Bulk Arrival Two Phase Retrial Queue with Two Types Service and Extended Bernoulli Vacation

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ABSTRACT

This paper is concerned with the analysis of a single server batch arrival retrial queueing system with optional extended server vacation. Server provides two stages of heterogeneous service in succession. Each phase has two types of service and the customer has the option to choice any one of the two types at the time of service. After completion of the second phase service, the server takes Bernoulli vacation. After the vacation completion the server has the option to extend the vacation. The steady state distributions of the server state and the number of customers in the orbit are obtained along with other system characteristics. A general decomposition law for the model is established. Numerical results are calculated.

KEYWORDS : Retrial, Bernoulli vacation, Extended server vacation, Stochastic decomposition

INTRODUCTION

Recently, most of the studies have been devoted to batch arrival two phase retrial queue with vacation because of its interdisciplinary character. Numerous researchers, including Senthil Kumar and Arumuganathan [6,7], Arivudainambi and Godhandaraman[1] and Sumitha et al [9] have analyzed retrial queueing models in which the server provides, two phases of heterogeneous service in succession to each unit. The motivation for two phase of queueing models comes from computer networks and telecommunication systems where messages are processed in two stages by a single server.

In this paper, we consider $M^x/G/1$ retrial queue with two phases and two types of general heterogeneous service and extended Bernoulli vacations. Just before a service starts, a customer has the option to choose one of the two kinds of services at each phase. This type of queueing model may find application in many day to day real life queueing situations encountered at automobile stations, post offices, banks or computer centers. Many authors like Madan[4], Madan et al[5], Baruah [2] and Ebenesar Anna bagyam and Udaya chandrika[3] have studied two types of heterogeneous service.

MODEL DESCRIPTION AND NOTATIONS

In this section, batch arrival two phase retrial queue with two stages of heterogeneous service and extended Bernoulli vacation is described as follows:

The Arrival Process: The primary calls arrive in batches according to a compound Poisson process with rate λ. The number of individuals arriving in a batch is $k \geq 1$ with probability C_k. C(z) be the generating function of $\{ C_k \}$ with factorial moments $\overline{C}_k$.

Retrial Rule: If the server is idle then one of the customers in the batch starts its service time and the rest join the retrial group. Upon arrival, when the server is blocked (busy or on vacation), all the customers enter in to the retrial group. Successive inter retrial times are governed by an arbitrary probability distribution function A(x), density function a(x) and Laplace Stieltjes transform $A^*(s)$.

The Service Process: The server provides two stages of heterogeneous service in succession. There are two options in each stage, 1A and 1B for first stage, 2A and 2B for second stage. The arriving or retrial customer choses type 1A with probability r_1 or 1B with its complementary probability. After completion of the first stage service the customer enters second stage service and chooses 2A with probability r_2 or 2B with probability $(1 - r_2)$. It is assumed that the service times of type A and type B services in both the stages follow general distribution with distribution function $B_{iA}(x)$ and $B_{iB}(x)$, Laplace Stieltjes transform $B^*_{iA}(s)$ and $B^*_{iB}(s)$ and finite moments μ_{iAn} and μ_{iBn}, $n \geq 1$ (i=1,2).

The Vacation Process: As soon as the second stage service of a customer is completed, the server may go for a vacation with probability θ_1 or may continue to staying in the system with complementary probability. The vacation time follows a general distribution with distribution function $V_1(x)$, Laplace Stieltjes transform $V_1^*(s)$ and finite moments v_{1n}, $n \geq 1$. After a vacation period the server may extend the vacation with probability θ_2 or rejoin the system immediately with probability $(1 - \theta_2)$. The extended vacation time follows a general distribution with distribution function $V_2(x)$, Laplace Stieltjes transform $V_2^*(s)$ and finite moments v_{2n}, $n \geq 1$.

