Single Term Haar Wavelet Series for Fuzzy Differential Equations

S. SEKAR[#1], S. SENTHILKUMAR[*2]

[#]*Department of Mathematics, Government Arts College (Autonomous), Cherry Road, Salem – 636 007, Tamil Nadu, India.*

**Department of Mathematics, A.V.C. College (Autonomous), Mannampandal, Mayiladuthurai – 609 305, Tamil Nadu, India.*

***Abstract*— In this paper numerical method for solving fuzzy ordinary differential equations using Single Term Haar Wavelet Series (STHWS) method [9-15] is considered. The obtained discrete solutions using STHWS are compared with the exact solutions of the fuzzy differential equations and Runge-Kutta method of order five [7]. Tables and graphs are presented to show the efficiency of this method. This STHW can be easily implemented in a digital computer and the solution can be obtained for any length of time.**

***Keywords*— Runge-Kutta method, Haar wavelets, Single-term Haar wavelet series, Fuzzy differential equations.**

I. INTRODUCTION

Fuzzy differential equations are a natural way to model dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations, which appear in many applications [5]. In the recent years, the topic of FDEs has been investigated extensively [1]. The concept of a fuzzy derivative was first introduced by S. L. Chang and L. A. Zadeh . In this paper, we have introduced and studied a new technique for getting the solution of fuzzy initial value problem. The organized paper is as follows: In the first three sections, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. In sections four, we present STHWS [9 – 15] method and its iterative solution for solving Fuzzy differential equations. The proposed algorithm is illustrated by an example in the last section.

II. PRELIMINARY

A parallelogram fuzzy number u is defined by four real numbers $k < l < m < n$, where the base of the parallelogram is the interval $[k, n]$ and its vertices at $x = l$, $x = m$. Parallelogram fuzzy number will be written as $u = (k , l , m , n)$. The membership function for the parallelogram fuzzy number $u = (k , \ell , m , n)$ is defined as the following :

$$u(x)=\begin{cases}\dfrac{x-k}{l-k}, & k\le x\le l\\ 1, & l\le x\le m \quad \text{u}\\ \dfrac{x-n}{m-n}, & m\le x\le n\end{cases} \tag{1}$$

we will have : *(1) $u > 0$ if $k > 0$; (2) $u > 0$ if > 0; (3) $u > 0$ if $m > 0$ & (4) $u > 0$ if $n > 0$*. Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \to [0, 1]$) satisfying the following properties:

(i) $\forall u \in R_F, u$ is normal, i.e. $\exists x_0 \in R$ with $u(x_0)=1$.

(ii) $\forall u \in R_F, u$ is convex fuzzy set,

i.e. $u(tx+(1-t)y) \ge \min\{u(x,u(y))\}, \forall t \in [0,1], x, y \in R$.

(iii) $\forall u \in R_F, u$ is upper semi continuous on R.

(iv) $\overline{\{x \in R; u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then R_F is called the space of fuzzy numbers .Obviously $R \subset R_F$. Here $R \subset R_F$ is understood as

R = {$\chi_{\{x\}}$; x is usual real number}.
We define the r-level set, $x \in R$;
$[u]r = \{x \setminus u(x) \geq r\}, 0 \leq r \leq 1;$ (2)
Clearly, $[u]_0 = \{ x \setminus u(x) > 0\}$ is compact, which is a closed bounded interval and we denote by $[u]r = [\underline{u}(r), \overline{u}(r)]$. It is clear that the following statements are true.

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over [0,1],
2. $\overline{u}(r)$ is a bounded right continuous non increasing function over [0,1],
3. $\underline{u}(r) \leq \overline{u}(r)$ for all $r \in (0,1]$, for more details see [2],[3].

