The Local Well-posedness of The Higher-order Camassa-Holm Equation

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Abstract—In this paper, the local well-posedness of the Cauchy problem for the higher-order Camassa-Holm equation is studied with the initial data in $H^s(R)$, $s \geq k$ by using Bourgain technology.

Key word—higher-order Camassa-Holm equation, local well-posedness, Fourier transformation.

1. INTRODUCTION

The formulation of the higher-order C-H equation is

$$u_t = B_k(u,u)\,, \tag{1.1}$$

where $B_k(u,u) = A_k^{-1}C_k(u) - uu_x$, $A_k(u) = \sum_{j=0}^{k}(-1)^j \partial_x^{2j} u$, $C_k(u) = -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u)$.

The equation is first derived by Adrian Constantin and Boris Kolev as an Euler equation in [3], but first studied by G.M. Coclite, H. Holden and K.H. Karlsen as an independent equation in [4].

Rewrite the equation (1.1) as

$$u_t + u_{xxx} + A_k^{-1}(uA_k(u_x)) + 2A_k^{-1}(u_x A_k(u)) - u_{xxx} = 0\,. \tag{1.2}$$

In this paper we consider the Cauchy problem of the case of $k \geq 2$ for the equation (1.2). By using Bourgain technology in [5,6], the local well-posedness of the equation (1.2) is established with the initial data in $H^s, s \geq k$.

In this paper, we consider the Cauchy problem of the higher-order Camassa-Holm equation

$$\begin{cases} u_t + u_{xxx} + A_k^{-1}(uA_k(u_x)) + 2A_k^{-1}(u_x A_k(u)) - u_{xxx} = 0 \\ u(x,0) = \varphi(x) \end{cases}. \tag{1.3}$$

The equation (1.3) has the equivalent integral equation $u(\cdot,t) = W(t)\varphi - \int_0^t W(t-\tau)\omega(\tau)d\tau$, where $W(t) = e^{-t\partial_x^3}$ and $\omega = A_k^{-1}(uA_k(u_x)) + 2A_k^{-1}(u_x A_k(u)) - u_{xxx}$. Then we shall apply a fixed point argument to the following truncated version

$$u(x,t) = \psi_1(t)\sum_{\xi\in Z}\hat{\varphi}(\xi)e^{i(\langle x,\xi\rangle+\xi^3 t)} + \psi_1(t)\sum_{\xi\in Z}e^{i(\langle x,\xi\rangle+\xi^3 t)}\int_{-\infty}^{\infty}\hat{\omega}(\xi,\lambda)\frac{e^{i(\lambda-\xi^3)t}-1}{\lambda-\xi^3}d\lambda\,,$$

where ψ_1 is a cutoff function, satisfying $0 \leq \psi_1 \leq 1$, supp$(\psi_1) \subset [-2\delta, 2\delta]$, and $\psi_1 = 1$ on $[-\delta,\delta]$. Here δ is a positive real number. Let ψ_2 is another cutoff function, which satisfying $0 \leq \psi_2 \leq 1$, $\psi_2 = 1$ on $[-T,T]$, and supp$\psi \subset [-2T, 2T]$.

Define the norm $|||u|||=\left\{\sum_{n\in\mathbf{Z}\setminus\{0\}}|n|^{2s}\int_{-\infty}^{\infty}(1+|\lambda-n^3|)|\hat{u}(n,\lambda)|^2\,d\lambda\right\}^{1/2}$, $s\geq 0$, where $u=u(x,t), x\in R, t\in[0,\infty)$.

Let $\chi^s=\{u\in L_2 \,|\, |||u|||<\infty\}$, then we obtain the main result.

Theorem 1.1. For $\varphi\in H^s(R)$, $s\geq k$, $k\geq 2$, there exists a $T>0$ and a unique solution $u=u(x,t)$ satisfying the Cauchy problem (1.3) such that $u\in C([0,T];H^s(R))\bigcap C^1([0,T];H^{s-1}(R))$.

2. SOME LEMMAS

Lemma 2.1[5,6] Following estimate holds

$$\|f\|_{L^4(R\times(0,\infty))}\leq c\left(\sum_{m,n\in\mathbf{Z}}(1+|n-m^3|)^{2/3}|\hat{f}(m,n)|^2\right)^{1/2},$$

where $f=f(x,t), x\in R, t\in[0,\infty)$.

