Tripled Fixed Point for Compatible mappings in Partially Ordered Fuzzy Metric Spaces

Animesh Gupta*, **Rohit Narayan****, **R.N.Yadava*****

* Department of Mathematics,
Sagar Institute of Engineering, Technology and Research,
Ratibad Bhopal (M.P.), India
** Department of Electrical Engineering,
University of Waterloo, Watarloo, Ontario,
Canada - N2L 3G1.
*** Director,
Environment Management and Human Welfare Council
Bhopal - INDIA.
*** Ex. Director,
Advance Material Process Research Institute,
AMPRI-CSIR
Bhopal - INDIA

2000 Mathematics subject classification: 54A40, 54E50, 54D35.
keywords: Fuzzy metric space, Tripled Fixed point, Mixed monotone property, g-monotone property.

Abstract

In this paper we prove a tripled coincidence point theorem for compatible mapping in fuzzy metric space. Our aim of this paper is to improve the result of "A. Roldan, J. M. Moreno, C. Roldan," Tripled fixed point theorem in fuzzy metric spaces and applications", Fixed point theory and applications, doi:10.1186/1687-1812-2013-29." Our technique for the proof of the theorem is different. We also give an example in support of our theorem.

Introduction

Fixed point theorems have been studied in many contexts, one of which is the fuzzy setting. the concept of fuzzy sets was initially introduced by Zadeh [1] in 1965. To use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and its applications. One of the most interesting research topics in fuzzy topology us to find an appropriate definition of fuzzy metric space for its possible applications in several areas. It is well known that a fuzzy metric space is anb important generalization of the metric space. Many authors have considered this problem and have introduced it in different ways. For instance, George and Veeramani [2] modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [3] and defined the Hausdorff topology of a fuzzy metric space. There exists considerable literature about fixed point properties for mappings defined on fuzzy metric psaces, which have been studied by many authors (see [4-13]).

Very recently, tripled fixed point problems belong to a category of problems in fixed point theory in which much interest has been generated recently after the publication of a tripled contraction mapping theorem by Berinde and Borcut [14]. In their manuscript, some new tripled fixed point theorems are obtained using the mixed g-monotone mapping. By using same concept Roldan et al [15] introduced the notion of tripled fixed point theorem in fuzzy metric space.

Our aim of this article to improve the result of Roldan et al [15] and use a new fuzzy contractive inequality and prove tripled coincidence point theorem in fuzzy metric space defined by George and Veeramani [2] by using natural technique for the proof of the theorem. We assume that the associated $t-norm$ is a Hadzic type $t-norm$. We give an example in support of our theorem also we give an application of our theorem top obtained tripled fixed point theorem in partially ordered metric space. The organization of the paper is as follows. In Section 2, we give mathematical preliminaries which include relevant definitions and some other results given as lemmas which will be used to deduce our results. In Section 3, tripled coincidence point and fixed point results in fuzzy metric spaces are established. An illustrative example is also given. In Section 4, a result in metric spaces is deduced. This result is obtained by an application of the maing result of Section 3.

Prilimaries

Definition 1: *A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is continuous $t - norm$ if $\star$ is satisfying the following conditions:*

i. $\star$ *is commutative and associative;*
ii. $\star$ *is continuous;*
iii. $a \star 1 = a$ *for all* $a \in [0,1]$;
iv. $a \star b \leq c \star d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a,b,c,d \in [0,1]$.

Some examples of continuous $t-norm$ are $a \star_1 b = min\{a,b\}$, $a \star_2 b = \frac{ab}{max\{a,b,\lambda\}}$ $for\ 0 < \lambda < 1$, $a \star_3 b = ab$ and $a \star_4 b = max\{a+b-1,0\}$. Several aspects of the theory of $t-norms$ with examples are given comprehensively by Klement et al. in their book [12].

George and Veeramani in their paper [2] introduced the following definition of fuzzy metric space. We will be concerned only with this definition of fuzzy metric space.

Definition 2: *A 3 - tuple $(X,M,\star)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $\star$ is a continuous t- norm, and M is a fuzzy set on $X^2 \times (0,+\infty)$ satisfying the following conditions, for each $x,y,z \in X$ and $t,s > 0$,*

$(F-1)$ $M(x,y,t) > 0$,

$(F-2)$ $M(x,y,t) = 1$ *if and only if* $x = y$,

$(F-3)$ $M(x,y,t) = M(y,x,t)$,

$(F-4)$ $M(x,y,t) \star M(y,z,s) \leq M(x,z,t+s)$,

$(F-5)$ $M(x,y,.) : (0,\infty) \to [0,1]$ *is continuous.*

Let $(X,M,\star)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x,r,t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x,r,t) = \{ y \in X : M(x,y,t) > 1 - r\}.$$

A subset $A \subset X$ is called open if, for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 3: *Let (X,d) be a metric space. Define $t-norm$ $a \star b = ab$ and for all $x,y \in X$ and $t > 0$, $M(x,y,t) = \frac{t}{t+d(x,y)}$. Then $(X,M,\star)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.*

Example 4: *Let (X,d) be a metric space and ψ be an increasing and continuous function of $(0,\infty)$ into $(0,1)$ such that $lim_{t\to\infty} \psi(t) = 1$. Three typical examples of these functions are $\psi(t) = \frac{t}{t+1}$, $\psi(t) = sin\left(\frac{\pi t}{2t+1}\right)$ and $\psi(t) = 1 - e^{-t}$. Let $\star$ be any continuous $t-norm$. For each $t > 0, x,y \in X$, let*

$M(x,y,t) = \psi(t)^{d(x,y)}$.

Then $(X, M, \star)$ is a fuzzy metric space.

Definition 5: Let $(X, M, \star)$ be a fuzzy metric space. Then,

i. A sequence $\{ x_n \}$ in X is said to be convergent to x if,

$$lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for all } t > 0.$$

ii. A sequence $\{ x_n \}$ in X is said to be Cauchy sequence if for any $\epsilon > 0$, there exists $n_0 \in N$, such that

$$lim_{m,n\to\infty} M(x_m, x_n, t) > 1-\epsilon$$

for all $t > 0$ and $n, m \geq n_0$.

iii. A fuzzy metric space $(X, M, \star)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

The following lemma was proved by Grabiec [10] for fuzzy metric spaces defined by Kramosil et al [3] . The proof is also applicable to the fuzzy metric space given in Definition 2.