Let $D: R_F \times R_F \rightarrow R+ \cup\{0\}$,
$D(u, v) = \mathrm{Sup}_{r\in[0,1]} \max \{ | \underline{u}(r) - \underline{v}(r)|, | \overline{u}(r) - \overline{v}(r) | \}$, be Hausdorff distance between fuzzy numbers, where $[u]_r = [\underline{u}(r), \overline{u}(r)], [v]_r = [\underline{v}(r), \overline{v}(r)]$. The following properties are well-known :
$D(u + w, v + w) = D(u, v), \ \forall u,v,w \in R_F$, ,
$D(k.u, k.v) = |k| D(u, v), \ \forall k \in R, u,v \in R_F$,
$D(u + v, w + e) \leq D(u,w) + D(v,e), \ \forall u,v,w,e \in R_F$ and (R_F, D) is a complete metric space.

Lemma 2.1

If the sequence of non-negative numbers $\{W_n\}_{n=0}^N$ satisfy $|W_{n+1}| \leq A|W_n| + B$, $0 \leq n \leq N-1$, for the given positive constants A and B, then $|W_n| \leq A^n |W_0| + B\frac{A^n - 1}{A-1}$, $0 \leq n \leq N$.

Lemma 2.2

If the sequence of numbers $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ satisfy $|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B$, $|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B$, for the given positive constants A and B, then denoting $U_n = |W_n| + |V_n|, \ 0 \leq n \leq N$, we have, $U_n \leq \overline{A^n} U_0 + \overline{B}\frac{\overline{A^n} - 1}{A-1}, \ 0 \leq n \leq N$, where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

Lemma 2.3

Let F(t,u,v) and G(t,u,v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$, $D(y(t_{n+1}), y^{(0)}(t_{n+1})) \leq h^2 L(1+2C)$, where L is a bound of partial derivatives of F and G, and $C = \max\{G[t_N, \underline{y}(t_N; r), \overline{y}(t_{N-1}; r)] \mid r \in [0,1]\} < \infty$.

Theorem 2.4

Let F(t,u,v) and G(t,u,v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$, the numerical solutions of $\underline{y}(t_{n+1}; r)$ and $\overline{y}(t_{n+1}; r)$ converge to the exact solutions $\underline{Y}(t; r)$ and $\overline{Y}(t; r)$ uniformly in t.

Theorem 2.5

Let F(t,u,v) and G(t,u,v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh<1$. Then for arbitrarily fixed $0 \leq r \leq 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n; r)$ and $\overline{y}^{(j)}(t_n; r)$ converge to the numerical solutions $\underline{y}(t_n; r)$ and $\overline{y}(t_n; r)$ in $t_0 \leq t_n \leq t_N$, when $j \rightarrow \infty$.

III. Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$\left.\begin{aligned} & y'(t) = f(t, y(t)),\ t \in [t_0, T], \\ & y(t_0) = y_0, \end{aligned}\right\} \quad (3)$$

where y is a fuzzy function of t, f (t, y) is a fuzzy function of the crisp variable t and the fuzzy variable *y*, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a parallelogram or a parallelogram shaped fuzzy number. We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of y(*t*) for $t \in [t_0, T]$ is

$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)], [y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)], r \in (0,1]$

we write $f(t;y) = [\underline{f}(t;y), \overline{f}(t;y)]$ and $\underline{f}(t;y) = F[t, \underline{y}, \overline{y}], \overline{f}(t;y) = G[t, \underline{y}, \overline{y}]$. Because of $y'(t) = f(t,y)$ we have

$$\underline{f}(t; y(t;r)) = F[t; \underline{y}(t;r), \overline{y}(t;r)] \quad (4)$$

$$\overline{f}(t; y(t;r)) = G[t; \underline{y}(t;r), \overline{y}(t;r)] \quad (5)$$

By using the extension principle, we have the membership function

$$f(t; y(t))(s) = Sup\{y(t)(\tau)\backslash s = f(t, \tau)\},\ s \in R \quad (6)$$

so fuzzy number *f(t; y(t))*. From this it follows that

$$[f(t; y(t))]_r = [\underline{f}(t, y(t;r)), \overline{f}(t, y(t;r))],\ r \in [0;1] \quad (7)$$

where $\underline{f}(t, y(t;r)) = \min\{f(t,u) \backslash u \in [y(t)]_r\}$ (8)

$$\overline{f}(t, y(t;r)) = \max\{f(t,u) \backslash u \in [y(t)]_r\} \quad (9)$$

Definition 3.1

A function f: R → R_F is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\in > 0, \delta > 0$ such that $|t - t_0 < \delta \Rightarrow D[f(t), f(t_0)] < \in$ exists.

