A Related Fixed Point Theorem of Integral Type on Two Complete Fuzzy Metric Spaces

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Abstract — **A related fixed point theorem for two pairs of mappings on two complete fuzzy metric spaces satisfying integral type inequalities is obtained. The result extends a result of R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas [7].**

Keywords — **Fuzzy metric space, fixed point, related fixed point, integral type inequality.**

I. Introduction

For the past many years, fixed point theorems have been studied and developed by many mathematicians. The concept of fuzzy set was introduced by L. Zadeh [8] in 1965. Fuzzy metric space was introduced by Kramosil and Michalek [6] in 1975. Then, it was modified by George and Veeramani [4] in 1994. Related fixed points are studied in [1, 2, 3, 5, 7] and many others.

The following was proved in [7].

1.1. Theorem : Let (X, d) and (Y, ρ) be complete metric spaces. Let T be a mapping of X into Y and S be a mapping of Y into X satisfying the inequalities

$$d(Sy, Sy')\, d(STx, STx') \le c \max\{ d(Sy, Sy')\, \rho(Tx, Tx'),\ d(x', Sy)\, \rho(y', Tx)\, d(x, x')\, d(Sy, Sy'),\ d(Sy, STx)\, d(Sy', STx') \}$$

$$\rho(Tx, Tx')\, \rho(TSy, TSy') \le c \max\{ d(Sy, Sy')\, \rho(Tx, Tx'),\ d(x', Sy)\, \rho(y', Tx), \rho(y, y')\, \rho(Tx, Tx'),\ \rho(Tx, TSy)\, \rho(Tx', TSy') \}$$

for all x, x' in X and y, y' in Y, where $0 \le c < 1$. If either S or T is continuous, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Now, theorem 1.1 is extended to two pairs of mappings in integral and fuzzy metric space settings as follows.

II. MAIN RESULT

2.1 Theorem **:** Let (X, μ, t) and (Y, υ, t) be two complete fuzzy metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities

$$\int_1^{k\mu(Sy, Ty', t)\mu(SAx, TBx', t)} \varphi(s)\, ds \ge \int_1^{\min\{ \mu(Sy, Ty', t)\upsilon(Ax, Bx', t),\ \mu(x', Sy, t)\upsilon(y', Ax, t),\ \mu(x, x', t)\mu(Sy, Ty', t),\ \mu(Sy, SAx, t)\mu(Ty', TBx', t) \}} \varphi(s)\, ds \qquad (1)$$

$$\int_1^{k\upsilon(Ax, Bx', t)\upsilon(BSy, ATy', t)} \varphi(s)\, ds \ge \int_1^{\min\{ \mu(Sy, Ty', t)\upsilon(Ax, Bx', t),\ \mu(x', Sy, t)\upsilon(y', Ax, t),\ \upsilon(y, y', t)\upsilon(Ax, Bx', t),\ \upsilon(Ax, BSy, t)\upsilon(Bx', ATy', t) \}} \varphi(s)\, ds \qquad (2)$$

for all x, x' in X and y, y' in Y, where $k \in (0, 1)$. If A and S or B and T are continuous, then SA and TB have a unique common fixed point z in X and BS and AT have unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

***Proof*:** Let x be any arbitrary point in X. We define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows:

$$S y_{2n-1} = x_{2n-1},\ \ Bx_{2n-1} = y_{2n},\ \ Ty_{2n} = x_{2n},\ \ Ax_{2n} = y_{2n-1},\ \text{ for } n = 1, 2, 3, \ldots$$

Applying inequality (1), we get

$$\int_1^{k\mu(Sy_{2n-1}, Ty_{2n}, t)\,\mu(SAx_{2n}, TBx_{2n-1}, t)} \varphi(s)\,ds = \int_1^{k\mu^2(x_{2n-1}, x_{2n}, t)} \varphi(s)\,ds$$

$$\geq \int_1^{\min\{\mu(Sy_{2n-1}, Ty_{2n}, t)\upsilon(Ax_{2n}, Bx_{2n-1}, t),\ \mu(x_{2n-1}, Sy_{2n-1}, t)\upsilon(y_{2n}, Ax_{2n}, t),\ \mu(x_{2n}, x_{2n-1}, t)\,\mu(Sy_{2n-1}, Ty_{2n}, t),\ \mu(Sy_{2n-1}, SAx_{2n}, t)\mu(Ty_{2n}, TBx_{2n-1}, t)} \varphi(s)\,ds$$

$$= \int_1^{\min\{\mu(x_{2n-1}, x_{2n}, t)\,\upsilon(y_{2n-1}, y_{2n}, t),\ \mu(x_{2n-1}, x_{2n-1}, t)\upsilon(y_{2n}, y_{2n-1}, t),\ \mu(x_{2n}, x_{2n-1}, t)\,\mu(x_{2n-1}, x_{2n}, t),\ \mu(x_{2n-1}, x_{2n-1}, t)\,\mu(x_{2n}, x_{2n}\ t)} \varphi(s)\,ds$$

from which it follows that

$$\int_1^{k\mu(x_{2n-1},\ x_{2n}, t)} \varphi(s)\,ds \geq \int_1^{\min\{\upsilon(y_{2n-1}, y_{2n}, t),\ \mu(x_{2n-1}, x_{2n}, t)\}} \varphi(s)\,ds \tag{3}$$