Corollary 2.2[6] $\left(\sum_{m,n\in\mathbf{Z}}(|n-m^3|+1)^{-2/3}|\hat{f}(m,n)|^2\right)^{1/2}\leq c\|f\|_{L^{4/3}(R\times(0,\infty))}$

Corollary 2.3[6] Assume $\lambda=(\lambda_{m,n})_{m,n\in\mathbf{Z}}$ is a multiplier satisfying $|\lambda_{m,n}|\leq(1+|n-m^3|)^{-2/3}$ for all m,n. The λ acts boundedly from $L^{4/3}(R\times(0,\infty))$ to $L^4(R\times(0,\infty))$.

Rewrite ω as $\omega=-A_k^{-1}(uA_k(u_x))+2A_k^{-1}((uA_k(u)_x)-u_{xxx}$. From the properties of Fourier transformation, the estimate holds $|\hat{\omega}(n,\lambda)|\leq\omega_1+\omega_2+\omega_3$, where

$$\omega_1=\sum_{n_1}\frac{|n_1|\sum_j n_1^{2j}}{\sum_j n^{2j}}\int d\lambda_1|u(n_1,\lambda_1)||u(n-n_1,\lambda-\lambda_1)|,$$

$$\omega_2=\sum_{n_1}\frac{2|n|\sum_{j=0}^{k}n_1^{2j}}{\sum_{j=0}^{k}n^{2j}}\int d\lambda_1|u(n_1,\lambda_1)||u(n-n_1,\lambda-\lambda_1)|,$$

$$\omega_3=|n^3||\hat{u}(n,\lambda)|.$$

Denote $c_s(n,\lambda)=|n^s|(1+|\lambda-n^3|)^{1/2}|\hat{u}(n,\lambda)|$. Thus $|||u|||$ is the L^2-norm of $\{c_s(n,\lambda)\}$.

Lemma 2.4 For ω defined as above, we have

$$\left(\sum_{n\neq 0}|n|^{2s}\int\frac{|\hat{\omega}(n,\lambda)|^2}{1+|\lambda-n^3|}d\lambda\right)^{1/2}\leq c\,|||u|||^2,\ s\geq k$$

where c is a constant depending on s and k.

Proof: One can get $\left(\sum_{n\neq 0}|n|^{2s}\int\frac{|\hat{\omega}(n,\lambda)|^2}{1+|\lambda-n^3|}d\lambda\right)^{1/2}\leq c(I_1+I_2+I_3)$, where

$$I_1=\left(\sum_{n\neq 0}|n|^{2s}\int\frac{|\omega_1|^2}{1+|\lambda-n^3|}d\lambda\right)^{1/2},$$

$$I_2=\left(\sum_{n\neq 0}|n|^{2s}\int\frac{|\omega_2|^2}{1+|\lambda-n^3|}d\lambda\right)^{1/2},$$

$$I_3=\left(\sum_{n\neq 0}|n|^{2s}\int\frac{|\omega_3|^2}{1+|\lambda-n^3|}d\lambda\right)^{1/2}.$$

The estimation for I_1. From ω_1,

$$Y_1=\frac{|n|^s|\omega_1|}{(1+|\lambda-n^3|)^{1/2}}\leq\sum_{n_1}\int d\lambda_1\frac{|n_1||n|^s\,(\sum_{j=0}^{k}n_1^{2j})|n_1|^{-s}|n-n_1|^{-s}\,c_s(n_1,\lambda_1)c_s(n-n_1,\lambda-\lambda_1)}{(\sum_{j=0}^{k}n^{2j})(1+|\lambda-n^3|)^{1/2}(1+|\lambda_1-n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}},\tag{2.1}$$

By assumption on u, $c_s(0,\lambda)=0$, so that we may assume $n_1\neq 0$, $n-n_1\neq 0$ (and $n\neq 0$). Obviously, $|n|^s\leq 2^{s+1}|n_1|^s|n-n_1|^s$. Then we consider two cases: $A=\{|n|\geq|n_1|\}$ and $B=\{|n|<|n_1|\}$.

Case A. It yields that $Y_1\leq\sum_{n_1}\int d\lambda_1\frac{2^{s+1}|n|c_s(n_1,\lambda_1)c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)^{1/2}(1+|\lambda_1-n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}$.

Also it holds that $|(\lambda-n^3)-[(\lambda_1-n_1^3)+(\lambda-\lambda_1-(n-n_1)^3)]|=|3n_1(n-n_1)n|>2n^2$. This results in that one of following cases happens

$$|\lambda-n^3|>1/2\,n^2,\tag{2.2}$$

$$|\lambda_1-n_1^3|>1/2\,n^2,\tag{2.3}$$

$$|\lambda-\lambda_1-(n-n_1)^3|>1/2\,n^2.\tag{2.4}$$

Case (2.2). It yields that $(2.1)\leq\sum_{n_1\neq 0,n}\int d\lambda_1\frac{C(s)c_s(n_1,\lambda_1)c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda_1-n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}$, where $C(s)$ is a constant depending on s.