Lemma 6: Let $(X, M, \star)$ be a fuzzy metric space. Then $M(x, y, .)$ is non decreasing for all $x, y \in X$.

Lopez and Romaguera [13] give the following result,

Lemma 7 : M is continuous function on $X^2 \times (0, \infty)$.

Our purpose in this paper is to prove tripled coincidence point theorem for two mappings in complete fuzzy metric space which has a partial order defined on it.

Let $(X, \preccurlyeq)$ be a partially ordered set and F be a mapping from X to itself. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X, x_1 \preccurlyeq x_2$ implies $F(x_1) \preccurlyeq F(x_2)$ and is said to be non-increasing if for all $x_1, x_2 \in X, x_1 \preccurlyeq x_2$ implies $F(x_1) \succcurlyeq F(x_2)$.

Definition 8: Let $(X, \preccurlyeq)$ be a partially ordered set, $F: X^3 \to X$ mapping. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$,

i. $x_1, x_2 \in X,\ x_1 \preccurlyeq x_2 \Rightarrow F(x_1, y, z) \preccurlyeq F(x_2, y, z)$,

ii. $y_1, y_2 \in X,\ y_1 \preccurlyeq y_2 \Rightarrow F(x, y_1, z) \succcurlyeq F(x, y_2, z)$,

iii. $z_1, z_2 \in X,\ z_1 \preccurlyeq z_2 \Rightarrow F(x, y, z_1) \preccurlyeq F(x, y, z_2)$.

Definition 9: An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F: X^3 \to X$ if

$$F(x, y, z) = x,\ F(y, x, y) = y,\ and\ F(z, y, x) = z.$$

Definition 10: Let $(X, \preccurlyeq)$ be a partially ordered set, $F: X^3 \to X$ and $g: X \to X$ be two mappings. The mapping F is said to have the mixed $g-$monotone property if for any $x, y, z \in X$,

i. $x_1, x_2 \in X,\ g(x_1) \preccurlyeq g(x_2) \Rightarrow F(x_1, y, z) \preccurlyeq F(x_2, y, z)$,

ii. $y_1, y_2 \in X,\ g(y_1) \preccurlyeq g(y_2) \Rightarrow F(x, y_1, z) \succcurlyeq F(x, y_2, z)$,

iii. $z_1, z_2 \in X,\ g(z_1) \preccurlyeq g(z_2) \Rightarrow F(x, y, z_1) \preccurlyeq F(x, y, z_2)$.

Definition 11: An element $(x, y, z) \in X^3$ is called a tripled coincidence point of the mappings $F: X^3 \to X$ and $g: X \to X$ if

$$F(x, y, z) = gx,\ F(y, x, y) = gy\ \ and\ \ F(z, y, x) = gz.$$

Definition 12 : An element $(x, y, z) \in X^3$ is called a tripled common fixed point of the mappings $F: X^3 \to X$ and $g: X \to X$ if

$$F(x, y, z) = gx = x,\ F(y, x, y) = gy = y\ \ and\ F(z, y, x) = gz = z.$$

Definition 13 : An element $x \in X$ is called a common fixed point of the mappings $F: X^3 \to X$ and $g: X \to X$ if

$$F(x, x, x) = gx = x.$$

Definition 14: Let X be a non empty set. The mappings $F: X^3 \to X$ and $g: X \to X$ are commuting if for all $x, y, z \in X$,

$$g\big(F(x, y, z)\big) = F\big(g(x), g(y), g(z)\big).$$

Definition 15: Let (X,d) be a metric space. The mappings F and g where $F: X^3 \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible if

$$lim_{n\rightarrow\infty} d\left(g\big(F(x_n,y_n,z_n)\big), F\big(g(x_n),g(y_n),g(z_n)\big)\right) = 0$$

$$lim_{n\rightarrow\infty} d\left(g\big(F(y_n,x_n,y_n)\big), F\big(g(y_n),g(x_n),g(y_n)\big)\right) = 0$$

and

$$lim_{n\rightarrow\infty} d\left(g\big(F(z_n,y_n,x_n)\big), F\big(g(z_n),g(y_n),g(x_n)\big)\right) = 0$$

whenever $\{x_n\},\{y_n\}$ and $\{z_n\}$ are sequences in X such that $lim_{n\rightarrow\infty} F(x_n,y_n,z_n) = lim_{n\rightarrow\infty} g(x_n) = x$, $lim_{n\rightarrow\infty} F(y_n,x_n,y_n) = lim_{n\rightarrow\infty} g(y_n) = y$ and $lim\_(n\rightarrow\infty) F(z_n,y_n,x_n) = lim_{n\rightarrow\infty} g(z_n) = z$ for some $x,y,z \in X$.

Intuitively we can think that the function F and g commute in the limit in the situations where the functional values tend to the same point.

Definition 16: The mappings $F: X^3 \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible if

$$lim_{n\rightarrow\infty} M(gF(x_n,y_n,z_n), F(gx_n,gy_n,gz_n), t = 1$$

$$lim_{n\rightarrow\infty} M(gF(y_n,x_n,y_n), F(gy_n,gx_n,gy_n), t) = 1$$

$$lim_{n\rightarrow\infty} M(gF(z_n,y_n,z_n), F(gz_n,yx_n,gx_n), t) = 1$$

for all $t > 0$ whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in X, such that

$$lim_{n\rightarrow\infty} F(x_n,y_n,z_n) = lim_{n\rightarrow\infty} gx_n = x,$$

$$lim_{n\rightarrow\infty} F(y_n,x_n,y_n) = lim_{n\rightarrow\infty} gy_n = y,$$

$$lim_{n\rightarrow\infty} F(z_n,y_n,x_n) = lim_{n\rightarrow\infty} gz_n = z,$$

for some $x,y,z \in X$.

Given a metric space (X,d), consider the fuzzy metric space $(X,M,\star)$ constructed in Example 3. Then $\{x_n\}$ converges to x in the metric space (X,d) if and only if $\{x_n\}$ converges to x in the fuzzy metric space $(X,M,\star)$. The equivalence between completeness of (X,d) and $(X,M,\star)$ was established by George and Veeramani in Result 2.9 of their paper [2].