The state of the system at time t can be described by the Markov process $\{X(t), t \geq 0\} = \{(J(t), N(t), \xi_i(t), i = 0, 1, 2, 3, 4; t \geq 0\}$, where J(t) denotes the server state 0, 1, 2, 3 or 4 according as the server being free, busy in first stage service , busy in second stage service, on vacation or on extended vacation respectively and N(t) corresponds to the number of customer in the orbit at time t. If J(t) = 0 and N(t) > 0, then $\xi_0(t)$ represents the elapsed retrial time; if J(t) = 1 and N(t) ≥ 0, then $\xi_1(t)$ corresponds to the elapsed time of the customer being served in first stage of service; if J(t) = 2 and N(t) ≥ 0, then $\xi_2(t)$ corresponds to the elapsed time of the customer being served in second stage of service; if J(t) = 3 and N(t) ≥ 0 then $\xi_3(t)$ represents the elapsed vacation time; if J(t) = 4 and N(t) ≥ 0 then $\xi_4(t)$ represents the elapsed extended vacation time at time t.

The hazard rate functions of repeated attempts, first stage service 1A and 1B, second stage service 2A and 2B, server vacation and of extended vacation are respectively.

$$\eta(x) = \frac{a(x)}{1-A(x)};\quad \mu_{iA}(x) = \frac{b_{iA}(x)}{1-B_{iA}(x)};\quad \mu_{iB}(x) = \frac{b_{iB}(x)}{1-B_{iB}(x)} \text{ and } \beta_i(x) = \frac{v_i(x)}{1-V_i(x)},\ i = 1, 2$$

The condition for the system to be stable is $\overline{C}_1[\,1 - A^*(\lambda)\,] + \lambda \overline{C}_1[r_1\,\mu_{1A1} + (1-r_1)\,\mu_{1B1} + r_2\,\mu_{2A1} + (1-r_2)\,\mu_{2B1} + \theta_1\,v_{11} + \theta_1\,\theta_2\,v_{21}] < 1$.

STEADY STATE DISTRIBUTION

Define the probabilities

$I_0(t) = P\{ J(t) = 0, N(t) = 0\}$ and the probability densities

$I_n(t,x)\,dx = P\{ J(t) = 0, N(t) = n, x \leq \xi_0(t) < x + dx, t \geq 0, x \geq 0, n \geq 1\}$

$W_{1j}(t,x)\,dx = P\{ J(t) = 1, N(t) = n, x \leq \xi_1(t) < x + dx, t \geq 0, x \geq 0, n \geq 0, j = A, B\}$

$W_{2j}(t,x)\,dx = P\{ J(t) = 2, N(t) = n, x \leq \xi_2(t) < x + dx, t \geq 0, x \geq 0, n \geq 0, j = A, B\}$

$V_1(t,x)\,dx = P\{ J(t) = 3, N(t) = n, x \leq \xi_3(t) < x + dx, t \geq 0, x \geq 0, n \geq 0\}$

$V_2(t,x)\,dx = P\{ J(t) = 4, N(t) = n, x \leq \xi_4(t) < x + dx, t \geq 0, x \geq 0, n \geq 0\}$

Let $I_0 = \lim_{t\to\infty} I_0(t)$, $I_n(x) = \lim_{t\to\infty} I_n(t,x)$, $W_{ij}(x) = \lim_{t\to\infty} W_{ij}(t,x)$, i=1,2; j =A, B, $V_i(x) = \lim_{t\to\infty} V_i(t,x)$, i = 1, 2

The steady state equations that governs the dynamics of the system behavior are given below

$$\lambda I_0 = (1 - \theta_2)\int_0^\infty V_{1,0}(x)\beta_1(x)\,dx + (1 - \theta_1)\int_0^\infty W_{2A,0}(x)\mu_{2A}(x)\,dx$$

$$+\int_0^\infty V_{2,0}(x)\beta_2(x)\,dx + (1-\theta_1)\int_0^\infty W_{2B,0}(x)\mu_{2B}(x)\,dx \quad (1)$$

$$\frac{d}{dx}I_n(x) = -(\lambda+\eta(x))I_n(x),\ n\geq 1 \quad (2)$$

$$\frac{d}{dx}W_{iA,n}(x) = -(\lambda+\mu_{iA}(x))W_{iA,n}(x)+\lambda(1-\delta_{0n})\sum_{k=1}^{n}c_k W_{iA,n-k}(x),\ i=1,2;\ n\geq 0 \quad (3)$$

