Performance Measures for Two Machine Production Line with Finite Intermediate Buffer

S.B. Kulshreshtha[#1], A.K. Singh[*2]

[#1]Associate Professor, [#2]Assistant Professor

Department of Mathematics

RBS Engineering Technical Campus, Agra (India)

Abstract**—** The present paper deals with two machine production lines system with state dependent rate. We have used matrix geometric method to determine the state probabilities. These probabilties are used to find out steady state output and average inventory level of production line, which is separated by a finite intermediate buffer. The processing time, inter failure time and repair times of each machine are exponentially distributed. The machines are subject to the blocking as well as starvation due to variability of processing time. The method proposed can be used to develop a suitable algorithm for implementation purpose.

Keywords**—** Production line, Matrix-Geometric method, Blocking, starvation, Buffer, Inventory level.

I. INTRODUCTION

Queueing networks have been successfully to model real life systems. Network is a set of interconnected nodes. Each node consists of a queue, where customers wait for service, There may be one or more servers. Queueing networks with blocking provides a useful modelling tool for evaluating the performance of various systems such as computers, communication network, production line etc. A production line consists of a set of machine (P_1 to P_n), arranged in series. Suppose in front of each machine P_i there is a storage space ST_i. Work-parts visit each machine before they finally leave the production line.

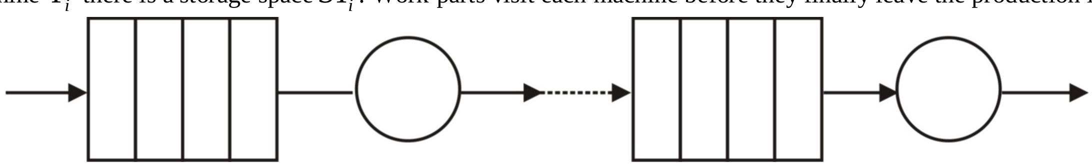

Fig.1

In particular, they move from machine P_i to storage space ST_{i+1} and then to machine P_{i+1}. Each storage space can hold only a finite number of work parts. Machine P_i gets blocked i.e. it cannot process any work-parts in its storage space ST_i, when the down stream storage space ST_{i+1} becomes full. The machine remains blocked until some space becomes available in the down stream storage space. Such a production line is typically modeled by a tandem configuration of finite capcity node.

A production line having many unreliable machines with intermediate buffer capacity was analyzed by numerous queueing theorists. A less general was considered by **Suzuki (1964), Avi-Itzhak and Yadin (1965)** and **Prabhu (1967**) analyzed its transient behaviour. **Neuts (1968)** considered a two node open queuing network model assuming general service times at the first server, exponential service time at the second server and blocking after service. He analyzed the model in terms of semi markov process imbedded at instant of service completion at the first server.

Pinedo and Wolf (1982) considered two nodes in tandem with and without blocking. They compared the expected waiting time when a customers has independent service times at the two servers with the expected waiting time when a customers has equal service time at the two servers. **Tsiotras and Badr (1990)** reported a recursive procedure for obtaining the blocking probability for various special cases. **Shiue and Alitok (1993)** analyzed approximately a two node multi product production/inventory system with setups. Phase type services and phase type set up times were assumed. **Dadune (1997)** presented some results for steady state and sojourn time distribution in open and closed linear network Bernoulli serves with state dependent services and arrival rates. **He and Jewkes (2000)** established performance measures for inventory production system. **Yue et al. (2006)** studied a state dependent $M/E_k/1$ queue with Balking and an unlimited waiting room. **Meerkov et al. (2010)** provided a characterization of transients in two machine geometric production line.

The purpose of the present study is to analyze two-machine production line system with state dependent rate and finite buffer capacity. The rest of the paper is organized as follow. The model description and equation are provided in section 1.2 and 1.3 respectively. The dependence of processing time on state is considered. By using matrix geometric method steady state probabilities are obtained in section 1.4. Some important results such as steady state output and average inventory level of the buffer are presented in section 1.5. To validate the algorithm developed, numerical results are provided in section 1.6. Finally discussion is given in section 3.7.

