Inextensible flows of curves in the equiform geometry of the simple isotropic space $\mathscr{I}_3^{(1)}$

Nevin Gürbüz

Eskişehir Osmangazi University, Mathematics and Computer Sciences Department

Abstract--In this paper, evolution equations for inextensible flows of curves in the simple isotropic space are investigated and necessary and sufficient conditions for simple isotropic inextensible curve flow are expressed as a partial differential equation involving the equiform curvature and equiform torsion in the equiform geometry of the simple isotropic space $\mathscr{I}_3^{(1)}$.

Keywords-Inextensible flow, simple isotropic space

I. INTRODUCTION

The flow of curve is called to be simple isotropic inextensible if the arclength is preserved in the equiform geometry of the simple isotropic space $\mathscr{I}_3^{(1)}$. Inextensible curve flows is used computer vision, computer animation and structural mechanic .

Inextensible flow of curves and developable surfaces for plane and space curves is investigated by Kwon and Park [1] . Later author studied inextensible flow of curves and developable surfaces and space curves in Minkowski 3-space [2].

In this work, we derive evolution equations for simple isotropic inextensible flows of curves in the equiform geometry of the simple isotropic space $\mathscr{I}_3^{(1)}$. Also, we give necessary and sufficient conditions for simple isotropic inextensible curve flows in the equiform geometry of the simple isotropic space $\mathscr{I}_3^{(1)}$.

In this section we give fundamental definitions relation simple isotropic space $\mathscr{I}_3^{(1)}$ [3],[4].

The simple isotropic geometry is one of the real Kayley-Klein geometries. The scalar product of two vectors $a=(a_1,a_2,a_3)\,,b=(b_1,b_2,b_3) \in \mathscr{I}_3^{(1)}$ is given by

$$\langle a,b\rangle_{I_3^{(1)}} = \begin{cases} a_1b_1+a_2b_2 & a_i \neq 0, \quad b_i \neq 0, \quad i=1,2 \\ a_3b_3 & a_1=a_2=b_1=b_2=0 \end{cases}$$

The formulas analogous to the Frenet's frame in the equiform geometry of the simple isotropic space are

$$\begin{aligned} \frac{\partial \hat{T}}{\partial \eta} &= \hat{\kappa}\hat{T} + \hat{N} \\ \frac{\partial \hat{N}}{\partial \eta} &= -\hat{T} + \hat{\kappa}\hat{N} + \hat{\tau}\hat{B} \\ \frac{\partial \hat{B}}{\partial \eta} &= \hat{\kappa}\hat{B} \end{aligned} \tag{1.1}$$

where η is an equiform invariant parameter. The equiform curvature and the equiform torsion of a curve is defined by $\hat{\kappa}$ and $\hat{\tau}$ [4].

II. INEXTENSIBLE FLOWS OF CURVES IN THE SIMPLE ISOTROPIC SPACE $\mathcal{I}_3^{(1)}$

Let $H:[0,l]\times[0,t)\rightarrow \mathcal{I}_3^{(1)}$ be a family of differentiable curves in the equiform geometry of the simple isotropic space $\mathcal{I}_3^{(1)}$. Any flow of H is given by

$$\frac{\partial H}{\partial t} = p\hat{T} + q\hat{N} + r\hat{B}.$$

The curve speed $v = \left| \left\langle \frac{\partial H}{\partial \xi}, \frac{\partial H}{\partial \xi} \right\rangle^{1/2} \right|$ and $\frac{\partial}{\partial \eta} = \frac{1}{v}\frac{\partial}{\partial \xi}$.