$$\frac{d}{dx}W_{iB,n}(x) = -(\lambda+\mu_{iB}(x))W_{iB,n}(x)+\lambda(1-\delta_{0n})\sum_{k=1}^{n}c_k W_{iB,n-k}(x),\ i=1,2;\ n\geq 0 \quad (4)$$

$$\frac{d}{dx}V_{i,n}(x) = -(\lambda+\beta_i(x))V_{i,n}(x)+\lambda(1-\delta_{0n})\sum_{k=1}^{n}c_k V_{i,n-k}(x),\ i=1,2;\ n\geq 0 \quad (5)$$

with boundary conditions

$$I_n(0) = (1-\theta_2)\int_0^\infty V_{1,n}(x)\beta_1(x)\,dx + (1-\theta_1)\int_0^\infty W_{2A,n}(x)\mu_{2A}(x)\,dx$$

$$+\int_0^\infty V_{2,n}(x)\beta_2(x)\,dx + (1-\theta_1)\int_0^\infty W_{2B,n}(x)\mu_{2B}(x)\,dx,\ n\geq 1 \quad (6)$$

$$W_{1A,0}(0) = \lambda r_1 c_1 I_0 + r_1\int_0^\infty I_1(x)\eta(x)\,dx \quad (7)$$

$$W_{1A,n}(0) = \lambda r_1 c_{n+1} I_0 + r_1\int_0^\infty I_{n+1}(x)\eta(x)\,dx + r_1\lambda\int_0^\infty \sum_{k=1}^{n}c_k I_{n-k+1}(x)\,dx,\ n\geq 1 \quad (8)$$

$$W_{1B,0}(0) = \lambda(1-r_1)c_1 I_0 + (1-r_1)\int_0^\infty I_1(x)\eta(x)\,dx \quad (9)$$

$$W_{1B,n}(0) = \lambda(1-r_1)c_{n+1} I_0 + (1-r_1)\int_0^\infty I_{n+1}(x)\eta(x)\,dx$$

$$+(1-r_1)\lambda\int_0^\infty \sum_{k=1}^{n}c_k I_{n-k+1}(x)\,dx,\ n\geq 1 \quad (10)$$

$$W_{2A,n}(0) = r_2\int_0^\infty W_{1A,n}(x)\mu_{1A}(x)\,dx + r_2\int_0^\infty W_{1B,n}(x)\mu_{1B}(x)\,dx,\ n\geq 0 \quad (11)$$

$$W_{2B,n}(0) = (1-r_2)\int_0^\infty W_{1A,n}(x)\mu_{1A}(x)\,dx + (1-r_2)\int_0^\infty W_{1B,n}(x)\mu_{1B}(x)\,dx,\ n\geq 0 \quad (12)$$

$$V_{1,n}(0) = \theta_1\int_0^\infty W_{2A,n}(x)\mu_{2A}(x)\,dx + \theta_1\int_0^\infty W_{2B,n}(x)\mu_{2B}(x)\,dx,\ n\geq 0 \quad (13)$$

$$V_{2,n}(0) = \theta_2\int_0^\infty V_{1,n}(x)\beta_1(x)\,dx,\ n\geq 0 \quad (14)$$

Define the probability generating functions for $|z| < 1$:

$I(z,x) = \sum_{n=1}^{\infty} I_n(x)z^n$, $W_{ij}(z,x) = \sum_{n=0}^{\infty} W_{ij,n}(x)z^n$, i= 1,2; j = A, B and $V_i(z,x) = \sum_{n=0}^{\infty} V_{i,n}(x)z^n$, i= 1,2

Theorem 1

Under the steady state, the joint steady state distributions of $\{(J(t), N(t)), t \geq 0\}$ are obtained as

$$I(z) = I_0 [z - C(z) T(z)] [1 - A^*(\lambda)] / D(z) \quad (15)$$

$$W_{1A}(z) = I_0 \, r_1 A^*(\lambda) [1 - B_{1A}^*(\lambda(1-C(z)))] / D(z) \quad (16)$$

$$W_{1B}(z) = I_0 (1-r_1) A^*(\lambda) [1 - B_{1B}^*(\lambda(1-C(z)))] / D(z) \quad (17)$$