In the following lemma we established that the compatibility in a metric space implies that the compatibility in the corresponding fuzzy metric space of Example \ref(exam2.3). We use it to obtain a result in metric spaces in Section 4.

Lemma 17: Let (X,d) be a metric space. It the mappings F and g where $F: X^3 \rightarrow X$ and $g: X \rightarrow X$ are compatible in (X,d) (according to Definition \ref(2.13)), then F and g are also compatible (according to Definition 16)in the corresponding fuzzy metric space $(X,M,\star)$ as described above.

Proof: As noted above, in the corresponding fuzzy metric space, for all $x,y \in X, t > 0$,

$$M(x,y,t) = \frac{t}{t+d(x,y)} \qquad (2.1)$$

and $\quad a \star b = min\{a,b\}$.

Let $\{x_n\},\{y_n\}$ and $\{z_n\}$ be sequences in (X,d) such that $lim_{n\rightarrow\infty} F(x_n,y_n,z_n) = lim_{n\rightarrow\infty} g(x_n) = x$, $lim_{n\rightarrow\infty} F(y_n,x_n,y_n) = lim_{n\rightarrow\infty} g(y_n) = y$ and $lim_{n\rightarrow\infty} F(z_n,y_n,x_n) = lim\_(n\rightarrow\infty) g(z_n) = z$.

Then the same limit also hold in $(X,M,\star)$, we have

$$lim_{n\rightarrow\infty} d\left(g\big(F(x_n,y_n,z_n)\big), F\big(g(x_n),g(y_n),g(z_n)\big)\right) = 0$$

$$lim_{n\rightarrow\infty} d\left(g\big(F(y_n,x_n,y_n)\big), F\big(g(y_n),g(x_n),g(y_n)\big)\right) = 0$$

and

$$lim_{n\rightarrow\infty} d\left(g\big(F(z_n,y_n,x_n)\big), F\big(g(z_n),g(y_n),g(x_n)\big)\right) = 0.$$

Now from 2.1, we have for all $t > 0$

$$M(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n)), t) = \frac{t}{t + d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n)))}$$

$$M(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n)), t) = \frac{t}{t + d(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n)))}$$

and

$$M(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n)), t) = \frac{t}{t + d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n)))}.$$

Taking $n \to \infty$ on the both sides of the above three equalities, for all $t > 0$, we have

$$lim_{n\to\infty} M(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), t) = 1$$

$$lim_{n\to\infty} M(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n), t) = 1$$

and

$$lim_{n\to\infty} M(gF(z_n, y_n, z_n), F(gz_n, yx_n, gx_n), t) = 1$$

Therefore F and g are compatible in $(X, M, \star)$.

We use continuous Hadzic type t-norm in our theorem.

Definition 18: *A $t - norm$ is said to be Hadzic type $t - norm$ if the family $\{\star^p\}_{p\geq 0}$ of its iterates defined for each $s \in [0,1]$ by*

$$\star^0 (s) = 1,\ \star^{p+1} (s) = \star(\star^p (s))\ for\ all\ p \geq 0,$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0,1)$ such that

$$1 \geq s > \eta(\lambda) \Rightarrow \star^p (s) > 1 - \lambda\ for\ all\ p \geq 0.$$

For an example of a non-trivial Hadzic type $t - norm$, we refer to [11].

The reason why we use continuous Hadzic type $t - norm$ is that with this choice we can ensure the existence of a function used in the statement of our main theorem. Also the proof depends on certain properties if this $t - norm$. These points are elaborated in the following.

We will require the result of the following lemma to establish our main theorem.

The speciality of its proof is that is utilizes equi-continuity of the iterates.

Lemma 19: *Let $(X, M, \star)$ be a fuzzy metric space with a Hadzic type $t - norm$ $\star$ such that $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. If the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X are such that, for all $n \geq 1, t > 0$,*

$$M(x_n, x_{n+1}, t) \star M(y_n, y_{n+1}, t) \star M(z_n, z_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right) \star M\left(y_{n-1}, y_n, \frac{t}{k}\right) \star M\left(z_{n-1}, z_n, \frac{t}{k}\right) \qquad (2.2)$$

where $0 < k < 1$, then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Proof : *By successive applications of (2.2) it follows that for all $t > 0, q \geq 0$ and each $i \geq 1$,*

$$M(x_{q+i}, x_{q+i+1}, t) \star M(y_{q+i}, y_{q+i+1}, t) \star M(z_{q+i}, z_{q+i+1}, t) \geq M\left(x_q, x_{q+1}, \frac{t}{k^i}\right) \star M\left(y_q, y_{q+1}, \frac{t}{k^i}\right) \star M\left(z_q, z_{q+1}, \frac{t}{k^i}\right)$$

(2.3)

Let $\epsilon > 0$ and $0 < \lambda < 1$ be given. Let p be another integer such that $p > q$. Then

$$\epsilon = \epsilon\frac{(1-k)}{(1-k)} > \epsilon(1-k)(1 + k + k^2 + \dots + k^{p-q-1}).$$

Then by Lemma \ref(lem2.6), for all $p > q$, we have

$$M(x_q, x_p, \epsilon) \star M(y_q, y_p, \epsilon) \star M(z_q, z_p, \epsilon) \geq M\left(x_q, x_{q+1}, \epsilon(1-k)\right) \star M\left(y_q, y_{q+1}, \epsilon(1-k)\right) \star M\left(z_q, z_{q+1}, \epsilon(1-k)\right)$$

$$\star\ M\left(x_{q+1}, x_{q+2}, \epsilon k(1-k)\right) \star M\left(y_{q+1}, y_{q+2}, \epsilon k(1-k)\right) \star M\left(z_{q+1}, z_{q+2}, \epsilon k(1-k)\right)$$

$$\star \dots \star M\left(x_{p-1}, x_p, \epsilon k^{(p-q-1)(1-k)}\right) \star M\left(y_{p-1}, y_p, \epsilon k^{p-q-1}(1-k)\right) \star M\left(z_{p-1}, z_p, \epsilon k^{p-q-1}(1-k)\right).$$