2.1 Definition . A curve evolution $H(\xi,t)$ in $\mathcal{I}_3^{(1)}$ and its flow is called to be simple isotropic inextensible if

$$\frac{\partial}{\partial t}\sqrt{\left\langle \frac{\partial H}{\partial \xi}, \frac{\partial H}{\partial \xi} \right\rangle} = 0.$$

2.1 Lemma .

$$\frac{\partial v}{\partial t} = \frac{\partial p}{\partial \xi} + vp\hat{\kappa} \tag{2.1}$$

Proof .

$$v^2 = \left\langle \frac{\partial H}{\partial \xi}, \frac{\partial H}{\partial \xi} \right\rangle_{I_3^{(1)}} \tag{2.2}$$

From (2.2), we have

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left\langle\frac{\partial H}{\partial \xi},\frac{\partial H}{\partial \xi}\right\rangle_{I_3^{(1)}}=2\left\langle\frac{\partial H}{\partial \xi},\frac{\partial}{\partial \xi}(p\hat{T}+q\hat{N}+r\hat{B}\right\rangle_{I_3^{(1)}}$$

$$=2\left\langle v\hat{T},(\frac{\partial p}{\partial \xi}+vp\hat{\kappa}-qv)\hat{T}+(pv+\frac{\partial q}{\partial \xi}+qv\hat{\kappa})\hat{N}+(vq\hat{\tau}+\frac{\partial r}{\partial \xi}+rv\hat{\kappa})\hat{B}\right\rangle_{I_3^{(1)}}$$

$$=\quad 2v(\frac{\partial p}{\partial \xi}+vp\hat{\kappa}-qv) \qquad (2.3)$$

From (2.3), we obtain

$$\frac{\partial v}{\partial t}=\frac{\partial p}{\partial \xi}+vp\hat{\kappa}-qv\,.$$

2.1 Theorem . (Necessary and sufficient conditions for simple isotropic inextensible flow of curves.)

Let $\frac{\partial H}{\partial t}=p\hat{T}+q\hat{N}+r\hat{B}$ be a differentiable flow in the equiform geometry of simple isotropic space. The curve flow is simple isotropic inextensible if and only if

$$\frac{\partial p}{\partial \eta}=-p\hat{\kappa}+q \qquad (2.4)$$

Proof.

$$\frac{\partial s(\xi,t)}{\partial t}=\int_0^\xi\frac{\partial v}{\partial t}d\xi=\int_0^\xi(\frac{\partial p}{\partial \xi}+vp\hat{\kappa}-qv)d\xi=0\,.$$

Thus, we obtain (2.4).

2.1 Corallary . An simple isotropic curve flow is independent of r binormal component in $I_3^{(1)}$**.**

3.1 Lemma.

$$\frac{\partial \hat{T}}{\partial t}=q\hat{T}+(\frac{\partial q}{\partial \eta}+q\hat{\kappa}+p)\hat{N}+(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa})\hat{B}$$

$$\frac{\partial \hat{N}}{\partial t}=-(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa})\hat{T}+\Theta\hat{B}$$

$$\frac{\partial \hat{B}}{\partial t}=-(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa})\hat{T}-\Theta\hat{N}$$

Proof.

$$\frac{\partial \hat{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial H}{\partial \eta}=\frac{\partial}{\partial \eta}\frac{\partial H}{\partial t}=\frac{\partial}{\partial \eta}(p\hat{T}+q\hat{N}+r\hat{B})$$

$$=(\frac{\partial p}{\partial \eta}+p\hat{\kappa})\hat{T}+(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa})\hat{N}+(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa})\hat{B}$$

From (2.1), we obtain

$$\frac{\partial \hat{T}}{\partial t}=q\hat{T}+(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa})\hat{N}+(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa})\hat{B} \qquad (2.5)$$

Also, we have

$$\frac{\partial}{\partial \eta}\left\langle \hat{T},\hat{N}\right\rangle_{I_3^{(1)}}=0 \Rightarrow p+\frac{\partial q}{\partial \eta}+q\hat{\kappa}+\left\langle \hat{T},\frac{\partial \hat{N}}{\partial t}\right\rangle=0 \qquad (2.6)$$

$$\frac{\partial}{\partial \eta}\left\langle \hat{T},\hat{B}\right\rangle_{I_3^{(1)}}=0 \Rightarrow q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa}+\left\langle \hat{T},\frac{\partial \hat{B}}{\partial t}\right\rangle_{I_3^{(1)}}=0 \qquad (2.7)$$

$$\frac{\partial}{\partial \eta}\left\langle \hat{N},\hat{B}\right\rangle_{I_3^{(1)}}=0 \Rightarrow \left\langle \frac{\partial \hat{N}}{\partial t},\hat{B}\right\rangle_{I_3^{(1)}}+\left\langle \hat{N},\frac{\partial \hat{B}}{\partial t}\right\rangle_{I_3^{(1)}}=\Theta+\left\langle \hat{N},\frac{\partial \hat{B}}{\partial t}\right\rangle_{I_3^{(1)}}=0 \qquad (2.8)$$