II. Model Description

We address a production line having two unreliable machines, with intermediate finite buffer capacity K (see fig. 2). The parts of each machine may fail due to common cause failure which can be repaired simultaneously by removing the common fault.

Suppose that when a machine fails, it is repaired immediately. As soon as the machine is repaired, it continues the processing of the same item. The point to be noted is that the two machine production line system can be considered as a continuous time Markov process.

Each machine can live in one of the two states operational and under repair. The system state is represented by (β_1, β_2, x) where x: storage level of buffer.

$\beta_i (i = 1, 2)$ can be defined as given below:

$\beta_i =$ 0, If M_i is to be repaired.

1, If M_i is processing

B, If M_i is in operational state and blocked.

S, If M_i is in operational state and starved.

So that $x = K$, If M_1 is blocked.

And $x = 0$, If M_2 is starved.

For the sake of convenience, the state distribution vector of the system can be obtained as follows:

$$P = (P_S, P_0, P_1, ...P_x ..., P_K, P_B)$$

Where,

$$P_S = [P(0, S, 0), P(1, S, 0)]$$

$$P_x = [P(0, 0, x), P(0,1, x), P(1, 0, x), P(1,1, x)],\ x = 0,1, 2,..., K$$

$$P_B = [P(B, 0, K), P(B,1, K)]$$

Here P_S and P_B represent the vectors corresponding to 'starvation' and 'blocked' state respectively.

To develop the mathematical model let us introduce the following state dependents rates:

$\rho_K(\beta_1, \beta_2, x)$: State dependent processing rate of machine when system state is (β_1, β_2, x)

$\lambda_K(\beta_1, \beta_2, x)$: State dependent failure rate of machine when system state is (β_1, β_2, x)

$\mu_K(\beta_1, \beta_2, x)$: Repair rate of each machine when system state is (β_1, β_2, x)

For various states, let us define characterizing processing time, failure and repair time, respectively as follows:

$$\rho_K = \begin{cases} \rho_{10} & (1,0,x) \to (1,0,x+1), x = 0,1,\dots,K-1 \\ \rho_{11} & (1,1,x) \to (1,0,x+1), x = 0,1,\dots,K-1; (1,S,0) \to (1,1,0) \\ \rho_{20} & (0,1,x) \to (1,1,x-1), x = 1,2,\dots,K \\ \rho_{21} & (1,1,x) \to (1,1,x-1), x = 1,2,\dots,K, (B,1,K) \to (1,1,K) \\ \rho_{10}{}' & (1,0,K) \to (B,0,K) \\ \rho_{11}{}' & (1,1,K) \to (B,1,K) \\ \rho_{20}{}' & (0,1,0) \to (0,S,0) \\ \rho_{21}{}' & (1,1,0) \to (1,S,0) \end{cases}$$

$$\lambda_K = \begin{cases} \lambda_1 & (i,1,x) \to (i,0,x); x = 0,1,2,\dots,K; i = 0,1 \\ \lambda_2 & (B,1,K) \to (B,0,K) \\ \lambda_C & (1,1,x) \to (0,0,x); x = 0,1,\dots,K \\ \lambda_S & (0,S,0) \to (0,0,0) \\ \lambda_B & (B,0,K) \to (0,0,K) \end{cases}$$

$$\mu_K = \begin{cases} \mu_1 & (0,0,x) \to (1,0,x), x = 0,1,\dots,K \\ \mu_1' & (0,1,x) \to (1,1,x), x = 0,1,\dots,K \\ \mu_2 & (0,0,x) \to (0,1,x), x = 0,1,\dots,K \\ \mu_2' & (1,0,x) \to (1,1,x), x = 0,1,\dots,K \\ \mu_S & (0,0,0) \to (0,S,0) \\ \mu_S{}' & (0,S,0) \to (1,0,K) \\ \mu_B & (0,0,K) \to (B,0,K) \\ \mu_B{}' & (B,0,K) \to (B,1,K) \end{cases}$$

