Numerical Comparison of multi-step iterative methods for finding roots of nonlinear equations

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Abstract—In this paper, we compare different multi-step Newton like methods for solving nonlinear equations. Results are shown in form of iteration tables. Numerical results show that the Modified Shamanskii Method performs either similarly or better in some cases with respect to some other Newton like multi -step iterative methods.

***Keywords*-Shamanskii Method, Ujević method, Numerical examples, nonlinear equations, Newton's method**

I. INTRODUCTION

One of the oldest numerical computation problems is of finding the roots of the nonlinear equation $f(x)=0$. It has many applications in applied sciences. Several numerical methods have been developed to compute the roots of nonlinear equation $f(x)=0$ including Newton's method. Most of these methods have been developed using Taylor's series expansion (see [1-13]). In this paper we compare the iterative methods Newton Method in Regula Falsi Method [NRF], Regula Falsi Method in Newton Method [RFN], Ujevic Method [NUM], Modified Ujevic Method [MNUM], Shamanskii Method[SM], and Modified Shamanskii Method [MSM] in terms of number of iterations. Numerical results show that the Modified Shamanskii Method [MSM] is very effective with respect to some other Newton like iterative methods for finding roots of nonlinear equations.

II. Different Newton like Iterative Methods

In this section, we present different Newton like multi step iterative methods in algorithmic form.

A. *Newton Method in Regula Falsi Method [NRF] (see [3])*

Step 1: For a given interval [a, b], compute $x_1, x_2, x_3, \ldots$ such that

$$x_{k+1} = \left\{ \frac{bf(a) - af(b)}{f(a) - f(b)} \right\}$$

Step 2: If $|x_{k+1} - x_k| < e$, then stop.

Step 3: If $f(a)f(x_{k+1}) < 0$, then set $b = x_{k+1}$ and

$$a = a - \alpha \frac{f(a)}{f'(a)}$$

else set $a = x_{k+1}$ and $b = b - \alpha \dfrac{f(b)}{f'(b)}$

Step 4: Set $k = k+1$ and go to step 1.

B. Regula Falsi Method in Newton Method [RFN] (see [3])

Step 1: For a given interval [a, b], compute $x_1, x_2, x_3, \ldots$ such that

$$x_k = \left\{ \frac{bf(a) - af(b)}{f(a) - f(b)} \right\}$$

$$x_{k+1} = x_k - \alpha \frac{f(x_k)}{f'(x_k)}$$

Step 2: If $|x_{k+1} - x_k| < e$, then stop

Step 3: If f(a)f(x_{k+1}) < 0 then set b = x_{k+1} else set a x_{k+1}.

Step 4: Set $k = k+1$ and go to step 1.

C. UJEVIĆ METHOD [NUM] (see [6, 3])

The algorithm for this method with weighting factor $\alpha \in (0,1]$ is as follows

For given x_0, compute $x_1, x_2, x_3, \ldots$ such that

$$z_k = x_k - \alpha \left\{ \frac{f(x_k)}{f'(x_k)} \right\}$$

$$x_{k+1} = x_k + 4(z_k - x_k) \left\{ \frac{f(x_k)}{3f(x_k) - 2f(z_k)} \right\}$$

D. MODIFIED UJEVIĆ METHOD [MNUM] (see [5])

In Ujevic method, Newton's method acts as predictor method. For modification of Ujević method, we take third order convergent Halley's method (see [12, 13]) as predictor formula. The algorithm for this method with weighting factor $\alpha \in (0,1]$ is as follows

For given x_0, compute $x_1, x_2, x_3, \ldots$ such that

$$z_k = x_k - \frac{2\alpha f(x_k) f'(x_k)}{2\{f'(x_k)\}^2 - f(x_k) f''(x_k)},$$

$$x_{k+1} = x_k + 4(z_k - x_k) \left\{ \frac{f(x_k)}{3f(x_k) - 2f(z_k)} \right\}$$

E. Shamanskii Method {m=4} (see [1])

The algorithm for this method with weighting factor $\alpha \in (0,1]$ is as follows

For given x_0, compute $x_1, x_2, x_3, \ldots$ such that

$$y_1 = x_k - \frac{\alpha f(x_k)}{f'(x_k)},$$

$$y_{j+1} = y_j - \frac{f(y_j)}{f'(x_k)}, \; for 1 \le j \le m-1,$$

$$x_{k+1} = y_m.$$

F. Modified Shamanskii Method {m=4} (see [2])

For modification of Shamanskii method, we take third order convergent Halley's method (see [13, 14]) as predictor formula. The algorithm for this method with weighting factor $\alpha \in (0,1]$ is as follows

For given x_0, compute $x_1, x_2, x_3, \ldots$ such that

$$y_1 = x_k - \frac{\alpha f(x_k) f'(x_k)}{\{f'(x_k)\}^2 - f(x_k) f''(x_k)},$$

$$y_{j+1} = y_j - \frac{f(y_j)\{f'(x_k)\}^2}{f'(y_j)[\{f'(x_k)\}^2 - f(x_k) f''(x_k)]}, \; for 1 \le j \le m-1,$$

$$x_{k+1} = y_m.$$

III. NUMERICAL EXPERIMENTS

In all of our examples, the maximum number of iteration is $n = 10^3$ and the examples are tested with precision $\varepsilon = 1 \times 10^{-10}$. We have checked the algorithms given above for 10 different values of α including 0.5. The following stopping criteria are used for our computer programs

(i) $|x_{k+1} - x_k| < \varepsilon$

(ii) $|f(x_{k+1})| < \varepsilon$

Example 1. Let $f(x) = x^3 - 3x + 2$ and $x_0 = 0.5$ in [0.5, 1.2].Then number of iterations for the methods NRF, RFN, NUM, MNUM, SM and MSM for different values of α is given in Table 1.