From (2.6), (2.7) and (2.8), we find

$$\frac{\partial \hat{N}}{\partial t}=-(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa})\hat{T}+\Theta\hat{B} \qquad (2.9)$$

$$\frac{\partial \hat{B}}{\partial t}=-(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa})\hat{T}-\Theta\hat{N} \qquad (2.10)$$

We derive equations for

2.2 Theorem. (Inextensible evolution equations in equiform geometry of simple isotropic space .)

Let the curve flow $\frac{\partial H}{\partial t}=p\hat{T}+q\hat{N}+r\hat{B}$ be simple isotropic inextensible in the equiform geometry of the simple isotropic space. The following system of partial differential equations are satisfied

$$\frac{\partial \hat{\kappa}}{\partial t}=-(\frac{\partial^2 r}{\partial \eta^2}+3\hat{\tau}\frac{\partial q}{\partial \eta}+\frac{\partial \hat{\kappa}}{\partial \eta}+p\hat{\tau}+q\hat{\tau}\hat{\kappa})\hat{\tau}$$

$$\frac{\partial \hat{\tau}}{\partial t} = \frac{\partial \Theta}{\partial \eta} + (q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa})$$

$$\Theta = \frac{\partial^2 r}{\partial \eta^2} + 3\hat{\tau}\frac{\partial q}{\partial \eta} + \frac{\partial \hat{\kappa}}{\partial \eta} + p\hat{\tau} + q\hat{\tau}\hat{\kappa}$$

***Proof*.**

$$\frac{\partial}{\partial \eta}\frac{\partial \hat{T}}{\partial t} = \frac{\partial}{\partial \eta}[q\hat{T} + (p + \frac{\partial q}{\partial \eta} + q\hat{\kappa})\hat{N} + (q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa})\hat{B}$$

$$= -p\hat{T} + (\frac{\partial}{\partial \eta}(p + \frac{\partial q}{\partial \eta} + q\hat{\kappa}) + q + \hat{\kappa}(\frac{\partial q}{\partial \eta} + p + \hat{\kappa}q))\hat{N} \quad (2.11)$$

$$+ ((\frac{\partial q}{\partial \eta} + p + \hat{\kappa}q)\hat{\tau} + \frac{\partial}{\partial \eta}(q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa}) + \hat{\kappa}(q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa}))\hat{B}$$

$$\frac{\partial}{\partial t}\frac{\partial \hat{T}}{\partial \eta} = \frac{\partial}{\partial t}(\hat{\kappa}\hat{T} + \hat{N})$$

$$=(\frac{\partial \hat{\kappa}}{\partial t} - (p + \frac{\partial q}{\partial \eta}))\hat{T} + (\hat{\kappa}\frac{\partial q}{\partial \eta} + p\hat{\kappa} + q\hat{\kappa}^2)\hat{N} + (\hat{\kappa}q\hat{\tau} + \hat{\kappa}\frac{\partial r}{\partial \eta} + r\hat{\kappa}^2 + \Theta)\hat{B} \quad (2.12)$$

$$\frac{\partial}{\partial t}\frac{\partial \hat{T}}{\partial \eta} = \frac{\partial}{\partial \eta}\frac{\partial \hat{T}}{\partial t} \quad (2.13)$$

From (2.11), (2.12), (2.13), we obtain

$$\frac{\partial}{\partial t}\frac{\partial \hat{T}}{\partial \eta} = (p + \frac{\partial q}{\partial \eta} + q\hat{\kappa})\hat{\tau} + \frac{\partial}{\partial \eta}(q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa}) \quad (2.14)$$