$$W_{2A}(z) = I_0 \, A^*(\lambda) [1 - B_{2A}^*(\lambda(1-C(z)))] \, r_2 [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))] / D(z) \quad (18)$$

$$W_{2B}(z) = I_0 \, A^*(\lambda) [1 - B_{2B}^*(\lambda(1-C(z)))](1-r_2)[r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))] / D(z) \quad (19)$$

$$V_1(z) = I_0 \, A^*(\lambda) [1 - V_1^*(\lambda(1-C(z)))] \, \theta_1 [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]$$
$$[r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))] / D(z) \quad (20)$$

$$V_2(z) = I_0 \, A^*(\lambda) [1 - V_2^*(\lambda(1-C(z)))] \, \theta_1 \theta_2 V_1^*(\lambda(1-C(z))) [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1)$$
$$B_{1B}^*(\lambda(1-C(z)))] [r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))] / D(z) \quad (21)$$

where $I_0 = T_3 / A^*(\lambda)$ (22)

$$D(z) = T(z) [A^*(\lambda) + C(z)(1 - A^*(\lambda))] - z$$

$$T(z) = [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))][r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))]$$
$$[1 - \theta_1 + \theta_1 (1-\theta_2) V_1^*(\lambda(1-C(z))) + \theta_1 \theta_2 V_1^*(\lambda(1-C(z))) \, V_2^*(\lambda(1-C(z)))]$$

$$T_1 = \lambda \overline{C}_1 [r_1 \mu_{1A1} + (1-r_1)\mu_{1B1} + r_2 \mu_{2A1} + (1-r_2)\mu_{2B1} + \theta_1 v_{11} + \theta_1 \theta_2 v_{21}]$$

$$T_2 = \lambda^2 \overline{C}_1^{\,2} \{[r_1 \mu_{1A1} + (1-r_1)\mu_{1B1}][r_2 \mu_{2A1} + (1-r_2)\mu_{2B1} + \theta_1 v_{11} + \theta_1 \theta_2 v_{21}] + [r_1 \mu_{1A2}$$
$$+ (1-r_1)\mu_{1B2} + r_2 \mu_{2A2} + (1-r_2)\mu_{2B2} + \theta_1 v_{12} + 2\theta_1 \theta_2 v_{11} v_{21} + \theta_1 \theta_2 v_{22}]/2$$
$$+ \theta_1 [r_2 \mu_{2A1} + (1-r_2)\mu_{2B1}][v_{11} + \theta_2 v_{21}]\}$$

$$T_3 = 1 - \overline{C}_1 [1 - A^*(\lambda)] - T_1$$

Proof:

Applying usual procedure, equations (2) – (14) yield

$$I(z, x) = I(z, 0) e^{-\lambda x} [1 - A(x)] \quad (23)$$

$$W_{iA}(z, x) = W_{iA}(z, 0) \, e^{-\lambda(1-C(z))x} [1 - B_{iA}(x)], \; i = 1, 2 \quad (24)$$

$$W_{iB}(z, x) = W_{iB}(z, 0) \, e^{-\lambda(1-C(z))x} [1 - B_{iB}(x)], \; i = 1, 2 \quad (25)$$

$$V_i(z, x) = V_i(z, 0) \, e^{-\lambda(1-C(z))x} [1 - V_i(x)], \; i = 1, 2 \quad (26)$$

$$I(z, 0) = \lambda I_0 [z - C(z) T(z)] / D(z) \quad (27)$$

$$W_{1A}(z, 0) = \lambda I_0 \, r_1 A^*(\lambda) [1 - C(z)] / D(z) \quad (28)$$

$$W_{1B}(z, 0) = \lambda I_0 (1-r_1) A^*(\lambda) [1 - C(z)] / D(z) \quad (29)$$

$$W_{2A}(z, 0) = \lambda I_0 \, A^*(\lambda) [1 - C(z)] r_2 [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))] / D(z) \quad (30)$$

$$W_{2B}(z, 0) = \lambda I_0 \, A^*(\lambda) [1 - C(z)](1-r_2)[r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))] / D(z) \quad (31)$$

$$V_1(z, 0) = \lambda I_0 \, A^*(\lambda) [1 - C(z)] \, \theta_1 [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]$$
$$[r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))] / D(z) \quad (32)$$

$$V_2(z,0) = \lambda I_0 A^*(\lambda)[1-C(z)]\ \theta_1\theta_2\ V_1^*(\lambda(1-C(z)))\ [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]\ [r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))] / D(z) \quad (33)$$