Putting $t = (1-k)\epsilon k^i$ in (2.3), we get, for all $q \geq 0, i \geq 1$

$$M(x_{q+i}, x_{q+i+1}, (1-k)\epsilon k^i) \star M(y_{q+i}, y_{q+i+1}, (1-k)\epsilon k^i) \star M(z_{q+i}, z_{q+i+1}, (1-k)\epsilon k^i)$$
$$\geq M(x_q, x_{q+1}, (1-k)\epsilon) \star M(y_q, y_{q+1}, (1-k)\epsilon) \star M(z_q, z_{q+1}, (1-k)\epsilon). \quad (2.4)$$

Then, from (2.4) and the above inequality, for all $p > q$, we have

$$M(x_q, x_p, \epsilon) \star M(y_q, y_p, \epsilon) \star M(z_q, z_p, \epsilon) \geq M\left(x_q, x_{q+1}, \epsilon(1-k)\right) \star M\left(y_q, y_{q+1}, \epsilon(1-k)\right) \star M\left(z_q, z_{q+1}, \epsilon(1-k)\right)$$
$$\star M\left(x_{q+1}, x_{q+2}, \epsilon(1-k)\right) \star M\left(y_{q+1}, y_{q+2}, \epsilon(1-k)\right) \star M\left(z_{q+1}, z_{q+2}, \epsilon(1-k)\right)$$
$$\star \ldots . \star M\left(x_{p-1}, x_p, \epsilon(1-k)\right) \star M\left(y_{p-1}, y_p, \epsilon(1-k)\right) \star M\left(z_{p-1}, z_p, \epsilon(1-k)\right).$$

That is

$$M(x_q, x_p, \epsilon) \star M(y_q, y_p, \epsilon) \star M(z_q, z_p, \epsilon) \geq \star^{p-q} M\left(x_q, x_{q+1}, \epsilon(1-k)\right) \star M\left(y_q, y_{q+1}, \epsilon(1-k)\right) \star M\left(z_q, z_{q+1}, \epsilon(1-k)\right).$$

(2.5)

Since, the $t-norm$ $\star$ is of Hadzic type, the family of iterates $\{\star^p\}$ is equi-continuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\star^{(m-n)}(s) > 1 - \lambda, \quad (2.6)$$

whenever $1 \geq s > \eta(\lambda)$, where $0 < \lambda < 1$, as mentioned above, is given.

Since $M(x_0, x_1, t) \to 1$ as $t \to \infty$ and $0 < k < 1$, there is a positive integer $N(\epsilon,\lambda)$ such that

$$M\left(x_0, x_1, \frac{(1-k)\epsilon}{k^n}\right) \star M\left(y_0, y_1, \frac{(1-k)\epsilon}{k^n}\right) \star M\left(z_0, z_1, \frac{(1-k)\epsilon}{k^n}\right) > \eta(\lambda) \ for all\ n \geq N(\epsilon,\lambda). \quad (2.7)$$

From (2.3) and (2.7) with $q = 0, i = n > N(\epsilon,\lambda)$ and $t = (1-k)\epsilon$, we get

$$M(x_n, x_{n+1}, (1-k)\epsilon) \star M(y_n, y_{n+1}, (1-k)\epsilon) \star M(z_n, z_{n+1}, (1-k)\epsilon) \geq \eta(\lambda).$$

Then, from (2.6), with $s = M(x_n, x_{n+1}, (1-k)\epsilon) \star M(y_n, y_{n+1}, (1-k)\epsilon) \star M(z_n, z_{n+1}, (1-k)\epsilon)$ and $m > n \geq N(\epsilon,\lambda)$, we have

$$\star^{(m-n)}\left(M(x_n, x_{n+1}, (1-k)\epsilon) \star M(y_n, y_{n+1}, (1-k)\epsilon) \star M(z_n, z_{n+1}, (1-k)\epsilon)\right) > 1 - \lambda,$$

Then, by (2.5), for all $m > n \geq N(\epsilon,\lambda)$, we have

$$M(x_n, x_m, (1-k)\epsilon) \star M(y_n, y_m, (1-k)\epsilon) \star M(z_n, z_m, (1-k)\epsilon) > 1 - \lambda.$$

The above inequality implies that

$$M(x_n, x_m, (1-k)\epsilon) > 1 - \lambda,\ M(y_n, y_m, (1-k)\epsilon) > 1-\lambda \text{ and } M(z_n, z_m, (1-k)\epsilon) > 1-\lambda$$

for all $m > n \geq N(\epsilon,\lambda)$.

But $\epsilon > 0$ and $0 < \lambda < 1$ were chosen arbitrarily.

This shows that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

We will require the following lemma to ensure the existence of the function \gamma which we use in our theorem in the next section.

Lemma 20: Let $\star$ be a $t-norm$ such that the function $c(x) = x \star x \star x \in [0,1]$ in right continuous on an interval $[b,1)$ for $b < 1$. Then $\star$ is a $t-norm$ of Hadzic type if and only if there exists a sequence $\{b_n\}_{n \in N}$ from the interval $(0,1)$ of idempotents of $\star$ such that $\lim_{n\to\infty} b_n = 1$.

Main result

Theorem 21: *Let $(X, M, \star)$ be a complete fuzzy metric space with a Hadzic type $t - norm$ such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $\preccurlyeq$ be a partial order defined on X. Let $F: X^3 \to X$ and $g: X \to X$ be two mappings such that F has mixed g –monotone property and satisfies the following conditions;*

i. $F(X^3) \subseteq g(X)$,

ii. *g is continuous and monotonic increasing,*

iii. *(g, F) is a compatible pair,*

and

$$M(F(x,y,z), F(u,v,w), kt) \geq \gamma\big(M(gx, gu, t) \star M(gy, gv, t) \star M(gz, gw, t)\big) \quad (3.1)$$

for all $x, y, z, u, v, w \in X, t > 0$. with $g(x) \preccurlyeq g(u)$, $g(y) \succcurlyeq g(v)$ and $g(z) \preccurlyeq g(w)$, where $0 < k < 1$, $\gamma : [0,1] \to [0,1]$ is continuous function such that $\gamma(a) \star \gamma(a) \star \gamma(a) \geq a$ for each $0 \leq a \leq 1$. Also suppose that X has the following properties;

a. *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \geq 0$,*

b. *if a non-decreasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \geq 0$,*

c. *if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \preccurlyeq z$ for all $n \geq 0$.*

If there are $x_0, y_0, z_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0, z_0)$, $g(y_0) \succcurlyeq F(y_0, x_0, y_0)$ and $g(z_0) \preccurlyeq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$, that is, g and F have a tripled coincidence point in X.

