

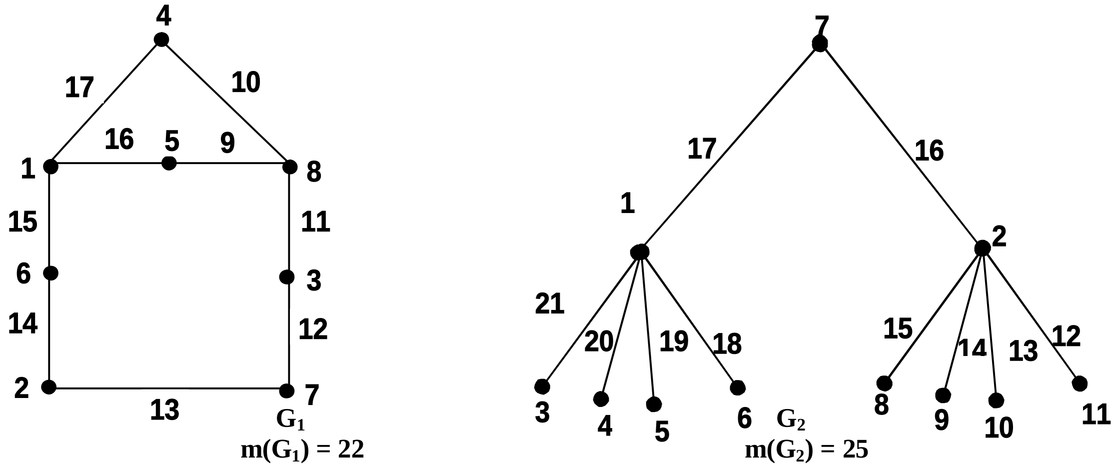


Figure 2

A magic labeling of a graph G (V, E) is called a *super magic labeling* of G, if f(V) = {1, 2,...., p} and so f(E) = {p+1, p+2,…., p+q}.

For example, a super magic labeling of a graph with c(f) = 29 is shown in Figure 3.

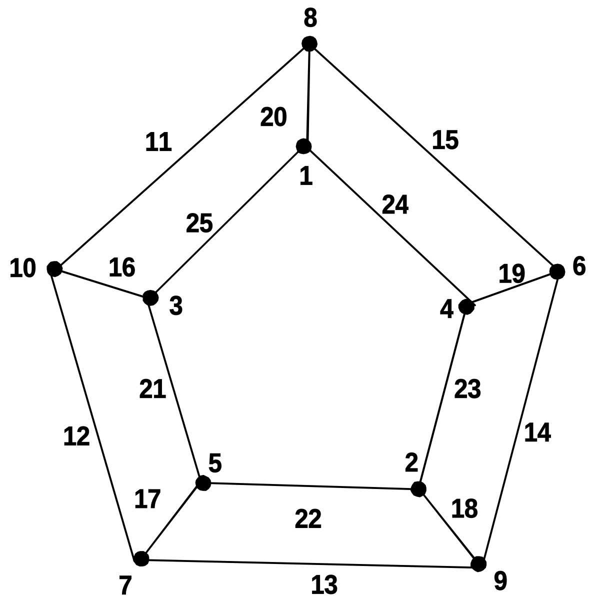


Figure 3

A graph is said to be *super magic* if it admits a super magic labeling.

In [5], the following results have been proved:

1. A cycle C_n is super magic if and only if n is odd.
2. A complete bipartite graph $K_{m,n}$ is super magic if and only if m=1 or n=1.

In 2001, Selvam Avadayappan et al. [3] introduced the concept of super magic strength of a graph. The *super magic strength* sm(G) of a graph G, is defined as the minimum of all c(f) where the minimum is taken over all super magic labelings f of G. That is,

$$sm(G) = \min\{\ c(f) : f \text{ is a super magic labeling of } G\ \}.$$

For example, consider the graphs G_1 and G_2 shown in Figure 4. Here $sm(G_1) = 21$ and $sm(G_2) = 28$ and the corresponding super magic labelings are shown.