Applying inequality (2), we get

$$\int_1^{k\upsilon(Ax_{2n},\ Bx_{2n-1}, t)\,\upsilon(BSy_{2n-1}, ATy_{2n}, t)} \varphi(s)\,ds = \int_1^{k\upsilon^2(y_{2n-1},\ y_{2n}, t)} \varphi(s)\,ds$$

$$\geq \int_1^{\min\{\mu(Sy_{2n-1}, Ty_{2n}, t)\upsilon(Ax_{2n}, Bx_{2n-1}, t),\ \mu(x_{2n-1}, Sy_{2n-1}, t)\upsilon(y_{2n}, Ax_{2n}, t),\ \upsilon(y_{2n-1}, y_{2n}, t)\upsilon(Ax_{2n}, Bx_{2n-1}, t),\ \upsilon(Ax_{2n}, BSy_{2n-1}, t)\upsilon(Bx_{2n-1}, ATy_{2n}, t)} \varphi(s)\,ds$$

$$= \int_1^{\max\{\ \mu(x_{2n-1}, x_{2n}, t)\,\upsilon(y_{2n-1}, y_{2n}, t),\ \mu(x_{2n-1}, x_{2n-1}, t)\,\upsilon(y_{2n}, y_{2n-1}, t),\ \upsilon(y_{2n-1}, y_{2n}, t)\upsilon(y_{2n-1}, y_{2n}, t),\ \upsilon(y_{2n-1}, y_{2n}, t)\upsilon(y_{2n}, y_{2n-1}, t)} \varphi(s)\,ds$$

from which it follows that

$$\int_1^{k\upsilon(y_{2n-1},\ y_{2n}, t)} \varphi(s)\,ds \geq \int_1^{\min\{\upsilon(y_{2n-1}, y_{2n}, t),\ \mu(x_{2n-1}, x_{2n}, t)\}} \varphi(s)\,ds \tag{4}$$

(3) and (4) can be written as

$$\int_1^{k\mu(x_{n-1},\ x_n, t)} \varphi(s)\,ds \geq \int_1^{\min\{\upsilon(y_{n-1}, y_n, t),\ \mu(x_{n-1}, x_n, t)\}} \varphi(s)\,ds$$

$$\int_1^{k\upsilon(y_{n-1},\ y_n, t)} \varphi(s)\,ds \geq \int_1^{\min\{\upsilon(y_{n-1}, y_n, t),\ \mu(x_{n-1}, x_n, t)\}} \varphi(s)\,ds$$

which can be again written as

$$\int_1^{k\mu(x_{n+1},\ x_n, t)} \varphi(s)\,ds \geq \int_1^{\min\{\upsilon(y_{n+1}, y_n, t),\ \mu(x_{n+1}, x_n, t)\}} \varphi(s)\,ds \tag{5}$$

$$\int_1^{k\upsilon(y_{n+1},\ y_n, t)} \varphi(s)\,ds \geq \int_1^{\min\{\upsilon(y_{n+1}, y_n, t),\ \mu(x_{n+1}, x_n, t)\}} \varphi(s)\,ds \tag{6}$$

From (5) and (6), by induction, we get

$$\int_1^{\mu(x_{n+1},\, x_n,\, t)} \varphi(s)\, ds \geq \int_1^{\frac{1}{k^n}\min\{\upsilon(y_1,\, y_2,\, t),\, \mu(x_1,\, x_2,\, t)\}} \varphi(s)\, ds$$

$$\int_1^{\upsilon(y_{n+1},\, y_n,\, t)} \varphi(s)\, ds \geq \int_1^{\frac{1}{k^n}\min\{\upsilon(y_1,\, y_2,\, t),\, \mu(x_1,\, x_2,\, t)\}} \varphi(s)\, ds$$