Throughout this paper we also consider fuzzy functions which are continuous in metric *D*. Then the continuity of *f(t, y(t); r*) guarantees the existence of the definition of *f(t, y(t); r*) for $t \in [t_0, T]$ and $r \in [0,1]$ [8]. Therefore, the functions *G* and *F* can be definite too.

IV. Single-Term Haar Wavelet Series Method

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1 in some intervals and zeros elsewhere [12]. In general,

$$\left.\begin{aligned} & h_n(t) = h_1(2^j t - k), n = 2^j + k, \\ & j \ge 0, 0 \le k < 2^j, n, j, k \in Z \end{aligned}\right\}$$

$$h_1(t) = \begin{cases} 1, 0 \le t < \frac{1}{2} \\ -1, \frac{1}{2} \le t < 1 \end{cases},$$

Namely, each Haar wavelet contains one and just one square wave, and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems. Any function *y(t),* which is square integrable in the interval [0,1). Can be expanded in a Haar series with an infinite number of terms

$$\left.\begin{array}{l} y(t)=\sum_{i=0}^{\infty} c_i h_i(t), i=2^j+k, \\ j\geq 0, 0\leq k<2^j, n, j, t\in[0,1] \end{array}\right\} \quad (10)$$

where the Haar coefficients

$$c_i = 2^j\int_0^1 y(t)h_i(t)dt$$

are determined such that the following integral square error ε is minimized:

$$\left.\begin{array}{l} \varepsilon=\int_0^1\left[y(t)-\sum_{i=0}^{m-1} c_i h_i(t)\right]^2 dt, \\ m=2^j, j\in\{0\}\cup N \end{array}\right\}$$

usually, the series expansion Equation (10) contains an infinite number of terms for a smooth *y(t)*. If *y(t)* is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (10) will be terminated after *m* terms, that is

$$\begin{array}{l} y(t)\approx\sum_{i=0}^{m-1} c_i h_i(t)=c_{(m)}^T h_{(m)}(t), t\in[0,1] \\ c_{(m)}(t)=[c_0 c_1 \ldots c_{m-1}]^T, \\ h_{(m)}(t)=[h_0(t)h_1(t)\ldots h_{m-1}(t)]^T, \end{array} \quad (11)$$

where "T" indicates transposition, the subscript *m* in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix *P[3]*.

$$\int h_{(m)}(\tau)d\tau\approx P_{(m\times m)}h_{(m)}(t), t\in[0,1]$$

where the m-square matrix *P* is called the operational matrix of integration and single-term $P_{(1\times 1)}=\frac{1}{2}$. Let us define [12]

$$h_{(m)}(t)h_{(m)}^T(t)\approx M_{(m\times m)}(t),$$

and $M_{(1\times 1)}(t)=h_0(t)$. Equation (3) satisfies

$$M_{(m\times m)}(t)c_{(m)}=C_{(m\times m)}h_{(m)}(t),$$

where $c_{(m)}$ is defined in Equation (11) and $C_{(1\times 1)}=c_0$.

V. Numerical Results

In this section, the exact solutions and approximated solutions obtained by STHWS method and Runge-Kutta method of order five (RK-5). To show the efficiency of the STHW, we have considered the following problem taken from [4] and [7], with step size $r=0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, STHW and the RK-5 methods; the absolute errors between them are tabulated and are presented in Table 1 - 3. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of "r"and are presented in Fig. 1 to Fig. 6 for the following problem, using three dimensional effects.