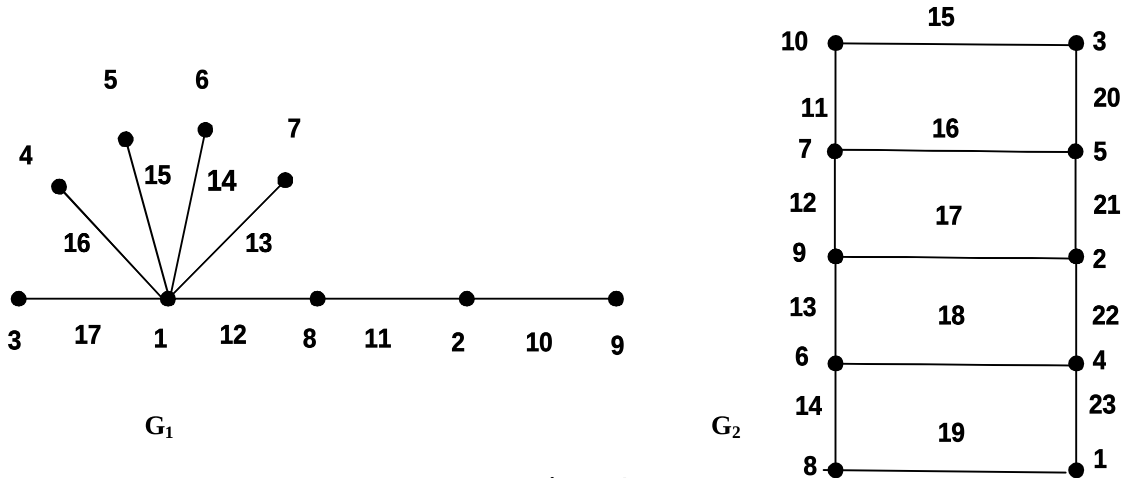


Figure 4

In a similar way, Swaminathan et al. [8] introduced the maximum super magic strength. The maximum *super magic strength* of a graph G, SM(G) is defined as the maximum of all c(f). That is,

$$SM(G) = \max\{c(f) : f \text{ is a super magic labeling of } G\}.$$

Also, they established a relation between the minimum super magic strength sm(G) and the maximum super magic strength SM(G) of any super magic graph G. In fact, they have proved that

Theorem A [8] A (p, q) - graph G has a super magic labeling with minimum strength sm(G) if and only if G has a super magic labeling with maximum strength SM(G) = 4p+q+3- sm(G).

Recently, D.G.Akka and Nanda S.Warad [1] have introduced a slight variation in the concept of super magic labeling and the super magic strength.

Though they have used the same terminology, to avoid confusion we call them respectively as super magic (special) labeling and super magic (special) strength.

A magic labeling f of G is called a *super magic (special) labeling* if f(E) = {1, 2,…., q} and thus f(V) = {q+1, q+2,…., p+q}.

For example, a super magic (special) labeling of a graph with c(f) = 53 is shown in Figure 5.

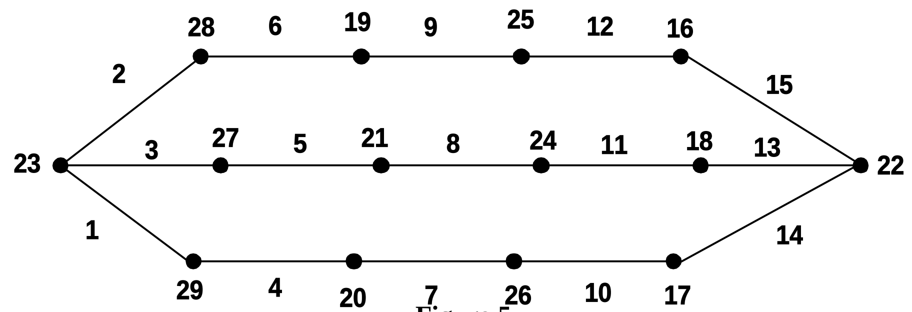


Figure 5

The *super magic (special) strength* sms(G) of a graph G is defined as the minimum of all constants c'(f) where the minimum is taken over all super magic (special) labelings f of G. That is,

$$sms(G) = \min\{ c'(f): f \text{ is a super magic (special) labeling of } G \}.$$

For example, a super magic (special) strength of < $K_{1,5}$: 2> with sms(G) = 49 is shown in Figure 6.