Using equations (27) – (33) in the equations (23) – (26), we have the following partial generating functions

$$I(z,x) = \lambda I_0 [z - C(z)\,T(z)]\, e^{-\lambda x}[1 - A(x)] / D(z) \quad (34)$$

$$W_{1A}(z,x) = \lambda I_0\, r_1\, A^*(\lambda)[1-C(z)]\ e^{-\lambda(1-C(z))x}[1 - B_{1A}(x)] / D(z) \quad (35)$$

$$W_{1B}(z,x) = \lambda I_0 (1-r_1) A^*(\lambda)[1-C(z)]\ e^{-\lambda(1-C(z))x}[1 - B_{1B}(x)] / D(z) \quad (36)$$

$$W_{2A}(z,x) = \lambda I_0 A^*(\lambda)[1-C(z)]\ r_2 [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]\ e^{-\lambda(1-C(z))x}[1 - B_{2A}(x)] / D(z) \quad (37)$$

$$W_{2B}(z,x) = \lambda I_0 A^*(\lambda)[1-C(z)]\ (1-r_2) [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]\ e^{-\lambda(1-C(z))x}[1 - B_{2B}(x)] / D(z) \quad (38)$$

$$V_1(z,x) = \lambda I_0 A^*(\lambda)[1-C(z)]\ \theta_1 [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]\ [r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))]\, e^{-\lambda(1-C(z))x}[1 - V_1(x)] / D(z) \quad (39)$$

$$V_2(z,x) = \lambda I_0 A^*(\lambda)[1-C(z)]\ \theta_1\theta_2\ V_1^*(\lambda(1-C(z)))\ [r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]\ [r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))]\, e^{-\lambda(1-C(z))x}[1 - V_2(x)] / D(z) \quad (40)$$

Integrating equations (34) – (40) from0 to ∞ we get the results in the equations (15) – (21). By using the normalizing condition $I_0 + \lim_{z\to 1}[I(z) + \sum_{i=1}^{2}(W_{iA}(z) + W_{iB}(z) + V_i(z))] = 1$ we obtain $I_0 = T_3 / A^*(\lambda)$.

Corollary-1

The mean number of customers in the orbit and system are

$$L_q = T_4 / T_3 \quad (41)$$

$$L_s = L_q + \lambda \overline{C}_1 [r_1 \mu_{1A1} + (1-r_1)\mu_{1B1} + r_2 \mu_{2A1} + (1-r_2)\mu_{2B1}] \quad (42)$$

where $T_4 = T_2 + [T_1 \overline{C}_1 + \overline{C}_2/2](1 - A^*(\lambda))$

Proof:

The probability generating function of the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + \sum_{i=1}^{2}[W_{iA}(z) + W_{iB}(z) + V_i(z)] = I_0 A^*(\lambda)(1-z)/D(z) \quad (43)$$

The probability generating function of the number of customers in the system is

$$P_s(z) = P_s(z) = I_0 + I(z) + \sum_{i=1}^{2}[z W_{iA}(z) + z W_{iB}(z) + V_i(z)]$$

$$= I_0 A^*(\lambda)(1-z)[r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1) B_{1B}^*(\lambda(1-C(z)))]\ [r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2) B_{2B}^*(\lambda(1-C(z)))] / D(z) \quad (44)$$

The results are obtained by differentiating equations (43) and (44) with respect to z and taking $z\to 1$.

The following theorem gives the decomposition of expected number of customer in the system and in the orbit for the model under consideration.

Theorem 2

$L_s = L_\psi + L_\phi$ and $L_q = L_\psi + L_\varphi$, where L_ϕ and L_φ are respectively the expected number of customers in the system and queue for the classical bulk arrival two phase queue with two types of heterogeneous service and extended Bernoulli vacation and L_ψ is the expected number of customers in the orbit during retrial time for the model under study.

