Eulerian integral associated with product of two multivariable I-functions, a generalized Lauricella function and a class of polynomials

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable I-functions defined by Prasad [1] a generalized Lauricella function and a class of multivariable polynomials with general arguments . Several particular cases are given .

Keywords: Eulerian integral, multivariable I-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, class of polynomials

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable I-functions defined by Prasad [1] and a class of polynomials with general arguments but of greater order. Several particular cases are given.
The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array} \right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \tag{1.3}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$$

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

Condider a second multivariable I-function defined by Panda [1]

$$I(z'_1, \cdots, z'_s) = I^{0,n'_2;0,n'_3;\cdots;0,n'_r:m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_s,q'_s:p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)}} \left(\begin{array}{c|c} z'_1 & (a'_{2j}; \alpha'^{(1)}_{2j}, \alpha'^{(2)}_{2j})_{1,p'_2}; \cdots ; \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (b'_{2j}; \beta'^{(1)}_{2j}, \beta'^{(2)}_{2j})_{1,q'_2}; \cdots ; \end{array} \right.$$

$$\left. \begin{array}{c} (a'_{sj}; \alpha'^{(1)}_{sj}, \cdots, \alpha'_{sj}{}^{(s)})_{1,p'_s} : (a'^{(1)}_j, \alpha'^{(1)}_j)_{1,p'^{(1)}}; \cdots ; (a'_j{}^{(s)}, \alpha'_j{}^{(s)})_{1,p'^{(s)}} \\ \\ (b'_{sj}; \beta'^{(1)}_{sj}, \cdots, \beta'_{sj}{}^{(s)})_{1,q'_s} : (b'^{(1)}_j, \beta'^{(1)}_j)_{1,q'^{(1)}}; \cdots ; (b'_j{}^{(s)}, \beta'_j{}^{(s)})_{1,q'^{(s)}} \end{array} \right) \tag{1.4}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \psi(t_1, \cdots, t_s) \prod_{i=1}^{s} \xi_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.5}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

where $|argz'_i| < \frac{1}{2}\Omega'_i\pi$,

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'^{(i)}_k - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'^{(i)}_k + \sum_{k=1}^{m'^{(i)}} \beta'^{(i)}_k - \sum_{k=m^{(i)}+1}^{q'^{(i)}} \beta'^{(i)}_k + \left(\sum_{k=1}^{n'_2} \alpha'^{(i)}_{2k} - \sum_{k=n_2+1}^{p'_2} \alpha'^{(i)}_{2k}\right)$$

$$+\cdots+\left(\sum_{k=1}^{n'_s} \alpha'^{(i)}_{sk} - \sum_{k=n'_s+1}^{p'_s} \alpha'^{(i)}_{sk}\right) - \left(\sum_{k=1}^{q'_2} \beta'^{(i)}_{2k} + \sum_{k=1}^{q'_3} \beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q'_s} \beta'^{(i)}_{sk}\right) \tag{1.6}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z'_1, \cdots, z'_s) = 0(\,|z'_1|^{\alpha'_1}, \cdots, |z'_s|^{\alpha'_s}\,), max(\,|z'_1|, \cdots, |z'_s|\,) \to 0$$

$$I(z'_1, \cdots, z'_s) \;= 0(\,|z'_1|^{\beta'_1}\;, \cdots, |z'_s|^{\beta'_s}\,), min(\,|z'_1|, \cdots, |z'_s|\,) \to \infty$$

where $k = 1, \cdots, z : \alpha''_k = min[Re(b'^{(k)}_j/\beta'^{(k)}_j)], j = 1, \cdots, m'_k$ and

$$\beta''_k = max[Re((a'^{(k)}_j - 1)/\alpha'^{(k)}_j)], j = 1, \cdots, n'_k$$

Srivastava and Garg [4] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1, \cdots, z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_vR_u \leqslant L} (-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots, R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.7}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\, {}_PF_Q\,[(A_P); (B_Q); -(x_1 + \cdots + x_r)]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L''_1}\cdots\int_{L''_r}\frac{\prod_{j=1}^{P}\Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^{Q}\Gamma(B_j + s_1 + \cdots + s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots, r$
In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[3,page 454] given by :