Example 5.1

Consider the initial value problem [4]

$$\left.\begin{array}{l} y'(t) = tf(t),\ t \in [0,1], \\ y(0) = \left(1.01 + 0.1r\sqrt{e}, 1.5 + 0.1r\sqrt{e}\right) \end{array}\right\}$$

The exact solution at $t = 0.1$ is given by $Y(0.1;r) = \left[\left(1.01 + 0.1r\sqrt{e}\right)e^{0.005}, \left(1.5 + 0.1r\sqrt{e}\right)e^{0.005}\right],\ 0 \le r \le 1$

Example 5.2

Consider the fuzzy initial value problem [8]

$$\left.\begin{array}{l} y'(t) = y(t),\ t \in I = [0,1], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), 0 < r \le 1. \end{array}\right\}$$

The exact solution is given by

$Y_1(t;r) = y_1(0;r)e^t$, $Y_2(t;r) = y_2(0;r)e^t$ which at *t = 1*

$Y_1(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e],\ 0 < r \le 1.$

Example 5.3

Consider the fuzzy initial value problem [6]

$$y'(t) = c_1 y^2(t) + c_2,\ y(0) = 0$$

where $c_i > 0$, for $i = 1,2$ are triangular fuzzy numbers.

The exact solution is given by

$Y_1(t;r) = l_1(r)\tan(w_1(r)t),$

$Y_2(t;r) = l_2(r)\tan(w_2(r)t),$

with

$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)},\quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)}$

$w_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)},\quad w_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)}$

where

$[c_1]_r = [c_{1,1}(r),\ c_{1,2}(r)]$ and $[c_2]_r = [c_{2,1}(r),\ c_{2,2}(r)]$

$c_{1,1}(r) = 0.5 + 0.5r,\ c_{1,2}(r) = 1.5 - 0.5r,$

$c_{2,1}(r) = 0.75 + 0.25r,\ c_{2,2}(r) = 1.25 - 0.25r,$

The r-level sets of $y'(t)$ are

$Y_1'(t;r) = c_{2,1}(r)\sec^2(w_1(r)t),$

$Y_2'(t;r) = c_{2,2}(r)\sec^2(w_2(r)t),$

which defines a fuzzy number. We have

$f_1(t,y;r) = \min\{c_1.u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)],\ c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$

$f_2(t,y;r) = \max\{c_1.u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)],\ c_2 \in [c_{2,1}(r), c_{2,2}(r)]\}.$

TABLE I

r	Example 5.1					
	Exact Solutions		RK-5 Error		STHWS Error	
	y_1	y_2	y_1	y_2	y_1	y_2
0	1.015063	1.507519	0.00035	0.00027	1.00E-07	1.00E-07
0.1	1.031633	1.524089	0.00035	0.00048	2.00E-07	2.00E-07
0.2	1.048202	1.540658	0.00036	0.00048	3.00E-07	3.00E-07
0.3	1.064772	1.557228	0.00038	0.00047	4.00E-07	4.00E-07
0.4	1.081342	1.573798	0.00038	0.00046	5.00E-07	5.00E-07
0.5	1.097912	1.590368	0.0004	0.00046	6.00E-07	6.00E-07
0.6	1.114482	1.606938	0.00041	0.00045	7.00E-07	7.00E-07
0.7	1.131052	1.623508	0.00042	0.00045	8.00E-07	8.00E-07
0.8	1.147622	1.640078	0.00043	0.00045	9.00E-07	9.00E-07
0.9	1.164191	1.656647	0.00045	0.000488	1.00E-06	1.00E-06
1.0	1.180761	1.673217	0.000455	0.000497	1.10E-06	1.10E-06