Let $F(x,t)=\sum_m\int d\mu\left\{e^{i(mx+\mu t)}\frac{c_s(m,\mu)}{(1+|\mu-m^3|)^{1/2}}\right\}$. Then $Y_1\leq C(s)\hat{F}^2(n,\lambda)$. Thus we get

$$I_1\sim\left[\sum_{n\neq 0}\int|\hat{F}^2(n,\lambda)|^2\,d\lambda\right]^{1/2}\sim\|F\|^2_{L^4(dxdt(loc))}.$$

Since $\frac{2}{3}<1$, Lemma 2.1 implies that $\|F\|_{L^4}\le c(\sum_m\int c_s(m,\mu)^2d\mu)^{1/2}=c\,|||u|||$.

Case (2.3). It yields that $Y_1\le\sum_{n_1}\int d\lambda_1\frac{C(s)c_s(n_1,\lambda_1)c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}$.

Let $G(x,t)=\sum_m\int d\mu\{e^{i(mx+\mu t)}c_s(m,\mu)\}$ which L_2-norm $\|G\|_2\sim(\sum_m\int c_s(m,\mu)^2d\mu)^{1/2}=|||u|||$.

From the above calculation, Corollary 2.2 yields that

$$I_1\le\left\{\sum_n\int d\lambda\left[\frac{1}{(1+|\lambda-n^3|)^{1/2}}(F\cdot G)^{\wedge}(n,\lambda)\right]^2\right\}^{1/2}\le c\,|FG\|_{L^{4/3}(dx\times dt(loc))}\le c\,\|F\|_4\|G\|_2\le c\,|||u|||^2 .$$

Case (2.4). It yields that $(2.1)\le\sum_{n_1}\int d\lambda_1\frac{C(s)c_s(n_1,\lambda_1)c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)^{1/2}(1+|\lambda_1-n_1^3|)^{1/2}}$.

Replacing $(1+|\lambda_1-n_1^3|)^{1/2}$ by $(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}$, the following process is similar to the case of (2.3).

Case B. The estimating process is similar to the case A.

The estimation for I_2.

$$Y_2=\frac{|n|^s|\omega_2|}{(1+|\lambda-n^3|)^{1/2}}\le\sum_{n_1}\int d\lambda_1\frac{|n||n|^s(\sum_{j=0}^k n_1^{2j})|n_1|^{-s}|n-n_1|^{-s}\,c_s(n_1,\lambda_1)c_s(n-n_1,\lambda-\lambda_1)}{(\sum_{j=0}^k n^{2j})(1+|\lambda-n^3|)^{1/2}(1+|\lambda_1-n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}. \tag{2.5}$$

The estimating process is similar to the estimation for I_1.

The estimation for I_3.

$$I_3=\left(\sum_{n\ne0}|n|^{2s}\int\frac{n^6|\hat{u}(n,\lambda)|^2}{1+|\lambda-n^3|}d\lambda\right)^{1/2}=\left(\sum_{n\ne0}|n|^{2s}\int\frac{n^6}{(1+|\lambda-n^3|)^2}(1+|\lambda-n^3|)|\hat{u}(n,\lambda)|^2d\lambda\right)^{1/2}\sim c\,|||u|||.$$

Lemma 2.4 is proved.

Lemma 2.5 For ω defined as above, we have

$$\left\{\sum_{n\ne0}|n|^{2s}\left(\int\frac{|\hat{\omega}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\right)^2\right\}^{1/2}\le c\,|||u|||^2,\ s\ge k,$$

where c is a constant depending on s and k.

Proof: By the computation, $\left\{\sum_{n\ne0}|n|^{2s}\left(\int\frac{|\hat{\omega}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\right)^2\right\}^{1/2}\le c(I_4+I_5+I_6)$, where

$$I_4=\left\{\sum_{n\ne0}|n|^{2s}\left(\int\frac{|\omega_1|}{1+|\lambda-n^3|}d\lambda\right)^2\right\}^{1/2},$$

$$I_5=\left\{\sum_{n\ne0}|n|^{2s}\left(\int\frac{|\omega_2|}{1+|\lambda-n^3|}d\lambda\right)^2\right\}^{1/2},$$

$$I_6 = \left\{ \sum_{n \neq 0} |n|^{2s} \left(\int \frac{|\omega_3|}{1+|\lambda - n^3|} d\lambda \right)^2 \right\}^{1/2}.$$