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+s_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+s_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-s_j)z_1^{s_1}\cdots z_l^{s_l}z_{l+1}^{s_{l+1}}\cdots,z_{l+k}^{s_{l+k}}\,\mathrm{d}s_1\cdots\mathrm{d}s_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [5, page 454].

3. Eulerian integral

In this section , we note :

$$\theta_i = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}, \zeta_j^{(i)} > 0 (i=1,\cdots,r)\,;\, \theta_i' = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}}, \zeta_j'^{(i)} > 0(i=1,\cdots,s)$$

$$\theta_i'' = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0 (i=1,\cdots,u) \tag{3.1}$$

$$U = p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};p_2',q_2';p_3',q_3';\cdots;p_{s-1}',q_{s-1}';0,0;\cdots;0,0;0,0;\cdots;0,0 \tag{3.2}$$

$$V = 0,n_2;0,n_3;\cdots;0,n_{r-1};0,n_2';0,n_3';\cdots;0,n_{s-1}';0,0;\cdots;0,0;0,0;\cdots;0,0 \tag{3.3}$$

$$X = m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)};m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)};1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.4}$$

$$Y = p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)};p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)};0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.5}$$

$$A = (a_{2k};\alpha_{2k}^{(1)},\alpha_{2k}^{(2)});\cdots;(a_{(r-1)k};\alpha_{(r-1)k}^{(1)},\alpha_{(r-1)k}^{(2)},\cdots,\alpha_{(r-1)k}^{(r-1)})\,;(a_{2k}';\alpha_{2k}'^{(1)},\alpha_{2k}'^{(2)});\cdots;$$

$$(a_{(s-1)k}';\alpha_{(s-1)k}'^{(1)},\alpha_{(s-1)k}'^{(2)},\cdots,\alpha_{(s-1)k}'^{(s-1)}) \tag{3.6}$$

$$;(b_{2k}';\beta_{2k}'^{(1)},\beta_{2k}'^{(2)});\cdots;B = (b_{2k};\beta_{2k}^{(1)},\beta_{2k}^{(2)});\cdots;(b_{(r-1)k};\beta_{(r-1)k}^{(1)},\beta_{(r-1)k}^{(2)},\cdots,\beta_{(r-1)k}^{(r-1)})$$

$$(b_{(s-1)k}';\beta_{(s-1)k}'^{(1)},\beta_{(s-1)k}'^{(2)},\cdots,\beta_{(s-1)k}'^{(s-1)}) \tag{3.7}$$

$$\mathfrak{A} = (a_{rk};\alpha_{rk}^{(1)},\alpha_{rk}^{(2)},\cdots,\alpha_{rk}^{(r)},0,\cdots,0,0\cdots,0,0,\cdots,0) \tag{3.8}$$

$$\mathfrak{A}' = (a_{sk}';0,\cdots,0,\alpha_{sk}'^{(1)},\alpha_{sk}'^{(2)},\cdots,\alpha_{sk}'^{(s)},0,\cdots,0,0,\cdots,0) \tag{3.9}$$

$$\mathfrak{B} = (b_{rk};\beta_{rk}^{(1)},\beta_{rk}^{(2)},\cdots,\beta_{rk}^{(r)},0,\cdots,0,0,\cdots,0,0,\cdots,0) \tag{3.10}$$

$$\mathfrak{B}' = (b_{sk}';0,\cdots,0,\beta_{sk}'^{(1)},\beta_{sk}'^{(2)},\cdots,\beta_{sk}'^{(s)},0,\cdots,0,0,\cdots,0) \tag{3.11}$$