Proof : *Let $x_0, y_0, z_0 \in X$ be three arbitrary points in X. Since $F(X^3) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that $g(x_1) = F(x_0, y_0, z_0)$, $g(y_1) = F(y_0, z_0, x_0)$ and $g(z_1) = F(z_0, y_0, x_0)$ continuing this way we can construct three sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in X such that*

$$g(x_{n+1}) = F(x_n, y_n, z_n),\ g(y_{n+1}) = F(y_n, x_n, y_n,),\ g(z_{n+1}) = F(z_n, y_n, x_n). \quad (3.2)$$

Next, we prove that for all $n \geq 0$,

$$g(x_n) \preccurlyeq g(x_{n+1}) \quad (3.3)$$

$$g(y_n) \succcurlyeq g(y_{n+1}) \quad (3.4)$$

and

$$g(z_n) \preccurlyeq g(z_{n+1}). \quad (3.5)$$

From the condition on x_0, y_0, z_0, we have

$g(x_0) \preccurlyeq F(x_0, y_0, z_0) = g(x_1), g(y_0) \succcurlyeq F(y_0, z_0, x_0) = g(y_1)$ *and* $g(z_0) \preccurlyeq F(z_0, y_0, x_0) = g(z_1)$.

Therefore (3.3), (3.4) and (3.5) hold for $n = 0$.

Let (3.3), (3.4) and (3.5) hold for some n = m. As F has the mixed monotone property and $g(x_m) \preccurlyeq g(x_{m+1})$, $g(y_m) \succcurlyeq g(y_{m+1})$ and $g(z_m) \preccurlyeq g(z_{m+1})$, it follows that

$$g(x_{m+1}) = F(x_m, y_m, z_m) \preccurlyeq F(x_{m+1}, y_m, z_m) \quad (3.6)$$

$$g(y_{m+1}) = F(y_m, x_m, y_m) \succcurlyeq F(y_{m+1}, x_m, y_m) \quad (3.7)$$

and

$$g(z_{m+1}) = F(z_m, y_m, x_m) \preccurlyeq F(z_{m+1}, y_m, x_m) \quad (3.8)$$

Also for the same reason, we have

$$F(x_{m+1}, y_m, z_m) \preccurlyeq F(x_{m+1}, y_{m+1}, z_{m+1}) = g(x_{m+2}) \quad (3.9)$$

$$F(y_{m+1}, x_m, y_m) \succcurlyeq F(y_{m+1}, x_{m+1}, y_{m+1}) = g(y_{m+2}) \quad (3.\text{..}$$

and

$$F(z_{m+1}, y_m, x_m) \preccurlyeq F(z_{m+1}, y_{m+1}, x_{m+1}) = g(z_{m+2})\,. \quad (3.\text{..}$$

Then, from (3.6)—(3.11),

$$g(x_{m+1}) \preccurlyeq g(x_{m+2}),\ \ g(y_{m+1}) \succcurlyeq g(y_{m+2}) \ \ and\ \ g(z_{m+1}) \preccurlyeq g(z_{m+2}). \qquad (3.12)$$

Then, by induction, (3.3) ---(3.5) hold for all $n \geq 0$.

Let for all $t > 0, n \geq 0$,

$$\delta_n(t) = M(g(x_n), g(x_{n+1}), t) \star M(g(y_n), g(y_{n+1}), t) \star M(g(z_n), g(z_{n+1}), t). \qquad (3.$$

Due to (3.1)---(3.5), for all $t > 0, n \geq 1$, we have

$$M(g(x_n), g(x_{n+1}), kt) = M(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n), kt)$$

$$\geq \gamma\big(M(g(x_{n-1}), g(x_n), t) \star M(g(y_{n-1}), g(y_n), t) \star M(g(z_{n-1}), g(z_n), t)\big)$$
$$= \gamma\left(\delta_{n-1}(t)\right). \qquad (3.14)$$

For similar reasons, for all $t > 0, n \geq 1$, we have

$$M(g(y_n), g(y_{n+1}), kt) = M(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n), kt)$$
$$\geq \gamma\big(M(g(y_{n-1}), g(y_n), t) \star M(g(x_{n-1}), g(x_n), t) \star M(g(y_{n-1}), g(y_n), t)\big)$$
$$= \gamma\left(\delta_{n-1}(t)\right). \qquad (3.15)$$

and

$$M(g(z_n), g(z_{n+1}), kt) = M(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n), kt)$$
$$\geq \gamma\big(M(g(z_{n-1}), g(z_n), t) \star M(g(y_{n-1}), g(y_n), t) \star M(g(x_{n-1}), g(x_n), t)\big)$$
$$= \gamma\left(\delta_{n-1}(t)\right). \qquad (3.16)$$

From (3.14)—(3.16), for all $t > 0, n \geq 0$, it follows that

$$M(g(x_n), g(x_{n+1}), kt) \star M(g(y_n), g(y_{n+1}), kt) \star M(g(z_n), g(z_{n+1}), kt) \geq \gamma\left(\delta_{n-1}(t)\right) \star \gamma\left(\delta_{n-1}(t)\right) \star \gamma\left(\delta_{n-1}(t)\right)$$
$$\geq \gamma\left(\delta_{n-1}(t)\right)$$

that is,

$$M(g(x_n), g(x_{n+1}), kt) \star M(g(y_n), g(y_{n+1}), kt) \star M(g(z_n), g(z_{n+1}), kt)$$
$$\geq M\left(g(x_{n-1}), g(x_n), \tfrac{t}{k}\right) \star M\left(g(y_{n-1}), g(y_n), \tfrac{t}{k}\right) \star M\left(g(z_{n-1}), g(z_n), \tfrac{t}{k}\right). \qquad (3.$$

From (3.17), by an application of Lemma 19 we conclude that $\{g(x_n)\}, \{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$lim_{n\to\infty} g(x_n) = x,\ \ lim_{n\to\infty} g(y_n) = y \ \ and\ \ lim_{n\to\infty} g(z_n) = z. \qquad (3.18)$$