Let $t_1 = \frac{t}{p}$. Now,

$$\int_1^{\mu(x_n,\, x_{n+p},\, t)} \varphi(s) ds = \int_1^{\mu(x_n,\, x_{n+p},\, t_1 + t_1 + \ldots\, p \text{ times})} \varphi(s) ds$$

$$\geq \int_1^{\mu(x_n,\, x_{n+1},\, t_1)} \varphi(s) ds * \int_1^{\mu(x_{n+1},\, x_{n+2},\, t_1)} \varphi(s) ds * \ldots * \int_1^{\mu(x_{n+p-1},\, x_{n+p},\, t_1)} \varphi(s) ds$$

$$\geq \int_1^{\frac{1}{k^n}\min\{\upsilon(y_1,\, y_2,\, t),\, \mu(x_1,\, x_2,\, t)\}} \varphi(s)\, ds * \ldots * \int_1^{\frac{1}{k^{n+p-1}}\min\{\upsilon(y_1,\, y_2,\, t),\, \mu(x_1,\, x_2,\, t)\}} \varphi(s)\, ds$$

which implies that

$$\lim \int_1^{\mu(x_n,\, x_{n+p},\, t)} \varphi(s) ds \geq 1$$

$$\Rightarrow \mu(x_n,\, x_{n+p},\, t) \geq 1$$

It follows that $\{x_n\}$ is a Cauchy sequence with a limit z in X.
Similarly, $\{y_n\}$ is a Cauchy sequence with a limit w in Y.

Now, on using the continuity of A and S respectively, we get

$$w = \lim y_{2n-1} = \lim Ax_{2n} = Az \quad \text{and} \quad z = \lim x_{2n} = \lim Sy_{2n} = Sw$$

so that we get

$$Az = w \tag{7}$$

$$Sw = z \tag{8}$$

From (7) and (8), we get

$$SAz = z \tag{9}$$

Again applying inequality (1), we get

$$\int_1^{k\mu(SAx_{2n},\, TBx_{2n-1},\, t)} \varphi(s)\, ds \geq \int_1^{\min\{\upsilon(Ax_{2n},\, Bx_{2n-1},\, t),\ \upsilon(y_{2n},\, Ax_{2n},\, t),\ \mu(x_{2n-1},\, x_{2n},\, t)\}} \varphi(s)\, ds \tag{10}$$

On letting $n \to \infty$, we have

$$\int_1^{k\mu(Sw, TBz, t)} \varphi(s)\, ds \geq \int_1^{\upsilon(Az, w, t)} \varphi(s)\, ds$$

By (7), we have

$$\int_1^{k\mu(Sw, TBz, t)} \varphi(s)\, ds \geq 0$$

$$\Rightarrow k\mu(Sw, TBz, t) \geq 1$$

which implies that

$$Sw = TBz$$

and from (8) , we get

$$z = TBz \tag{11}$$

From (9) and (11), we get

$$SAz = z = TBz \tag{12}$$

Now, (10) gives

$$\int_1^{k\mu(x_{2n-1}, Ty_{2n}, t)} \varphi(s)\, ds \geq \int_1^{\min\{\upsilon(Ax_{2n}, Bx_{2n-1}, t),\ \upsilon(y_{2n}, Ax_{2n}, t),\ \mu(x_{2n-1}, x_{2n}, t)\}} \varphi(s)\, ds$$

On letting $n \to \infty$, we get

$$\int_1^{k\mu(z, Tw, t)} \varphi(t)\, dt \geq 0$$

$$\Rightarrow k\mu(z, Tw, t) \geq 1$$

which implies that

$$z = Tw \tag{13}$$

Again applying inequality (2), we get

$$\int_1^{k\upsilon(BSy_{2n-1}, ATy_{2n}, t)} \varphi(s)\, ds \geq \int_1^{\min\{\mu(Sy_{2n-1}, Ty_{2n}, t),\ \mu(x_{2n-1}, Sy_{2n-1}, t),\ \upsilon(y_{2n-1}, y_{2n}, t),\ \upsilon(Ax_{2n}, Bx_{2n-1})\}} \varphi(s)\, ds \tag{14}$$

On letting $n \to \infty$, we get

$$\int_1^{k\upsilon(BSw, ATw, t)} \varphi(s)\, ds \geq 0$$

$$\Rightarrow k\upsilon(BSw, ATw, t) \geq 1$$

which implies that

$$BSw = ATw \tag{15}$$

Now, (14) gives

$$\int_1^{k\upsilon(y_{2n}, ATy_{2n}, t)} \varphi(s)\, ds \geq \int_1^{\min\{\mu(Sy_{2n-1}, Ty_{2n}, t),\ \mu(x_{2n-1}, Sy_{2n-1}, t), \upsilon(y_{2n-1}, y_{2n}, t),\ \upsilon(Ax_{2n}, Bx_{2n-1}, t)\}} \varphi(s)\, ds$$

On letting $n \to \infty$, we get

$$\int_1^{k\upsilon(w, ATw, t)} \varphi(s)\, ds \geq 0$$
$$\Rightarrow k\upsilon(w, ATw, t) \geq 1$$

which implies that

$$w = ATw \tag{16}$$

From (15) and (16), we get

$$BSw = w = ATw \tag{17}$$

From (8) and (17), we get

$$Bz = w \tag{18}$$

From (7) and (18), we get

$$Az = Bz = w \tag{19}$$

From (8) and (13), we get

$$Sw = Tw = z \tag{20}$$

Similarly, on using the continuity of B and T, the above results hold.