TABLE III

r	Example 5.2					
	Exact Solutions		RK-5 Error		STHWS Error	
	y_1	y_2	y_1	y_2	y_1	y_2
0.01	2.0394	3.0578	0.00611	0.00876	6.00E-07	6.00E-07
0.1	2.1067	3.0241	0.0063	0.00845	7.00E-07	7.00E-07
0.2	2.1746	2.9901	0.00649	0.00856	8.00E-07	8.00E-07
0.3	2.2426	2.9561	0.00669	0.00847	9.00E-07	9.00E-07
0.4	2.3105	2.9222	0.00689	0.00837	1.00E-06	1.00E-06
0.5	2.3785	2.8882	0.00709	0.00826	1.10E-06	1.10E-06
0.6	2.4465	2.8542	0.00728	0.00817	1.20E-06	1.20E-06
0.7	2.5144	2.8202	0.00748	0.00807	1.30E-06	1.30E-06
0.8	2.5824	2.7862	0.00768	0.00797	1.40E-06	1.40E-06
0.9	2.6503	2.7523	0.00788	0.00788	1.50E-06	1.50E-06
1.0	2.7183	2.7183	0.0080727	0.0078047	1.60E-06	1.60E-06

TABLE IIII

r	Example 5.3					
	Exact Solutions		RK-5 Error		STHWS Error	
	y_1	y_2	y_1	y_2	y_1	y_2
0.01	0.8650	4.3914	1.00E-06	1.00E-07	1.00E-07	1.00E-08
0.1	0.9079	3.7886	2.00E-06	2.00E-07	2.00E-07	2.00E-08
0.2	0.9585	3.2851	3.00E-06	3.00E-07	3.00E-07	3.00E-08
0.3	1.0129	2.8994	4.00E-06	4.00E-07	4.00E-07	4.00E-08
0.4	1.0715	2.5918	5.00E-06	5.00E-07	5.00E-07	5.00E-08
0.5	1.1348	2.3419	6.00E-06	6.00E-07	6.00E-07	6.00E-08
0.6	1.2038	2.1330	7.00E-06	7.00E-07	7.00E-07	7.00E-08
0.7	1.2793	1.9568	8.00E-06	8.00E-07	8.00E-07	8.00E-08
0.8	1.3625	1.8051	9.00E-06	9.00E-07	9.00E-07	9.00E-08
0.9	1.4545	1.6732	1.00E-05	9.90E-07	1.00E-06	9.90E-08
1.0	1.5574	1.5574	1.10E-05	1.10E-06	1.10E-06	1.10E-07