$$\mathfrak{A}_1 = (a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}};(a_k'^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k'^{(s)},\alpha_k'^{(s)})_{1,p'^{(s)}};$$

$$(1,0);\cdots;(1,0);(1.0);\cdots;(1.0) \tag{3.12}$$

$$\mathfrak{B}_1 = (b_k^{(1)},\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}};(b_k'^{(1)},\beta_k'^{(1)})_{1,q'^{(1)}};\cdots;(b_k'^{(s)},\beta_k'^{(s)})_{1,q'^{(s)}};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{3.13}$$

$$K_1=(1-\alpha-\sum_{i=1}^{u}R_ia_i;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1) \tag{3.14}$$

$$K_2=(1-\beta-\sum_{i=1}^{u}R_ib_i;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0) \tag{3.15}$$

$$K_j=[1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j''^{(i)};\lambda_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l} \tag{3.16}$$

$$K'_j=[1+\sigma_j-\sum_{i=1}^{u}R_i\lambda_j''^{(i)};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \tag{3.17}$$

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{u}R_i(a_i+b_i);\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_1,\cdots,\mu'_r+\rho'_r,$$

$$h_1,\cdots,h_l,1,\cdots,1) \tag{3.18}$$

$$L_j=[1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j''^{(i)};\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l} \tag{3.19}$$

$$L'_j=[1+\sigma_j-\sum_{i=1}^{u}R_i\lambda_j''^{(i)};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0,\cdots,0]_{1,k} \tag{3.20}$$

$$P_1=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}\right\} \tag{3.21}$$

$$P_u=(b-a)^{\sum_{i=1}^{u}(a_i+b_i)R_i}\left\{\prod_{j=1}^{h}(af_j+g_j)^{-\sum_{l=1}^{u}\lambda_j''^{(i)}R_i}\right\} \tag{3.22}$$

$$B_u=\frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{3.23}$$

We the following generalized Eulerian integral :

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} z_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ . \\ . \\ . \\ z_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ z_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t =$$

$$= P_1 \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_vR_u\leqslant L}\prod_{k=1}^{u} z_k''^{R_k}P_uB_u$$

$$I_{U:p_r+p_s'+l+k+2,q_r+q_s'+l+k+1;Y}^{V;0,n_r+n_s'+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K_j',\mathfrak{A},\mathfrak{A}';\mathfrak{A}_1 \\ \cdots & . \\ \cdots & . \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & . \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \tau_1(b-a)^{h_1} & \\ \cdots & \\ \cdots & \\ \tau_l(b-a)^{h_l} & \mathrm{B}\ ;\ \mathrm{L}_1,L_j,L_j',\mathfrak{B},\mathfrak{B}';\mathfrak{B}_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (3.24)$$

We obtain the I-function of $r+s+k+l$ variables. The quantities $U,V,X,Y,A,B,K_1,K_2,K_j,K'_j,\mathfrak{A},\mathfrak{A}',\mathfrak{A}_1,L_1,L_j,L'_j$, $\mathfrak{B},\mathfrak{B}',P_1,P_u,B_u$ and $\mathfrak{B}_1$ are defined above.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\mu'_u,\rho_i,\rho'_u,\lambda_j^{(i)},\lambda_j'^{(u)},h_v\in\mathbb{R}^+, f_i,g_j,\tau_v,\sigma_j,\lambda_v\in\mathbb{C}\ (i=1,\cdots,r;j=1,\cdots;k;$

$u=1,\cdots,s;v=1,\cdots,l),a_i,b_i,\lambda_j''^{(i)}\in\mathbb{R}^+,(i=1,\cdots,u;j=1,\cdots,k)$

(B) $a_{ij},b_{ik},\in\mathbb{C}\ (i=1,\cdots,r;j=1,\cdots,p_i;k=1,\cdots,q_i);\ a_j^{(i)},b_j^{(k)}\in\mathbb{C}$