Proof:

The probability generating function of the number of customer in the system for the classical bulk arrival two phase queue with two types of heterogeneous service and extended Bernoulli vacation is given by

$$\phi(z) = [1-T_1][1-z][r_1 B_{1A}^*(\lambda(1-C(z))) + (1-r_1)B_{1B}^*(\lambda(1-C(z)))]$$

$$[r_2 B_{2A}^*(\lambda(1-C(z))) + (1-r_2)B_{2B}^*(\lambda(1-C(z)))] / [T(z)-z] \tag{45}$$

$$L_\phi = \lim_{z\to1}\frac{d}{dz}\phi(z) = T_2/[1-T_1] + \lambda\overline{C}_1[r_1\mu_{1A1} + (1-r_1)\mu_{1B1} + r_2\mu_{2A1} + (1-r_2)\mu_{2B1}] \tag{46}$$

The probability generating function of the number of customer in the queue for the classical bulk arrival two phase queue with two types of heterogeneous service and extended Bernoulli vacation is

$$\varphi(z) = [1-T_1][1-z]/[T(z)-z] \tag{47}$$

$$L_\varphi = \lim_{z\to1}\frac{d}{dz}\varphi(z) = T_2/[1-T_1] \tag{48}$$

If the server is idle either due to retrial of customers from the orbit or due to empty system. Then

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)} = I_0 A^*(\lambda)[T(z)-z]/\{D(z)[1-T_1]\} \tag{49}$$

$$L_\psi = \lim_{z\to1}\frac{d}{dz}\psi(z) = T_4/T_3 - T_2/[1-T_1] \tag{50}$$

From equations (41), (42), (46), (48) and (50) it is clear that $L_s = L_\psi + L_\phi$ and $L_q = L_\psi + L_\varphi$

SYSTEM MEASURES

In this section, we provide explicit expressions for the system state probabilities and some important performance measures.

- The probability that the server is idle in the non empty system is

$$I = \lim_{z\to1} I(z) = [1 - A^*(\lambda)][T_1 + \overline{C}_1 - 1] / A^*(\lambda) \tag{51}$$

The mean system size when the server is in idle is

$$L_I = \lim_{z\to1} I'(z) = [1 - A^*(\lambda)]\{[\overline{C}_1 T_1 + \overline{C}_2/2 + T_2] + [T_1 - \overline{C}_1 + 1] T_4/T_3\} / A^*(\lambda) \tag{52}$$

- The probability that the server is busy is

$$W = \sum_{i=1}^{2}\sum_{j=A}^{B} W_{ij}(z) = \lambda\overline{C}_1[r_1\mu_{1A1} + (1-r_1)\mu_{1B1} + r_2\mu_{2A1} + (1-r_2)\mu_{2B1}] \tag{53}$$

The mean system size when the server is busy is

$$L_W = \lambda^2 \overline{C}_1^{\,2} [r_2 \mu_{2A2} + (1-r_2)\mu_{2B2} + r_1 \mu_{1A2} + (1-r_1)\mu_{1B2}] / 2 + \lambda \overline{C}_1 T_4 [r_2 \mu_{2A1} + (1-r_2)\mu_{2B1} + r_1 \mu_{1A1} + (1-r_1)\mu_{1B1}] / T_3 + \lambda^2 \overline{C}_1^{\,2} [r_1 \mu_{1A2} + (1-r_1)\mu_{1B2}][r_2 \mu_{2A1} + (1-r_2)\mu_{2B1}] \quad (54)$$

- The probability that the server is in on vacation is

$$V = \lim_{z\to 1} [V_1(z) + V_2(z)] = \theta_1 [v_{11} + \theta_2 v_{21}] \lambda \overline{C}_1 \quad (55)$$

The mean system size when the server is on vacation is

$$L_v = \lambda^2 \overline{C}_1^{\,2} \theta_1 [(r_1 \mu_{1A1} + (1-r_1)\mu_{1B1} + r_2 \mu_{2A1} + (1-r_2)\mu_{2B1})[v_{11} + \theta_2 v_{21}]] + \theta_1 \lambda \overline{C}_1 T_4 [v_{11} + \theta_2 v_{21}] / T_3 + \lambda^2 \overline{C}_1^{\,2} \theta_1 [v_{12} + v_{22} + 2 v_{11} v_{21}] / 2 \quad (56)$$