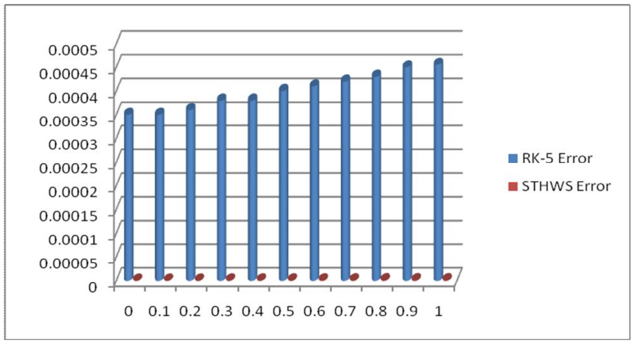

Fig. 1 Error estimation of Example 5.1 at y_1

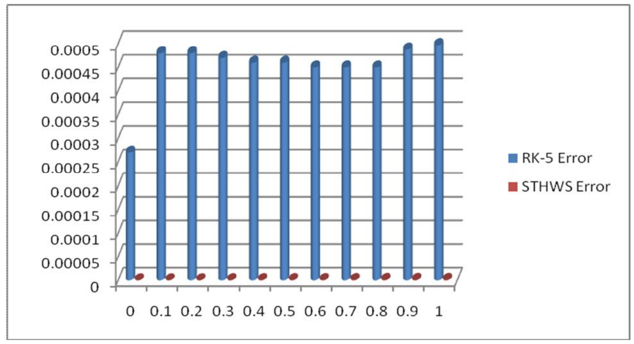

Fig. 2 Error estimation of Example 5.1 at y_2

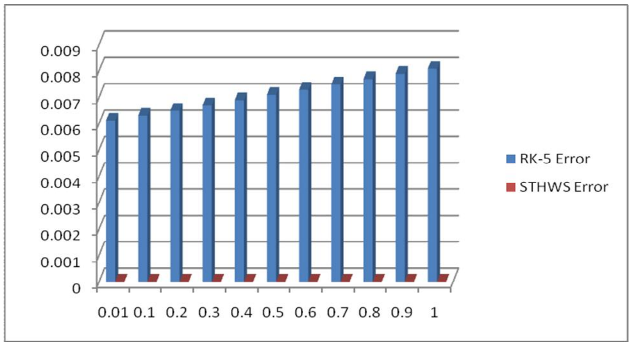

Fig. 3 Error estimation of Example 5.2 at y_1

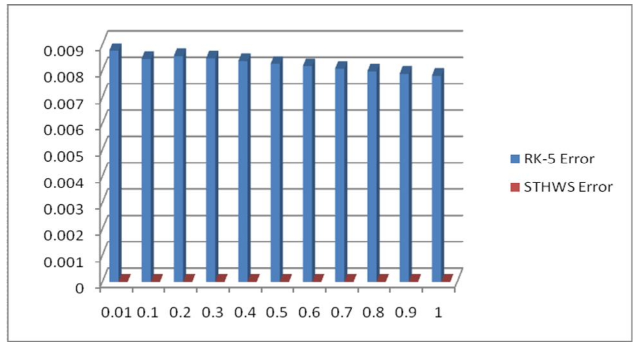

Fig. 4 Error estimation of Example 5.2 at y_2

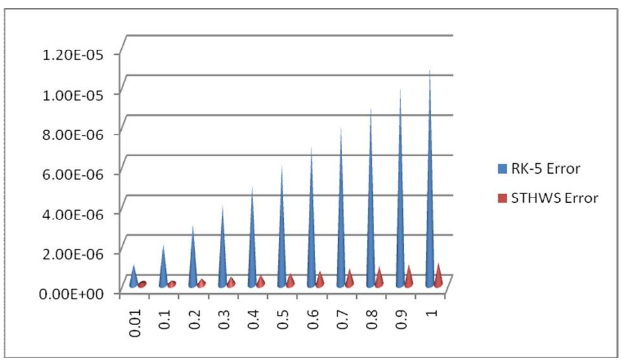

Fig. 5 Error estimation of Example 5.3 at y_1

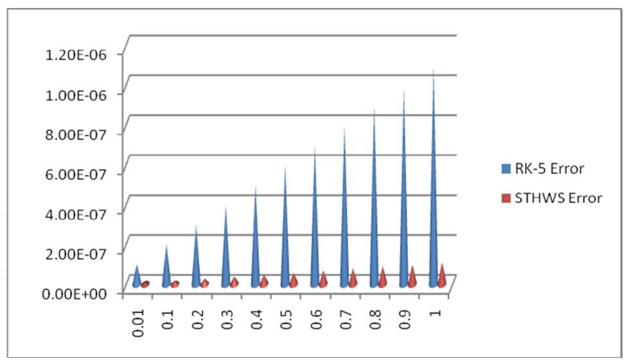

Fig. 6 Error estimation of Example 5.3 at y_2

VI. CONCLUSIONS

A simple and easy method is introduced in this paper to obtain discrete solutions of FDE using STHWS. The efficiency and the accuracy of the STHWS method have been illustrated by suitable examples. The solutions obtained are compared well with the exact solutions and RK-5 method. It has been observed that the solutions by our method show good agreement with the exact solutions. The present method is very convenient as it requires only simple computing systems, less computing time and less memory. The STHWS method is very simple and direct which provides the solutions for any length of time.

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