$(i=1,\cdots,r;j=1,\cdots,p^{(i)};k=1,\cdots,q^{(i)})$

$a'_{ij},b'_{ik},\in\mathbb{C}\ (i=1,\cdots,s;j=1,\cdots,p'_i;k=1,\cdots,q'_i);\ a'^{(i)}_j,b'^{(k)}_j,\in\mathbb{C}$

$(i=1,\cdots,r;j=1,\cdots,p'^{i)};k=1,\cdots,q'^{(i)})$

$\alpha_{ij}^{(k)},\beta_{ij}^{(k)}\in\mathbb{R}^+\ (\ (i=1,\cdots,r,j=1,\cdots,p_i,k=1,\cdots,r)\ ;\alpha_j^{(i)},\beta_i^{(i)}\in\mathbb{R}^+\ (i=1,\cdots,r;j=1,\cdots,p_i)$

$\alpha'^{(k)}_{ij},\beta'^{(k)}_{ij}\in\mathbb{R}^+\ (\ (i=1,\cdots,s,j=1,\cdots,p'_i,k=1,\cdots,s)\ ;\alpha'^{(i)}_j,\beta'^{(i)}_i\in\mathbb{R}^+\ (i=1,\cdots,s;j=1,\cdots,p'_i)$

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1$

(D) $Re\big[\alpha+\sum_{j=1}^{r}\mu_j\min_{1\leqslant k\leqslant m^{(i)}}\frac{b_k^{(j)}}{\beta_k^{(j)}}+\sum_{j=1}^{s}\mu'_i\min_{1\leqslant k\leqslant m'^{(i)}}\frac{b_k'^{(j)}}{\beta_k'^{(j)}}\big]>0$

$$Re\big[\beta+\sum_{j=1}^{r}\rho_j\min_{1\leqslant k\leqslant m^{(i)}}\frac{b_k^{(j)}}{\beta_k^{(j)}}+\sum_{j=1}^{s}\rho'_i\min_{1\leqslant k\leqslant m'^{(i)}}\frac{b_k'^{(j)}}{\beta_k'^{(j)}}\big]>0$$

(E) $Re\left(\alpha+\sum_{i=1}^{u}R_ia_i+\sum_{i=1}^{r}\mu_is_i+\sum_{i=1}^{s}t_i\mu'_i\right)>0\ ;Re\left(\beta+\sum_{i=1}^{u}R_ib_i+\sum_{i=1}^{r}v_is_i+\sum_{i=1}^{s}t_i\rho'_i\right)>0$

$$Re\left(\lambda_j+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\zeta_j^{(i)}+\sum_{i=1}^{s}t_i\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);$$

$$Re\left(-\sigma_j+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\lambda_j^{(i)}+\sum_{i=1}^{s}t_i\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(F) $\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$

$$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right)-\mu_i-\rho_i$$

$$-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}>0 \quad (i=1,\cdots,r)$$

$$\Omega'_i=\sum_{k=1}^{n'^{(i)}}\alpha'^{(i)}_k-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha'^{(i)}_k+\sum_{k=1}^{m'^{(i)}}\beta'^{(i)}_k-\sum_{k=m^{(i)}+1}^{q'^{(i)}}\beta'^{(i)}_k+\left(\sum_{k=1}^{n'_2}\alpha'^{(i)}_{2k}-\sum_{k=n_2+1}^{p'_2}\alpha'^{(i)}_{2k}\right)+$$

$$\cdots+\left(\sum_{k=1}^{n'_s}\alpha'^{(i)}_{sk}-\sum_{k=n'_s+1}^{p'_s}\alpha'^{(i)}_{sk}\right)-\left(\sum_{k=1}^{q'_2}\beta'^{(i)}_{2k}+\sum_{k=1}^{q'_3}\beta'^{(i)}_{3k}+\cdots+\sum_{k=1}^{q'_s}\beta'^{(i)}_{sk}\right)-\mu'_i-\rho'_i$$

$$-\sum_{l=1}^{k}\lambda'^{(i)}_j-\sum_{l=1}^{l}\zeta'^{(i)}_j>0 \quad (i=1,\cdots,s)$$