III. GOVERNING EQUATIONS

The transient state equation governing the model are as follows:

$$P'(0,S,0) = -(\lambda_S + \mu_S{}')P(0,S,0) + \lambda_S P(0,S,0) + \lambda_1 P(1,S,0) + \mu_S P(0,0,0) + \rho_{20}{}' P(0,1,0) \tag{1}$$

$$P'(1,S,0) = -(\lambda_1 + \rho_{11})P(1,S,0) + \mu_S{}' P(0,S,0) + \rho_{21}{}' P(1,1,0) \tag{2}$$

$$P'(0,0,0) = -(\mu_1 + \mu_2 + \mu_C + \mu_S)P(0,0,0) + \lambda_S P(0,S,0) + \lambda_2 P(0,1,0,) + \lambda_1 P(1,0,0) + \lambda_C P(1,1,0) \tag{3}$$

$$P'(0,1,0) = -(\lambda_2 + \rho_{20}{}' + \mu_1{}')P(0,1,0) + \mu_2 P(0,0,0) + \lambda_1 P(1,1,0,) + \rho_{20} P(0,1,1) \tag{4}$$

$$P'(1,0,0) = -(\lambda_1 + \mu_2{}' + \rho_{10})P(1,0,0) + \mu_1 P(0,0,0) + \lambda_2 P(1,1,0) \quad (5)$$

$$P'(1,1,0) = -(\rho_{21}{}' + \rho_{11} + \lambda_2 + \lambda_1 + \lambda_C)P(1,1,0) + \rho_{11}P(1,S,0) + \mu_C P(0,0,0,) + \mu_1{}' P(0,1,0) + \mu_2{}' P(1,0,0) + \rho_{21}P(1,1,1) \quad (6)$$

$$P'(0,0,x) = -(\mu_1 + \mu_2 + \mu_C)P(0,0,x) + \lambda_2 P(0,1,x) + \lambda_1 P(1,0,x) + \lambda_C P(1,1,x)$$
$$x = 0,1,2,...,K-1 \quad (7)$$

$$P'(0,1,x) = -(\rho_{20} + \lambda_2 + \mu_{1'})P(0,1,x) + \mu_2 P(0,0,x) + \lambda_1 P(1,1,x) + \rho_{20}P(0,1,K)$$
$$x = 0,1,2,...,K-1 \quad (8)$$

$$P'(1,0,x) = -(\rho_{10} + \lambda_1 + \mu_2{}')P(1,0,x) + \rho_{10}P(1,0,x) + \mu_1 P(0,0,x) + \lambda_2 P(1,1,x)$$

$$x = 0,1,2,...,K-1 \quad (9)$$

$$P'(1,1,x) = -(\rho_{21} + \rho_{11} + \lambda_2 + \lambda_1 + \lambda_C)P(1,1,x) + \rho_{11}P(1,0,x) + \mu_C P(0,0,x) + \mu_1{}' P(0,1,x) + \mu_2{}' P(1,0,x) + \rho_{21}P(1,1,K)$$

$$x = 0,1,2,...,K-1 \quad (10)$$

$$P'(0,0,K) = -(\mu_1 + \mu_2 + \mu_C + \mu_B)P(0,0,K) + \lambda_2 P(0,1,K) + \lambda_1 P(1,0,K) + \lambda_C P(1,1,K) + \lambda_B P(B,0,K) \quad (11)$$

$$P'(1,0,K) = -(\rho_{20} + \lambda_2 + \mu_1{}')P(1,0,K) + \mu_2 P(0,0,K) + \lambda_1 P(1,1,K) \quad (12)$$

$$P'(1,0,K) = -(\lambda_1 + \rho_{10}{}' + \mu_2{}')P(1,0,K) + \rho_{10}P(1,0,x) + \mu_1 P(0,0,K) + \lambda_2 P(1,1,K) \quad (13)$$

$$P'(1,1,K) = -(\rho_{21} + \lambda_1 + \lambda_2 + \lambda_C + \rho_{11}{}')P(1,1,K) + \rho_{11}P(1,1,x) + \mu_C P(0,0,K) + \mu_1{}' P(0,1,K) + \mu_2{}' P(1,0,K) + \rho_{21}P(B,1,K) \quad (14)$$

$$P'(B,0,K) = -(\lambda_B + \mu_B{}')P(B,0,K) + \rho_{10}{}' P(1,0,K) + \lambda_2 P(B,1,K) \quad (15)$$

$$P'(B,1,K) = -(\rho_{21} + \lambda_2)P(B,1,K) + \rho_{11}{}' P(1,1,K) + \mu_B{}' P(B,0,K) \quad (16)$$