Table 1.

NRF	RFN	NUM	MNUM	SM	MSM	α
177	177	75	60	14	3	0.1
100	100	43	35	14	3	0.2
71	71	32	25	14	3	0.3
55	55	25	20	14	3	0.4
45	45	21	16	14	3	0.5
38	38	18	14	14	3	0.6
32	32	16	12	14	3	0.7
28	27	14	10	14	3	0.8
24	24	12	8	14	3	0.9
11	21	10	6	14	3	1.0

Here we can see that MSM gives better accuracy than NRF, RFN, NUM, MNUM and SM for most of the values of α. Here the exact root of $f(x)=0$ is 1.

Example 2.Let $xe^{x}-1=0$ and $x_0=0.5$ in [0.5, 1.2].Then number of iterations for the methods NRF, RFN, NUM, MNUM, SM and MSM for different values of α is given in Table 2.

Table 2.

NRF	RFN	NUM	MNUM	SM	MSM	α
13	10*	32*	32*	8	5	0.1
11	10*	17	17	8	5	0.2
10	9*	11	10	8	5	0.3
9	9	7	8*	8	5	0.4
8	8	4	4	8	8	0.5
8	8	8*	8*	8	8	0.6
7	7	9	8	8	8	0.7
7	6	11	10	8	8	0.8
6	5	13	12	8	8	0.9
6	3	15	15	8	8	1.0

*The method stuck after these numbers of iterations.

Here we can see that MSM gives better accuracy than NRF, RFN, NUM, MNUM and SM for most of the values of α. Here the root is 0.5671432614, correct to 10 decimal places.

Example 3.Let $x^4+x^2-1=0$ and $x_0=0.5$ in [0.5, 1.2]. Then number of iterations for the methods NRF, RFN, NUM, MNUM, SM and MSM for different values of α is given in Table 3.

Table 3.

NRF	RFN	NUM	MNUM	SM	MSM	α
14	10	32*	32	6	3	0.1
12	10	17*	17	9	3	0.2
10	9	11*	11*	36	3	0.3
9	9	7	8*	11	3	0.4
9	8	4*	4	4	3	0.5
8	8	8	7	4	3	0.6
7	7	10	10	4	3	0.7
7	6	11	11	4	3	0.8
6	3	13	13	4	3	0.9
6	4	15	15	4	3	1.0

* The method stuck after these numbers of iterations.

Here we can see that MSM gives better accuracy than NRF, RFN, NUM, MNUM and SM for most of the values of α. Here the root is 0.7861513495, correct to 10 decimal places.

Example 4. Let $3x-\cos x-1=0$ and $x_0=0.5$ in [0.5, 1.2]. Then number of iterations for the methods NRF, RFN, NUM, MNUM, SM and MSM for different values of α is given in Table 4.

Table 4.

NRF	RFN	NUM	MNUM	SM	MSM	α
5	5	37	37	2	2	0.1
5	5	19	20	2	2	0.2
5	5	13	13	2	2	0.3
5	5	8	9	2	2	0.4
4	5	4	4	2	2	0.5
4	5	8	8	2	2	0.6
4	4	11	11	2	2	0.7
4	4	13	13	2	2	0.8
3	4	14	14	2	2	0.9
3	2	17	17	2	2	1.0

Here we can see that MSM gives better accuracy than NRF, RFN, NUM, MNUM and SM for most of the values of α. Here the root is 0.6071016192, correct to 10 decimal places.

Example 5. Let $x - \cos x = 0$ and $x_0 = 0.5$ in [0.5, 1.2]. Then number of iterations for the methods NRF, RFN, NUM, MNUM, SM and MSM for different values of α is given in Table 5.

Table 5.

NRF	RFN	NUM	MNUM	SM	MSM	α
7	4	31	31	3	3	0.1
7	4	17	17	3	3	0.2
7	4	12	12	3	3	0.3
6	3	8	8	3	3	0.4
6	3	4	3	3	3	0.5
6	3	7	7	3	3	0.6
6	3	9	9	3	3	0.7
5	3	11	11	3	3	0.8
5	3	13	13	3	3	0.9
4	3	14	14	3	3	1.0

Here we can see that MSM gives similar or better accuracy than NRF, RFN, NUM, MNUM and SM for some of the values of α. Here the root is **0.73908507824**, correct to 10 decimal places.

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