(G) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\Omega_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,r)$$

$$\left|arg\left(z'_i\prod_{j=1}^{l}\left[1-\tau'_j(t-a)^{h'_i}\right]^{-\zeta'^{(i)}_j}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(i)}_j}\right)\right|<\frac{1}{2}\Omega'_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,s)$$

Proof

To prove (3.24), first, we express in serie a class of multivariable polynomials defined by Srivastava et al [4], $S_L^{h_1,\cdots,h_u}[.]$ with the help of (1.7), espressing the I-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.2), the I-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.5). Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral to multivariable I-function of Prasad, we obtain the equation (3.24).

Remarks

If a) $\rho_1=\cdots,\rho_r=\rho'_1=\cdots,\rho'_s=0$; b) $\mu_1=\cdots,\mu_r=\mu'_1=\cdots,\mu'_s=0$

we obtain the similar formulas that (3.24) with the corresponding simplifications.

4. Particular cases

a) If $U=V=A=B=0$, the multivariable I-function defined by Prasad reduces to multivariable H-function defined by Srivastava et al [6] and we obtain :

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} \mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{dt}=$$

$$=P_1\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{k=1}^u z_k''^{R_k}P_uB_u$$

$$H_{p_r+p_s'+l+k+2,q_r+q_s'+l+k+1;Y}^{0,n_r+n_s'+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{K}_1,K_2,K_j,K_j',\mathfrak{A},\mathfrak{A}';\mathfrak{A}_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \\ \cdots & \\ \cdots & \\ \tau_l(b-a)^{h_l} & \mathrm{L}_1,L_j,L_j',\mathfrak{B},\mathfrak{B}';\mathfrak{B}_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad \textbf{(4.1)}$$

under the same notations and conditions that (3.24) with $U = V = A = B = 0$

b) If $$B(L;R_1,\cdots,R_u) = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi'_j+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}} \tag{4.2}$$

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [3].

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c}z_1\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ z_u\end{array}\middle|\begin{array}{c}[(\text{-L});R_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}]\\ [(c);\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}]\end{array}\right) \tag{4.3}$$

and we have the following formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c}z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda''^{(1)}_j}\\ \cdot\\ \cdot\\ \cdot\\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda''^{(u)}_j}\end{array}\right.$$

$$\left.\begin{array}{c}[(\text{-L});R_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}]\\ [(c);\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}]\end{array}\right)$$

$$I\left(\begin{array}{c}z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}}\end{array}\right)$$

$$I\begin{pmatrix} z'_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{pmatrix}\mathrm{d}t =$$

$$= P_1 \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} \prod_{k=1}^{u} z_k''^{R_k} P_u B'_u$$

$$I_{U:p_r+p'_s+l+k+2,q_r+q'_s+l+k+1;Y}^{V;0,n_r+n'_s+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \cdots & \mathrm{A} ; \mathrm{K}_1, K_2, K_j, K'_j, \mathfrak{A}, \mathfrak{A}'; \mathfrak{A}_1 \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \\ \tau_l(b-a)^{h_l} & \mathrm{B} ; \mathrm{L}_1, L_j, L'_j, \mathfrak{B}, \mathfrak{B}'; \mathfrak{B}_1 \\ \frac{(b-a)f_1}{af_1+g_1} & \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (4.4)$$

under the same conditions that (3.24)

and $B'_u = \dfrac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$; $B(L;R_1,\cdots,R_u)$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions of Prasad [1] and a class of multivariable polynomials defined by Srivastava et al [4].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions defined by Prasad [1] and a class of multivariable polynomials defined by Srivastava et al [4] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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