IV. MATRIX GEOMETRIC PROCEDURE

For steady state, the generator matrix of the continuous time Markov chain can be seen to have partitioned matrix-geometric structure as follows:

$$Q = \begin{bmatrix} S_1 & S_2 & 0 & 0 & . & . & . & . & 0 & 0 \\ Q_S & Q_0 & Q_3 & 0 & . & . & . & . & 0 & 0 \\ 0 & Q_1 & Q_2 & Q_3 & . & . & . & . & 0 & 0 \\ 0 & 0 & Q_1 & Q_2 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & Q_1 & Q_2 & Q_3 & 0 \\ 0 & 0 & . & . & . & . & . & Q_1 & Q_L & Q_B \\ 0 & 0 & . & . & . & . & . & . & B_1 & B_2 \end{bmatrix}.$$

Where

$$S_1 = \begin{bmatrix} -(\lambda_S + \mu_S') & \mu_S' \\ \lambda_1 & -(\lambda_1 + \rho_{11}) \end{bmatrix}$$

$$S_2 = \begin{bmatrix} \lambda_S & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{11} \end{bmatrix}$$

$$Q_S = \begin{bmatrix} \mu_S & 0 \\ \rho_{20}' & 0 \\ 0 & 0 \\ 0 & \rho_{21}' \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{20} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{21} \end{bmatrix}$$

$$Q_B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \rho_{10}' & 0 \\ 0 & \rho_{11}' \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{10} & 0 \\ 0 & 0 & 0 & \rho_{11} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \lambda_B & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{21} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -(\lambda_B + \mu_B') & \mu_B' \\ \lambda_2 & -(\lambda_2 + \rho_{21}) \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} -(\mu_1 + \mu_2 + \mu_C + \mu_S) & \mu_2 & \mu_1 & \mu_C \\ \lambda_2 & -(\lambda_2 + \rho_{20}' + \mu_1') & 0 & \mu_1' \\ \lambda_1 & 0 & -(\lambda_1 + \mu_2' + \rho_{10}) & \mu_2' \\ \lambda_C & \lambda_1 & \lambda_2 & -(\rho_{21} + \lambda_1 + \lambda_2 + \lambda_C + \rho_{11}') \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} -(\mu_1+\mu_2+\mu_C) & \mu_2 & \mu_1 & \mu_C \\ \lambda_2 & -(\lambda_2+\rho_{20}'+\mu_1') & 0 & \mu_1' \\ \lambda_1 & 0 & -(\lambda_1+\mu_2'+\rho_{10}) & \mu_2' \\ \lambda_C & \lambda_1 & \lambda_2 & -(\rho_{21}+\rho_{11}+\lambda_2+\lambda_1+\lambda_C) \end{bmatrix}$$

Suppose P is the steady state probability vector associated with Q such that:

$$P.Q = 0 \tag{17}$$

A normalizing condition can be written as:

$$P.e = 1 \tag{18}$$

Where e is a column vector of appropriate dimension with all elements equal to 1.