The estimation for I_4**.**

$$Y_4 = \frac{|n|^s |\omega_1|}{(1+|\lambda - n^3|)} \le \sum_{n_1} \int d\lambda_1 \frac{|n_1|\,|n|^s (\sum_{j=0}^{k} n_1^{2j}) |n_1|^{-s} |n-n_1|^{-s} c_s(n_1,\lambda_1) c_s(n-n_1,\lambda-\lambda_1)}{(\sum_{j=0}^{k} n^{2j})(1+|\lambda-n^3|)(1+|\lambda_1 - n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}. \tag{2.6}$$

Case A**.** It yields that $Y_4 \le \sum_{n_1} \int d\lambda_1 \frac{|n| c_s(n_1,\lambda_1) c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)(1+|\lambda_1-n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}$.

Case (2.2). To estimate I_4, replace $\frac{1}{1+|\lambda-n^3|}$ by $\frac{1}{n^2+|\lambda-n^3|}$ in I_4. Then consider a sequence $\{a_n\}$, satisfying $a_n \ge 0$, $\sum a_n^2 = 1$. Applying these conditions, we have $Y_4' = \sum_n \int d\lambda \left\{ \frac{a_n |n|}{n^2+|\lambda-n^3|} \hat{F}^2(n,\lambda) \right\}$.

Define the function $H(x,t) = \sum_n \int d\lambda e^{i(nx+\lambda t)} \frac{|n| a_n}{n^2+|\lambda-n^3|}$ which L^2-norm

$$\| H \|_2 \sim \left(\sum_n \int d\lambda \frac{|n|^2 a_n^2}{(n^2+|\lambda|)^2} \right)^{1/2} \sim \sum a_n^2 = 1.$$

Then by the Hölder inequality, it yields that $Y_4' \le c \| H \|_2 \| F \|_4^2 \le c \||u\||^2$.

Case (2.3). It obviously yields that $Y_4 \le c \sum_{n_1} \int d\lambda_1 \frac{c_s(n_1,\lambda_1) c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}$.

Choose $\frac{1}{3} < \rho < \frac{1}{2}$ and write $(1+|\lambda-n^3|)^{-1} = (1+|\lambda-n^3|)^{-(1-\rho)}(1+|\lambda-n^3|)^{-\rho}$.By using the Cauchy- Schwartz inequality, it yields that $I_4 \le c \left(\sum_n \int d\lambda \left\{ \sum_{n_1} \int d\lambda_1 \frac{c_s(n_1,\lambda_1) c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)^\rho (1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}} \right\}^2 \right)^{1/2} \le c \left\{ \sum_n \int d\lambda \left[\frac{1}{(1+|\lambda-n^3|)^\rho} (F \cdot G)^\wedge(n,\lambda) \right]^2 \right\}^{1/2}$.

Corollary 2.2 yield the estimate $I_4 \le c |FG\|_{L^{4/3}(dx \times dt(loc))} \le c \| F \|_4 \| G \|_2 \le c \||u\||^2$.

Case (2.4). It yields that $Y_4 \le c \sum_{n_1} \int d\lambda_1 \frac{c_s(n_1,\lambda_1) c_s(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-n^3|)(1+|\lambda_1-n_1^3|)^{1/2}}$.

Replacing $(1+|\lambda_1-n_1^3|)^{1/2}$ by $(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}$, the following process is similar to the case of (2.3) in this lemma.

Case B The process is similar to the case of A.

The estimation for I_5**.** It yields that $\frac{|n|^s |\omega_2|}{(1+|\lambda-n^3|)} \le \sum_{n_1} \int d\lambda_1 \frac{|n|\,|n|^s (\sum_{j=0}^{k} n_1^{2j}) |n_1|^{-s} |n-n_1|^{-s} c_s(n_1,\lambda_1) c_s(n-n_1,\lambda-\lambda_1)}{(\sum_{j=0}^{k} n^{2j})(1+|\lambda-n^3|)(1+|\lambda_1-n_1^3|)^{1/2}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}$.

The following process is similar to the estimate of I_4.

The estimation for I_6**.** From ω_3,

$$I_6=\left\{\sum_{n\neq 0}|n|^{2s}\left(\int\frac{n^3|\hat{u}|}{1+|\lambda-n^3|}d\lambda\right)^2\right\}^{1/2}\leq c\left(\sum_n|n|^{2s}\int(1+|\lambda-n^3|)|\hat{u}(n,\lambda)|^2\,d\lambda\right)^{1/2}=c|||u|||.$$

Lemma 2.5 is proved.