Therefore, $lim_{n\to\infty} g(x_{n+1}) = lim_{n\to\infty} F(x_n, y_n, z_n) = x$, $lim_{n\to\infty} g(y_{n+1}) = lim_{n\to\infty} F(y_n, x_n, y_n) = y$ and $lim_{n\to\infty} g(z_{n+1}) = lim_{n\to\infty} F(z_n, y_n, x_n) = z$. Since (g, F) is a compatible pair, using continuity of g and Definition \ref(2.14), we have

$$g(x) = lim_{n\to\infty} g\big(g(x_{n+1})\big) = lim_{n\to\infty} g\big(F(x_n, y_n, z_n)\big) = lim_{n\to\infty} F\big(g(x_n), g(y_n), g(z_n)\big) \qquad (3.19)$$
$$g(y) = lim_{n\to\infty} g\big(g(y_{n+1})\big) = lim_{n\to\infty} g\big(F(y_n, x_n, y_n)\big) = lim_{n\to\infty} F\big(g(y_n), g(x_n), g(y_n)\big) \qquad (3.20)$$

and

$$g(z) = lim_{n\to\infty} g\big(g(z_{n+1})\big) = lim_{n\to\infty} g\big(F(z_n, y_n, x_n)\big) = lim_{n\to\infty} F\big(g(z_n), g(y_n), g(x_n)\big). \qquad (3.21)$$

By (3.5)—(3.8) and (3.18) , we have that $\{g(x_n)\}$ and $\{g(z_n)\}$ are non-decreasing sequences with $g(x_n) \to x$ and $g(y_n) \to y$ respectively also $\{g(z_n)\}$ is non-increasing sequence with $g(y_n) \to y$ as $n \to \infty$. Then, by condition (a), (b) and (c) of the theorems, it follows that, for all $n \geq 0$,

$$g(x_n) \preccurlyeq x,\ g(y_n) \succcurlyeq y \text{ and } g(z_n) \preccurlyeq z. \tag{3.22}$$

Since g is monotone increasing,

$$g(g(x_n)) \preccurlyeq g(x),\ g(g(y_n)) \succcurlyeq g(y) \text{ and } g(g(z_n)) \preccurlyeq g(z). \tag{3.23}$$

Now, for all $t > 0, n \geq 0$, we have

$$M(F(x,y,z), g(x), t) \geq M(F(x,y,z), g(g(x_{n+1}), kt) \star M(g(g(x_{n+1}), g(x), (t-kt)).$$

Taking $n \to \infty$ on the both sides of the above inequality and using Lemma 7, for all $t > 0$,

$$\begin{aligned} M(F(x,y,z), g(x), t) &\geq lim_{n\to\infty}\left[M(F(x,y,z), g(g(x_{n+1})), kt) \star M\left(g(g(x_{n+1})), g(x), (t-kt)\right)\right] \\ &= lim_{n\to\infty}\left[M(F(x,y,z), g(F(x_n,y_n,z_n)), kt) \star M\left(g(g(x_{n+1})), g(x), (t-kt)\right)\right] \\ &= M(F(x,y,z), lim_{n\to\infty} g(F(x_n,y_n,z_n)), kt) \star M\left(lim_{n\to\infty} g(g(x_{n+1})), g(x), (t-kt)\right) \\ &= M\left(F(x,y,z), lim_{n\to\infty} F(g(x_n), g(y_n), g(z_n)), kt\right) \star M(g(x), g(x), (t-kt)) \\ &= M(F(g(x_n), g(y_n), g(z_n)), F(x,y,z), kt) \star 1 \\ &= M(F(g(x_n), g(y_n), g(z_n)), F(x,y,z), kt). \end{aligned}$$

From the above inequality, using (3.1) and (3.23) for all $t > 0$, we have

$$\begin{aligned} M(F(x,y,z), g(x), t) &\geq lim_{n\to\infty}\left[\gamma\left(M(g(x), g(g(x_n)), t) \star M(g(g(y_n)), g(y), t) \star M(g(g(z_n)), g(z), t)\right)\right] \\ &= \gamma\left(M\left(lim_{n\to\infty} g(g(x_n)), g(x), t\right) \star M\left(lim_{n\to\infty} g(g(y_n)), g(y), t\right) \star M\left(lim_{n\to\infty} g(g(z_n)), g(z), t\right)\right) \\ &= \gamma\left(M(g(x), g(x), t) \star M(g(y), g(y), t) \star M(g(x), g(x), t)\right) \\ &= \gamma(1 \star 1 \star 1) \\ &= \gamma(1) \\ &= 1, \end{aligned} \tag{3.24}$$

which implies that $g(x) = F(x,y,z)$.

Similarly, using (3.20) and (3.21), we can prove that $g(y) = F(y,x,y)$ and $g(z) = F(z,y,x)$ respectively.

This completes the proof of the theorem.

Note: *In the above Theorem 21, we defined a new fuzzy tripled contraction with the help of the function γ. the mapping γ in the statement of the Theorem 21 exists for the following reason. Since the $t-norm$ $\star$ is continuous Hadzic type, by Lemma 20, there exists an increasing sequence $\{b_n\}$ of distinct idempotents of $\star$ in $(0,1)$ with $lim_{n\to\infty} b_n = 1$. Then, $\gamma : [0,1] \to [0,1]$ defined as $\gamma(s) = b_{n+1}$ whenever $b_n < s < b_{n+1}$, for all n, $\gamma(s) = 1$, if $s = 1$, is a function having the desired properties of γ in Theorem 21. Thus the statement of the Theorem 21 is meaningful for arbitrary continuous Hadzic type $t-norms$. The proof of the theorem is different from the proof given by Roldan et al [15].*

Corollary 22 : *Let $(X, M, \star)$ be a complete fuzzy metric space with a Hadzic type $t-norm$ such that $M(x,y,t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $\preccurlyeq$ be a partial order defined on X. Let $F: X^3 \to X$ and $g: X \to X$ be two mappings such that F has mixed g –monotone property and satisfies the following conditions;*

i. $F(X^3) \subseteq g(X)$,
ii. g is continuous and monotonic increasing,
iii. (g, F) is a commuting pair,

and

$$M(F(x,y,z), F(u,v,w), kt) \geq \gamma(M(gx, gu, t) \star M(gy, gv, t) \star M(gz, gw, t)) \tag{3.25}$$

for all $x, y, z, u, v, w \in X, t > 0$. with $g(x) \preccurlyeq g(u)$, $g(y) \succcurlyeq g(v)$ and $g(z) \preccurlyeq g(w)$, where $0 < k < 1, \gamma : [0,1] \to [0,1]$ is continuous function such that $\gamma(a) \star \gamma(a) \star \gamma(a) \geq a$ for each $0 \leq a \leq 1$. Also suppose that X has the following properties;

a. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \geq 0$,
b. if a non-decreasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \geq 0$,
c. if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \preccurlyeq z$ for all $n \geq 0$.