Let us denote $P = (P_S, P_0, P_1, ...P_x..., P_K, P_B)$

Where $P_S = [P(0,S,0), P(1,S,0)]$

$P_B = [P(B,0,K), P(B,1,K)]$

$[P_x]_{1\times 4}$, for $K \geq x \geq 0$

Now we consider the existence of the solution:

$$P_x = P_1 R^{x-1}, x \geq 1 \tag{19}$$

$$P_x = P_{x-1} R, x \geq 1$$

Now we have

$$P_S S_1 + P_0 Q_S = 0 \tag{20}$$

$$P_S S_2 + P_0 Q_0 + P_1 Q_1 = 0 \tag{21}$$

$$P_{x-1} Q_3 + P_x Q_2 + P_{x+1} Q_1 = 0 \tag{22}$$

$$P_K Q_B + P_B B_2 = 0 \tag{23}$$

In equation (19), R is the square matrix having order 4 and is the unique minimal non-negative solution to the non-linear matrix equation

$$R^2 Q_1 + R Q_2 + Q_3 = 0 \tag{24}$$

Which is obtained by (19) and (22) by the successive substitution in the following recurrence relation,

$$R(x+1) = -Q_B B_2^{-1} - R^2(x) - B_1 Q_L^{-1} \text{ for } x \geq 0 \tag{25}$$

With $R_0 = 0$

In order to obtain vector $P = (P_S, P_0, P_1, ...P_x..., P_K, P_B)$ we use the equation (20) and (21) which can be written in the matrix from as follows:

$$\begin{bmatrix} P_S & P_0 \end{bmatrix} \begin{bmatrix} S_1 & Q_1 \\ S_2 & Q_0 + RQ_1 \end{bmatrix} \tag{26}$$

Where P_1 is determined by using

$$P_1 = P_0 R$$

Using normalizing condition, we get

$$P_S.e + P_0 \left(\frac{1-R^{K-1}}{1-R} \right).e + M_B(-B_2^{-1}) R^K P_0.e = 1 \tag{27}$$

This gives a unique solution for $P = (P_S, P_0, P_1, ...P_x..., P_K, P_B)$.

V. PERFORMANCE INDICES

The steady state output of the production line can be determined as follows:

$$E = \rho_{20}\left(\sum_{x=1}^{K} P(0,1,x) + P(B,1,K)\right) + \rho_{21}\sum_{x=1}^{K} P(1,1,x) + \rho_{20}{}' P(0,1,0) + \rho_{21}{}' P(1,1,0)$$

The inventory level is obtained as:

$$L = \sum\sum\sum xP(i,j,x) + \sum xP(B,j,K)$$

VI. NUMERICAL RESULTS

The numerical illustration is considered to provide sensitivity analysis. We fix the system parameters as follows:

$\lambda_1 = 0.1, \lambda_2 = 0.2, \mu_1 = 2.0, \mu_2 = 0.2, \rho_{11} = \rho_{10} = 1.0, \rho_2 = \rho_{20} = \rho_{21} = 1.5$

The graphical representation of inventory level (L) Vs buffer size (K) is shown in fig. (3). It is observed that L increases linearly as K increases. In fig. (5), we have plotted L against μ_2 for fix parameter given by. $\lambda_1 = 0.1, \lambda_2 = 0.2, \mu_1 = 1.0, \rho_1 = \rho_{11} = \rho_{10} = 1.0$ We notice that L decreases as μ_2 increases.

Fig. (4) displays that the effect of repair rate (μ_2) on the steady state output (E) for different processing rate (ρ_2). It is noted that E increases by increasing ρ_2 and μ_2 . Fig. (2) and (4) reveals the increasing trend of E with the increase in K and μ_2 .

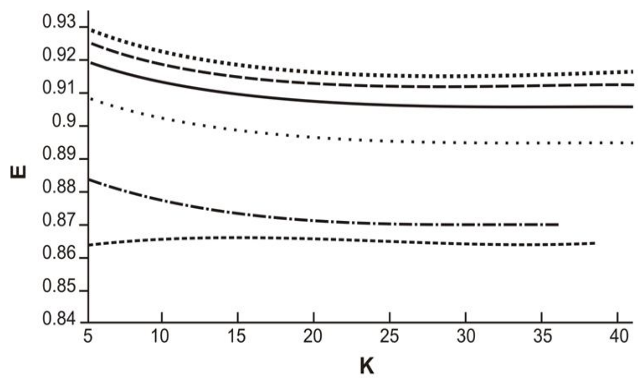

Fig. 2: Steady state Out put (E) and Buffer Size (K)