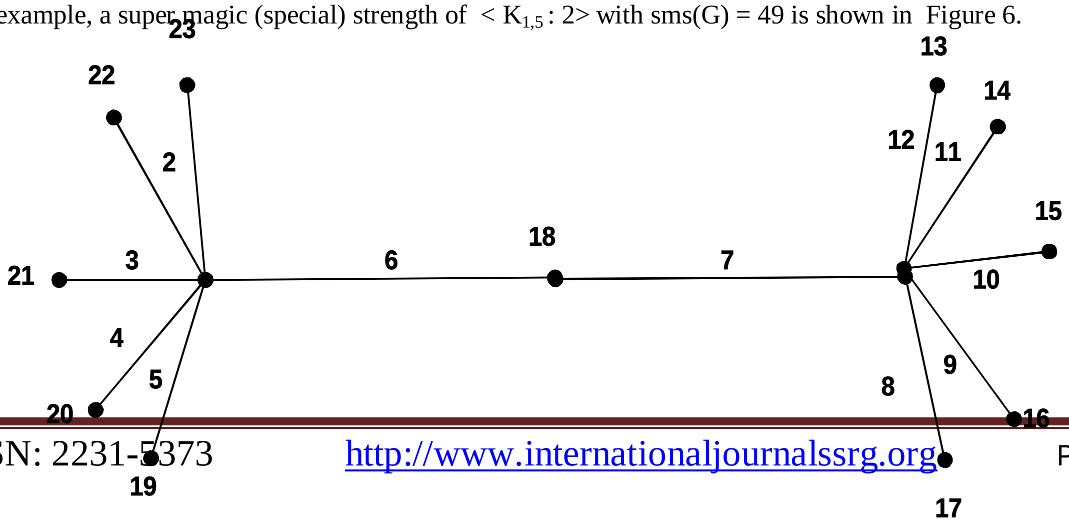
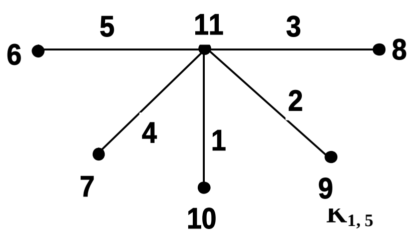


Figure 6

In this paper, we introduce an equivalent parameter, *the maximum super magic (special) strength* of a graph G. It is denoted by SMS(G) and is defined as the maximum of all c'(f) where the maximum is taken over all super magic (special) labelings f of G. That is,

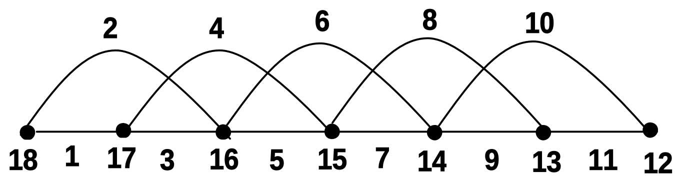
$$SMS(G) = \max \{c'(f) \text{ is a super magic (special) labeling of } G\}.$$

For example, the maximum super magic (special) strength of $K_{1,5}$ and P_5^2 are shown in Figure 7.



$K_{1,5}$

SMS ($K_{1,5}$) = 22



P_7^2

SMS (P_7^2) = 36

Figure 7

Here we prove that SMS(G) = 2p + 5q +3 - sms(G). Also, we prove that a graph G admits a super magic labeling if and only if G admits a super magic (special) labeling and hence we deduce that sms(G) = 2q - p + sm(G). This forces that studying about super magic (special) strength is equivalent to a study on super magic strength.

2. MAIN RESULTS

Theorem 1 A (p, q) - graph G has a super magic (special) labeling with minimum strength sms(G) if and only if G has a super magic (special) labeling with maximum strength SMS(G) = 2p + 5q + 3 - sms(G).

Proof Let G(V,E) be a (p, q) - super magic graph with minimum strength sms(G).

Then there exists a super magic (special) labeling $f : V \cup E \rightarrow \{1, 2, \dots, p+q\}$ of G such that for all $xy \in E$, $f(x) + f(y) + f(xy) = sms(G)$.

Define a labeling $f_1 : V \cup E \rightarrow \{1, 2, \dots, p+q\}$ by

$f_1(x) = p+ 2q+ 1- f(x)$ and $f_1(x) = q+ 1- f(xy)$.

We can easily verify that f_1 is bijective with $f_1(V) = \{q+1, q+2, \dots, p+q\}$ and $f_1(E) = \{1, 2, \dots, q\}$.
Also for all $xy \in E$ we get,

$$\begin{aligned} f_1(x)+ f_1(y)+ f_1(xy) &= p+2q+1- f(x)+ p+2q+1- f(y)+ q+1- f(xy) \\ &= 2p+5q+3 - (f(x) +f(y) + f(xy)) \\ &= 2p+5q+3 - sms(G) \\ &= \text{a constant.} \end{aligned}$$

Therefore, f_1 is a super magic (special) labeling of G with constant $c'(f_1) = 2p+5q+3 - sms(G)$.