If there are $x_0, y_0, z_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0, z_0)$, $g(y_0) \succcurlyeq F(y_0, x_0, y_0)$ and $g(z_0) \preccurlyeq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $g(x) = F(x, y, z), g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$, that is, g and F have a tripled coincidence point in X.

Proof : Since a commuting pair is also a compatible pair, the result of the Corollary 22 follows from Theorem 21.

The following corollary is a fixed point result.

Corollary 23: Let $(X, M, ,\star)$ be a complete fuzzy metric space with a Hadzic type $t-norm$ such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $\preccurlyeq$ be a partial order defined on X. Let $F: X^3 \to X$ be a mapping such that F has mixed monotone property and satisfies the following condition;

$$M(F(x,y,z), F(u,v,w), kt) \geq \gamma\big(M(x,u,t) \star M(y,v,t) \star M(z,w,t)\big) \tag{3.26}$$

for all $x, y, z, u, v, w \in X, t > 0$. with $x \preccurlyeq u$, $y \succcurlyeq v$ and $z \preccurlyeq w$, where $0 < k < 1, \gamma : [0,1] \to [0,1]$ is continuous function such that $\gamma(a) \star \gamma(a) \star \gamma(a) \geq a$ for each $0 \leq a \leq 1$. Also suppose that X has the following properties;

a. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \geq 0$,
b. if a non-decreasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \geq 0$,
c. if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \preccurlyeq z$ for all $n \geq 0$.

If there are $x_0, y_0, z_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0, z_0)$, $y_0 \succcurlyeq F(y_0, x_0, y_0)$ and $z_0 \preccurlyeq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $x = F(x, y, z), y = F(y, x, y)$ and $z = F(z, y, x)$, that is, F has a tripled fixed point in X.

Proof : The proof follows by putting g = I, the identity function, in Theorem 21.

Example 24: Let $X = [0,1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$. Let $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ for all $t > 0, x, y \in X$ and $a \star b = min\{a, b\}$ for all $a, b \in [0,1]$. It is easy to verify that M(x,y,t) is a complete fuzzy metric space. Let $F: X^3 \to X$, $F(x,y,z) = \frac{x^2 - y^2 - z^2}{4} + \frac{3}{4}$ $g: X \to X, g(x) = x^2$. Then $F: (X^3) \subseteq g(X)$ and F satisfies the mixed $g-$monotone property.

Let $\gamma : [0,1] \to [0,1]$ be defined as $\gamma(s) = s$ for each $s \in [0,1]$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences in X such that

$$lim_{n\to\infty} F(x_n, y_n, z_n) = a, \quad lim_{n\to\infty} g(x_n) = a,$$
$$lim_{n\to\infty} F(y_n, x_n, y_n) = b, \quad lim_{n\to\infty} g(y_n) = b,$$
$$lim_{n\to\infty} F(z_n, y_n, x_n) = c, \quad lim_{n\to\infty} g(z_n) = c,$$

Now, for all $n \geq 0$,

$$g(x_n) = x_n^2, \; g(y_n) = y_n^2 \text{ and } g(z_n) = z_n^2.$$

Then necessarily $a = b = c = \frac{3}{4}$.

It then follows from Lemma 7 that, for all $t > 0$,

$$lim_{n\to\infty} M\big(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n)), t\big) = 1,$$
$$lim_{n\to\infty} M\big(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n)), t\big) = 1,$$

and

$$lim_{n\to\infty} M(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n)), t) = 1.$$

Therefore the mappings F and g are compatible in X.

Let $k = \frac{3}{4}$. Then we show that 3.1 is satisfied with $k = \frac{3}{4}$, for all $t > 0$ and $x, y, z, u, v, w \in X$. If 3.1 does not hold, then there exists $t > 0$ such that

$$M(F(x,y,z), F(u,v,w), kt) < min\{M(g(x), g(u), t), M(g(y), g(v), t), M(g(z), g(w), t)\},$$

that is

$$M(F(x,y,z), F(u,v,w), kt) < M(g(x), g(u), t),$$
$$M(F(x,y,z), F(u,v,w), kt) < M(g(y), g(v), t),$$

and

$$M(F(x,y,z), F(u,v,w), kt) < M(g(z), g(w), t),$$

that is,

$$e^{-\frac{\left|\frac{x^2-y^2-z^2}{4} - \frac{u^2-v^2-w^2}{4} - \frac{3}{4}\right|}{kt}} < e^{-\frac{|(x^2-u^2)|}{t}},$$
$$e^{\frac{-\left|\frac{(x^2-y^2-z^2)}{4} - \frac{(u^2-v^2-w^2)}{4} - \frac{3}{4}\right|}{kt}} < e^{\frac{-|(y^2-v^2)|}{t}},$$

and

$$e^{-\frac{\frac{x^2-y^2-z^2}{4} - \frac{(u^2-v^2-w^2)}{4} - \frac{3}{4}}{kt}} < e^{-\frac{|(z^2-w^2)|}{t}},$$

that is,

$$\frac{1}{4k}\,|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,x^2 - u^2\,|,$$
$$\frac{1}{4k}\,|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,y^2 - v^2\,|,$$

and

$$\frac{1}{4k}\,|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,z^2 - w^2\,|.$$

Since we have assumed $k = \frac{3}{4}$, the three inequalities reduce to ,

$$\frac{1}{3}\,|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,x^2 - u^2\,|,$$
$$\frac{1}{3}\,|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,y^2 - v^2\,|,$$

and

$$\frac{1}{3}\,|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,z^2 - w^2\,|.$$

Combining the above three inequalities, we have $|\,(x^2 - y^2 - z^2) - (u^2 - v^2 - w^2)\,| > |\,x^2 - u^2\,| + |\,y^2 - v^2\,| + |\,z^2 - w^2\,|$, which contradiction. Hence (3.1) holds. Thus all the conditions of Theorem 21 satisfied. Then, by an application of Theorem 21, we conclude that g and F have a tripled coincidence point. Here $\left(\sqrt{\frac{3}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{3}{4}}\right)$ is tripled coincidence point of g and F in X.