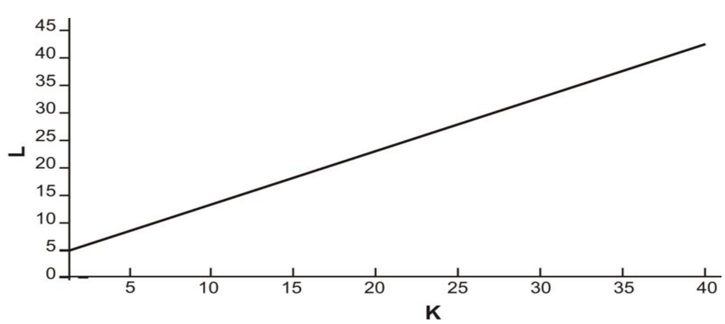

Fig. 3: Inventory Level (L) and Intermediate Buffer Capacity (K)

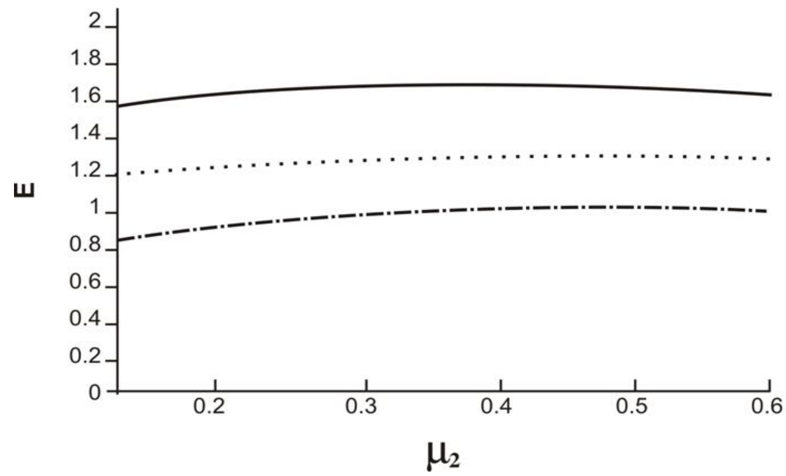

Fig. 4: Steady state Out put (E) and μ_2

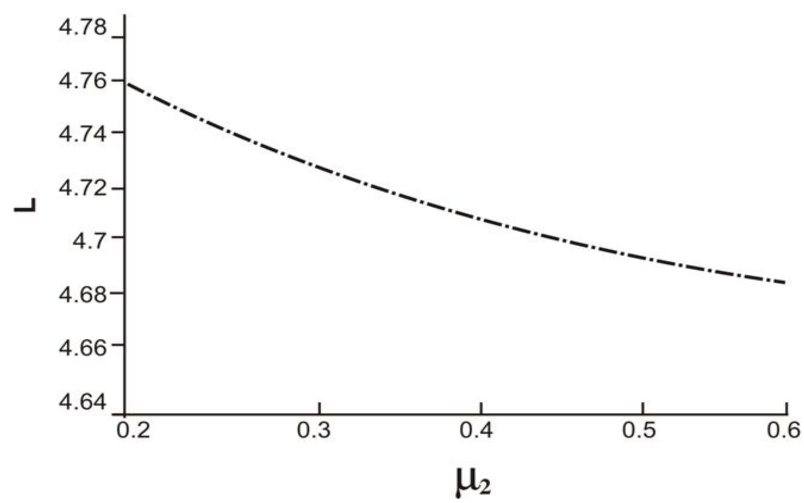

Fig. 5: Inventory Level (L) and μ_2

VII. Conclusion

We have investigated a production line system consisting of two unreliable machines with state dependent rates and finite buffer capacity. We have provided the implicit expressions for the steady state output and average inventory level in terms of steady state probabilities by using matrix-geometric method. Incorporation of state dependent processing rate of both machines, makes our model useful for more general situations in comparison to earlier existing models. The inclusion of common cause failure gives a new dimension to our model and is more realistic in many real time manufacturing systems.

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