To prove the uniqueness, let SA and TB have a second distinct common fixed point z' in X and BS and AT have a second distinct common fixed point w' in Y.

Applying inequality (1), we have

$$\int_1^{k\mu^2(z, z', t)} \varphi(s)\, ds$$
$$\geq \int_1^{\min\{\mu(z, z', t)\upsilon(Az, Bz', t),\ \mu(z', z', t)\upsilon(Bz', Az, t),\ \mu(z, z', t)\mu(z, z', t), \mu(z', z', t)\mu(z, z, t)\}} \varphi(s)\, ds$$
$$\Rightarrow \int_1^{k\mu(z, z', t)} \varphi(s)\, ds \geq \int_1^{\min\{\upsilon(Az, Bz', t),\ (Bz', Az, t)\}} \varphi(s)\, ds$$

$$\Rightarrow \int_1^{k\mu(z, z', t)} \varphi(s)\, ds \geq \int_1^{\upsilon(Az, Bz', t)} \varphi(s)\, ds \tag{21}$$

Applying inequality (2), we get

$$\int_1^{k\upsilon^2(Az, Bz', t)} \varphi(s)\, ds$$

$$\geq \int_1^{\min\{\mu(z, z', t)\upsilon(Az, Bz', t),\ \mu(z', z', t)\upsilon(Bz', Az, t),\ \upsilon(Az, Bz' t)\upsilon(Az, Bz', t),\ \upsilon(Az, Bz', t)\upsilon(Bz', Az, t)\}} \varphi(s)\,ds$$

$$\Rightarrow \int_1^{k\upsilon(Az, Bz', t)} \varphi(s)\,ds \geq \int_1^{\mu(z, z', t)} \varphi(s)\,ds \tag{22}$$

From (21) and (22), we get

$$\int_1^{k^2\mu(z, z', t)} \varphi(s)\,ds \geq \int_1^{\mu(z, z', t)} \varphi(s)\,ds$$

$$\Rightarrow \int_1^{\mu(z, z', t)} \varphi(s)\,ds \geq \int_1^{\frac{1}{k^2}\mu(z, z', t)} \varphi(s)\,ds$$

$$\Rightarrow \int_1^{\mu(z, z', t)} \varphi(s)\,ds \geq \int_1^{\frac{1}{k^2}\mu(z, z', t)} \varphi(s)\,ds \geq \ldots \geq \int_1^{\frac{1}{k^n}\mu(z, z', t)} \varphi(s)\,ds$$

$$\Rightarrow \int_1^{\mu(z, z', t)} \varphi(s)\,ds \geq \lim \int_1^{\frac{1}{k^n}\mu(z, z', t)} \varphi(s)\,ds \geq 1$$

$$\Rightarrow \mu(z, z', t) \geq 1$$

which implies that

$$z = z'.$$

This proves the uniqueness of z. Similarly, the uniqueness of w can be proved.

The following corollary is a fuzzy metric space version of theorem 1.1 in integral setting.

2.2 Corollary : Let (X, μ, t) and (Y, υ, t) be two complete fuzzy metric spaces. Let S be mappings of X into Y and T be mappings of Y into X satisfying the inequalities

$$\int_1^{k\mu(Ty, Ty', t)\,\mu(TSx, TSx', t)} \varphi(s)\,ds \geq \int_1^{\min\{\mu(Ty, Ty', t)\upsilon(Sx, Sx', t),\ \mu(x', Ty, t)\upsilon(y', Sx, t),\ \mu(x, x', t)\mu(Ty, Ty', t),\ \mu(Ty, TSx, t)\mu(Ty', TSx', t)\}} \varphi(s)\,ds$$

$$\int_1^{k\upsilon(Sx, Sx', t)\upsilon(STy, STy', t)} \varphi(s)\,ds \geq \int_1^{\min\{\mu(Ty, Ty', t)\upsilon(Sx, Sx', t),\ \mu(x', Ty, t)\upsilon(y', Sx, t),\ \upsilon(y, y', t)\upsilon(Sx, Sx', t),\ \upsilon(Sx, STy, t)\upsilon(Sx', STy', t)\}} \varphi(s)\,ds$$

for all x, x' in X and y, y' in Y, where $k \in (0, 1)$. If either S or T is continuous, then TS has a unique fixed point z in X and ST has a unique fixed point w in Y. Further, $Sz = w$ and $Tw = z$.

***Proof* :** The proof easily follows by putting $A = B = S$ and $S = T = T$ in theorem 2.1.

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