As similar,

$$\frac{\partial}{\partial \eta}\frac{\partial \hat{B}}{\partial t} = (-\frac{\partial}{\partial \eta}(q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa}) - \hat{\kappa}(q\hat{\tau} + \frac{\partial r}{\partial \eta} + r\hat{\kappa}) + \Theta)\hat{T}$$

$$-(q\hat{\tau} + \frac{\partial r}{\partial \eta} + \frac{\partial \Theta}{\partial \eta} + \Theta\hat{\kappa})\hat{N} - \Theta\hat{\tau}\hat{B} \quad (2.15)$$

$$\frac{\partial}{\partial t}\frac{\partial \hat{B}}{\partial \eta}=\frac{\partial}{\partial \eta}(\hat{\kappa}\hat{B})$$
$$=\frac{\partial \hat{\kappa}}{\partial t}\hat{B}-(\hat{\kappa}q\hat{\tau}+\hat{\kappa}\frac{\partial r}{\partial \eta}+r\hat{\kappa}^{2})\hat{T}-\Theta\hat{\kappa}\hat{N} \qquad (2.16)$$

$$\frac{\partial}{\partial t}\frac{\partial \hat{B}}{\partial \eta}=\frac{\partial}{\partial \eta}\frac{\partial \hat{B}}{\partial t} \qquad (2.17)$$

From (2.15), (2.16), (2.17), we obtain

$$\frac{\partial \hat{\kappa}}{\partial t}=-(\frac{\partial^{2} r}{\partial \eta^{2}}+3\hat{\tau}\frac{\partial q}{\partial \eta}+\frac{\partial \hat{\kappa}}{\partial \eta}+p\hat{\tau}+q\hat{\tau}\hat{\kappa})\hat{\tau}$$

$$\frac{\partial}{\partial t}\frac{\partial \hat{N}}{\partial \eta}=\frac{\partial}{\partial \eta}(-\hat{T}+\hat{\kappa}\hat{N}+\hat{\tau}\hat{B})$$
$$=((\frac{\partial \hat{\kappa}}{\partial t}-\Theta\hat{\tau}-(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa}))\hat{N}+((p-\hat{\kappa}(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa})-\hat{\tau}(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa}))\hat{T}$$
$$+((\frac{\partial \hat{\tau}}{\partial t}+\Theta\hat{\kappa}-(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa}))\hat{B} \qquad (2.18)$$

$$\frac{\partial}{\partial \eta}\frac{\partial \hat{N}}{\partial t}=(-\frac{\partial}{\partial \eta}(q\hat{\tau})-\hat{\kappa}q\hat{\tau}-\frac{\partial^{2} r}{\partial \eta^{2}}-\hat{\kappa}\frac{\partial r}{\partial \eta}-\frac{\partial (r\hat{\kappa})}{\partial \eta}-r\hat{\kappa}^{2})\hat{T}-(q\hat{\tau}+\frac{\partial r}{\partial \eta}-\frac{\partial \Theta}{\partial \eta}-\Theta\hat{\kappa}-r\hat{\kappa})\hat{N}-\Theta\hat{\tau}\hat{B}$$
$$=((\frac{\partial \hat{\kappa}}{\partial t}-\Theta\hat{\tau}-(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa}))\hat{N}+((p-\hat{\kappa}(p+\frac{\partial q}{\partial \eta}+q\hat{\kappa})-\hat{\tau}(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa}))\hat{T} \qquad (2.19)$$

$$\frac{\partial}{\partial t}\frac{\partial \hat{N}}{\partial \eta}=\frac{\partial}{\partial \eta}\frac{\partial \hat{N}}{\partial \eta} \qquad (2.20)$$

From (2.18), (2.19), (2.20), we obtain

$$\frac{\partial \hat{\tau}}{\partial t}=\frac{\partial \Theta}{\partial \eta}+(q\hat{\tau}+\frac{\partial r}{\partial \eta}+r\hat{\kappa})$$

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