Lemma 2.6 For limited constant $T>0$, we have that $|||\psi_2 u|||\leq X(T)|||u|||$, where $X(T)$ is a monotonically increasing function at argument T.

Proof: Obviously, it holds that

$$|||\psi_2 u|||=\left\{\sum_{n\in Z\backslash\{0\}}|n|^{2s}\int_{-\infty}^{\infty}(1+|\lambda-n^3|)|\widehat{\psi_2 u}(n,\lambda)|^2\,d\lambda\right\}^{1/2}$$

$$=\|\int_{-\infty}^{\infty}(1+|\lambda-n^3|)^{1/2}|n|^s\hat{\psi}_2(\eta)\hat{u}(n,\lambda-\eta)d\eta\|_{L^2(n,\lambda)}$$

$$\leq|||u|||\int_{-\infty}^{\infty}(1+|\eta|)^{1/2}|\hat{\psi}_2(\eta)|\,d\eta\,.$$

From the definition of Fourier transformation, it holds

$$|\hat{\psi}_2(\eta)|=|\int_{-2T}^{2T}e^{-ix\eta}\psi_2(x)dx|\leq|\frac{1}{i\eta}(e^{2iT\eta}-e^{-2iT\eta})|=|\frac{\sin 2T\eta}{\eta}|$$

Denote $X(T)=\int_{-\infty}^{\infty}(1+|\eta|)^{1/2}\frac{\sin 2T\eta}{\eta}d\eta$ .Then $X(T)$ is bounded. In fact it's known that

$$X(T)=2\int_0^{\pi/4T}(1+|\eta|)^{1/2}\frac{\sin 2T\eta}{\eta}d\eta+2\int_{\pi/4T}^{\infty}(1+|\eta|)^{1/2}\frac{\sin 2T\eta}{\eta}d\eta$$

Hence, $X(T)$ is convergent.

There exists a constant $M>0$, satisfying

$$X(T)=X_1(T)+X_2(T)\,,$$

where $X_1(T)=2\int_0^M(1+|\eta|)^{1/2}\frac{\sin 2T\eta}{\eta}d\eta$, $X_2(T)=2\int_M^{\infty}(1+|\eta|)^{1/2}\frac{\sin 2T\eta}{\eta}d\eta$, and $|X_2(T)|<\varepsilon$. Here ε is an arbitrarily small number. The first derivative of $X_1(T)$ is $X_1'(T)=4\int_0^M(1+|\eta|)^{1/2}\cos(2T\eta)d\eta$. So for $0<T<\frac{\pi}{4M}$, $0<\eta<M$, we have $X_1'(T)>0$. That is, $X_1(T)$ is a monotonically increasing function at argument T. So $X(T)$ is a monotonically increasing function at argument T.

3. LOCAL WELL-POSEDNESS

Now we give the proof of theorem 1.1.

Lemma (2.4) and Lemma(2.5)yield that $|||u|||$ is bounded, i.e. $u\in\chi^s$. Denote Γ the transformation $(\Gamma u)(x,t)=\psi_1(t)W(t)\varphi-\psi_1(t)\int_0^t W(t-\tau)\omega(\tau)d\tau$, where $\omega=A_k^{-1}(uA_k(u_x))+2A_k^{-1}(u_xA_k(u))-u_{xxx}$. From the preceding it's proved that $|||\Gamma u|||\leq c(\|\varphi\|_{H^s}+|||u|||^2)$. So Γ is a map from χ^s to χ^s .Then if consider $\Gamma u-\Gamma v$, ω is replaced by

$$A_k^{-1}(uA_k(u_x))+2A_k^{-1}(u_xA_k(u))-u_{xxx}-A_k^{-1}(vA_k(v_x))-2A_k^{-1}(v_xA_k(v))+v_{xxx}.$$

The preceding calculation yields that

$$|||\Gamma u-\Gamma v|||\leq c_1(|||\psi_2 u|||+|||\psi_2 v|||)|||u-v|||+c_2|||\psi_2(u-v)|||\leq[c_1X(T)(|||u|||+|||v|||)+c_2X(T)]|||u-v|||.$$

Hence, for small enough T , it holds that $|||\Gamma u-\Gamma v|||\leq\kappa(T)|||u-v|||$ where $\kappa(T)$ satisfies $0<\kappa(T)<1$.

So that Γ is a contraction and therefore Picard's theorem yields a function u satisfying $\Gamma u=u$. Moreover the solution is unique and persistent. By the Sobolev embedding theory, we prove the theorem 1.1.

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