- The availability of the server under the steady state condition is given by

$$A = I_0 + \lim_{z\to 1} \{ I(z) + \sum_{i=1}^{2} [W_{iA}(z) + W_{iB}(z)] \}$$

$$= \{ T_3 + [1 - A^*(\lambda)][T_1 + \overline{C}_1 - 1] \} / A^*(\lambda) + [r_1 \mu_{1A1} + (1-r_1)\mu_{1B1} + r_2 \mu_{2A1} + (1-r_2)\mu_{2B1}] \lambda \overline{C}_1 \quad (57)$$

NUMERICAL ILLUSTRATION

Assuming that the retrial time, service time and vacation time are exponentially distributed with parameters η, μ_{ij}, β_i, (i=1,2; j=A, B). Using MATLAB the performance measures are calculated and presented in table by taking the parameter values (θ_1, θ_2, μ_{1A}, μ_{1B}, μ_{2A}, μ_{2B}, r_1, r_2, c_1, c_2) = (0.5,0.5, 15, 14, 12, 10, 0.5, 0.5, 0.5, 0.5)

Table -1 Performance measures A, Lq and Ls

λ	H	β_1	β_2	A	L_q	L_s
0.5	10	10	10	0.9437	0.0796	0.2002
			15	0.9500	0.0757	0.1963
		15	10	0.9562	0.0725	0.1931
			15	0.9625	0.0690	0.1895
	30	10	10	0.9437	0.0450	0.1656
			15	0.9500	0.0420	0.1625
		15	10	0.9562	0.0396	0.1601
			15	0.9625	0.0368	0.1574
	50	10	10	0.9437	0.0383	0.1589
			15	0.9500	0.0355	0.1560
		15	10	0.9563	0.0332	0.1538
			15	0.9625	0.0306	0.1512
1	10	10	10	0.8875	0.3673	0.6083
		10	15	0.9000	0.3399	0.5810
		15	10	0.9125	0.3173	0.5584
		15	15	0.9250	0.2934	0.5345
	30	10	10	0.8875	0.2122	0.4533
		10	15	0.9000	0.1937	0.4348
		15	10	0.9125	0.1790	0.4201
		15	15	0.9250	0.1629	0.4040
	50	10	10	0.8875	0.1845	0.4256
		10	15	0.9000	0.1676	0.4086
		15	10	0.9125	0.1542	0.3952
		15	15	0.9250	0.1394	0.3804
1.5	10	10	10	0.8313	1.3859	1.7475
		10	15	0.8500	1.2230	1.5846

		15	10	0.8688	1.0933	1.4549
		15	15	0.8875	0.9698	1.3314
	30	10	10	0.8312	0.6841	1.0458
		10	15	0.8500	0.6069	0.9685
		15	10	0.8687	0.5460	0.9076
		15	15	0.8875	0.4840	0.8456
	50	10	10	0.8313	0.5833	0.9450
		10	15	0.8500	0.5164	0.8780
		15	10	0.8688	0.4639	0.8255
		15	15	0.8875	0.4097	0.7713
2	10	10	10	0.7750	14.8114	15.2936
		10	15	0.8000	8.7901	9.2722
		15	10	0.8250	6.0918	6.5739
		15	15	0.8500	4.4958	4.9780
	30	10	10	0.7750	2.3716	2.8537
		10	15	0.8000	1.9535	2.4357
		15	10	0.8250	1.6495	2.1317
		15	15	0.8500	1.3827	1.8648
	50	10	10	0.7750	1.8484	2.3306
		10	15	0.8000	1.5421	2.0242
		15	10	0.8250	1.3157	1.7978
		15	15	0.8500	1.1097	1.5919

CONCLUSION

In this paper $M^X/\begin{bmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{2B} \end{bmatrix}/1$ queue with general retrial times, Bernoulli vacation and extended server vacation is analysed. Performance measures are obtained analytically and numerically. Future investigation can be carried out to incorporate a more complex system having non- Markovian input, balking and reneging.

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