Hence $SMS(G) \geq 2p + 5q + 3 - sms(G)$.

Suppose $SMS(G) > 2p + 5q + 3 - sms(G)$, there exists a super magic (special) labeling $g : V \cup E \rightarrow \{1, 2, \dots, p+q\}$ such that for all edges $xy \in E$, $g(x) + g(y) + g(xy) > 2p + 5q + 3 - sms(G)$.

Again, defining a bijection $g_1 : V \cup E \rightarrow \{1, 2, \dots, p+q\}$ by

$g_1(x) = p+ 2q +1- f(x)$ and $g_1(x) = q+ 1- f(xy)$.

For all $xy \in E$ we get,
$g_1(x)+ g_1(y)+ g_1(xy) = p+2q+1- g(x)+ p+2q+1- g(y)+ q+1- g(xy)$

$= 2p+5q+3 - (g(x)+ g(y)+ g(xy))$
$< 2p+5q+3 - (f_1(x)+ f_1(y)+ f_1(xy))$
$= 2p+5q+3 - (2p+5q+3) + sms(G)$
$= sms(G)$.

Hence $c'(g_1) < sms(G)$, which is a contradiction to the fact that sms(G) is the minimum super magic (special) strength of a graph G.
Therefore, $SMS(G) = 2p + 5q + 3 - sms(G)$.
Hence, G has a super magic (special) labeling with maximum super magic (special) strength SMS(G).
Similarly, we can prove the converse.

Theorem 2 Let G be a graph and $f : V \cup E \rightarrow \{ 1, 2,\ldots, p+q \}$ be a bijection. Then f is a super magic labeling of G with constant c (f) if and only if there is a super magic (special) labeling g of G with constant $c'(g) = 3(p+q+1) - c(f)$.

Proof Let $V(G) = \{ v_1, v_2,\ldots, v_p\}$ and $E(G) = \{e_1, e_2,\ldots ,e_q\}$.

First, we assume that f is a super magic labeling of G with constant c(f).
Then $f(V) = \{1, 2,\ldots, p\}$ and thus $f(E) = \{p+1, p+2,\ldots, p+q\}$.
Now we define, $g(x) = p+q+1-f(x)$.
Clearly, $g(V) = \{q+1, q+2,\ldots, p+q\}$ and $g(E) = \{1, 2,\ldots, q\}$.
Now for any edge $xy \in E$,

$$\begin{aligned} g(x)+ g(y)+ g(xy) &= p+q+1- f(x)+ p+q+1- f(y)+ p+q+1- f(xy) \\ &= 3(p+q+1) - (f(x) +f(y) + f(xy)) \\ &= 3(p+q+1) - c(f). \end{aligned}$$

Thus g is a super magic (special) labeling of G with constant $c'(g) = 3(p+q+1) - c(f)$.
Similarly, we can prove the converse part.

Corollary $sms(G) = 2q- p+ sm(G)$.

Proof Let f be any super magic labeling of G. Then by Theorem 2, we have $c'(g) = 3(p+q+1)- c(f)$ where g is a super magic (special) labeling of G.
Thus $sms(G) \leq 3(p+q+1) - c(f)$ for any super magic labeling f of G.
$sms(G) \leq 3(p+q+1) - SM(G)$ and hence by Theorem A,
$sms(G) \leq 3(p+q+1) - (4p+q+3-sm(G)) = 2q - p + sm(G)$
Therefore, $sms(G) \leq 2q - p + sm(G)$. On the other hand,

Let g be any super magic (special) labeling of G. Then by Theorem 2,
We have $c(f) = 3(p+q+1) - c'(g)$ where f is a super magic labeling of G.
Thus $sm(G) \leq 3(p+q+1) - c'(g)$ for any super magic (special) labeling g of G.
$sm(G) \leq 3(p+q+1) - SMS(G)$ and hence by Theorem 1,
$sm(G) \leq 3(p+q+1) - (2p+5q+3-sms(G)) = p- 2q + sms(G)$
That is, $sms (G) \geq 2q - p + sm (G)$ and hence

$sms(G) = 2q- p+ sm(G)$.

The above theorem means that , super magic strength and super magic (special) strength are interrelated.

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