Remark 25: *In Example 24, the function g and F do not commute. Hence Corollary 22 cannot be applied to this example. This shows that Theorem 21 properly contains its Corollary 22. Theorem 21, and all its corollaries are valid for Hadzic $t - norm$ like minimum $t - norm$ as is used in Example 24. But Theorem 21 cannot be applied to other cases as for example, when $a \star b = ab$ which is not a Hadzic type $t - norm$.*

Application to a new result in metric spaces

In this section we present a tripled coincidence point result in partially ordered metric space. This is obtained by an application of the theorem established in the previous section.

Theorem 26: Let $(X, \preccurlyeq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F: X^3 \to X$ and $g: X \to X$ be two mappings such that F has the mixed g –monotone property and satisfies the following condition;

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w))] \quad (4.1)$$

for all $x, y, z, u, v, w \in X$, with $g(x) \preccurlyeq g(u)$, $g(y) \succcurlyeq g(v)$ and $g(z) \preccurlyeq g(w)$, where $0 < k < 1$. Suppose $F(X^3) \subseteq g(X)$, g is continuous and (g, F) is compatible pair. Also suppose that X has the following properties;

i. *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \geq 0$,*
iv. *if a non-decreasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \geq 0$,*
v. *if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \preccurlyeq z$ for all $n \geq 0$.*

If there are $x_0, y_0, z_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0, z_0)$, $g(y_0) \succcurlyeq F(y_0, x_0, y_0)$ and $g(z_0) \preccurlyeq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$, that is, g and F have a tripled coincidence point in X.

For all $x, y, z \in X$ and $t > 0$, we define

$$M(x, y, t) = \frac{t}{t + d(x, y)} \quad (4.2)$$

and $a \star b = min\{a, b\}$. Then, as noted earlier, $(X, M, \star)$ is a fuzzy metric space.

Further, from the above definition of M, $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y, z \in X$.

Using Lemma \ref(lem2.15), we conclude that (g, F) is a compatible pair in this fuzzy metric space. Next we show that the inequality (4.1) implies (3.1) with $\gamma(s) = s$ where $0 \leq s \leq 1$. If otherwise, from (3.1), for some $t > 0, x, y, z, u, v, w \in X$ we have

$$\frac{t}{t + \frac{1}{k}(d(F(x,y,z), F(u,v,w)))} < \min\left\{\frac{t}{t + d(g(x), g(u))}, \frac{t}{t + d(g(y), g(v))}, \frac{t}{t + d(g(z), g(w))}\right\},$$

that is,

$$t + \frac{1}{k}(d(F(x, y, z), F(u, v, w))) > t + d(g(x), g(u))$$

$$t + \frac{1}{k}(d(F(x, y, z), F(u, v, w))) > t + d(g(y), g(v))$$

and

$$t + \frac{1}{k}(d(F(x, y, z), F(u, v, w))) > t + d(g(z), g(w)).$$

Combining the above three inequalities, we have that

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w))]$$

which is contradiction with 4.1.

The proof is the completed by an application of Theorem 21.

Remark 27: *Theorem 26 is an special case of Berinde and Borcut [14] for $a = b = c = \frac{k}{3}$.*

Conclusion

In this paper we have proved tripled coincidence point results in partially ordered fuzzy metric spaces by assuming an inequality, certain conditions on the t-norm and compatibility condition between the mappings. There is no natural way *generating a partial ordered from a fuzzy metric. Fuzzy metric spaces and partially ordered metric spaces have essential differences. It remain to be seen whether the same result can be obtained under different sets of conditions. Particularly, whether conditions weaker than compatibility can be defined which can replace compatibility in Theorem \ref(thm1) is an interesting open problem.*

References

1. Zadeh LA. Fuzzy sets. Inform Control 1965;8:338-53.
2. A.George, P.Veeramani, On some result in fuzzy metric space,Fuzzy Sets Syst. 64(1994)395-399.
3. I.Kramosil, J.Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11(1975)326-334.
4. Deschrijver G, Kerre EE. On the relationship between some extensions of fuzzy set theory. Fuzzy Sets Syst 133(2003) 227-35.
5. Fang JX. On fixed point theorems in fuzzy metric spaces. Fuzzy Sets Syst 46(1992) 107-13.
6. Gregori V, Sapena A. On fixed-point theorem in fuzzy metric spaces. Fuzzy Sets Syst 125(2002) 245-52.
7. Jungck G. Commuting maps and fixed points. Am Math Mon 83(1976) 261-3.
8. Rodriguez Lopez J, Ramaguera S. The Hausdorff fuzzy metric on compact sets. Fuzzy Sets Syst 147(2004) 273-83.
9. Mihet D. A Banach contraction theorem in fuzzy metric spaces. Fuzzy Sets Syst 144(2004) 431-9.
10. M.Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets Syst .27(1988)385-389.
11. O. Hadzic, E.Pap, Fixed Point Theory in Probabilistic Metric Space, Kluwer Academic Publishers, Dordrecht, 2001.
12. E.P.Klement, R.Mesiar,E.Pap,Triangular Norms, Kluwer Academic Publishers,Dordrecht,2000.
13. J.Rodriguez Lopez, S.Romaguera, The Hausdorff fuzzy metric on compact sets,Fuzzy Sets Syst.147 (2004) 273-283.
14. V. Berinde, M. Borcut, "Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces," Nonlinear Anal. 74(2011)4889-4897.
15. A. Roldan, J. M. Moreno, C. Roldan," Tripled fixed point theorem in fuzzy metric spaces and applications", Fixed point theory and applications, doi:10.1186/1687-1812-2013-29.