Eulerian integral associated with product of two multivariable A-functions, a generalized Lauricella function and a class of polynomials

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable A-functions defined by Gautam et al [1] a generalized Lauricella function and a class of multivariable polynomials with general arguments . Several particular cases are given .

Keywords: Eulerian integral, multivariable A-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, class of polynomials.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable A-functions defined by Gautam et al [1] and a class of polynomials with general arguments but of greater order. Several particular cases are given.
The A-function is defined and represented in the following manner.

$$A(z_1,\cdots,z_r) = A^{m,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;A_j^{(1)},\cdots,A_j^{(r)})_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;B_j^{(1)},\cdots,B_j^{(r)})_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i),i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^m\Gamma(b_j-\sum_{i=1}^r B_j^{(i)}s_i)\prod_{j=1}^n\Gamma(1-a_j+\sum_{i=1}^r A_j^{(i)}s_j)}{\prod_{j=n+1}^p\Gamma(a_j-\sum_{i=1}^r A_j^{(i)}s_j)\prod_{j=m+1}^q\Gamma(1-b_j+\sum_{i=1}^r B_j^{(i)}s_j)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}s_i)\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}s_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}s_i)\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}-D_j^{(i)}s_i)} \tag{1.4}$$

Here $m,n,p,m_i,n_i,p_i,c_i \in \mathbb{N}^*; i=1,\cdots,r ; a_j,b_j,c_j^{(i)},d_j^{(i)},A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i)z_k| < \frac{1}{2}\eta_k\pi, \xi^* = 0, \eta_i > 0 \tag{1.5}$$

$$\Omega_i = \prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}}; i=1,\cdots,r \tag{1.6}$$

$$\xi_i^* = Im(\sum_{j=1}^{p}A_j^{(i)} - \sum_{j=1}^{q}B_j^{(i)} + \sum_{j=1}^{q_i}D_j^{(i)} - \sum_{j=1}^{p_i}C_j^{(i)}); i=1,\cdots,r \tag{1.7}$$

$$\eta_i = Re\left(\sum_{j=1}^{n}A_j^{(i)} - \sum_{j=n+1}^{p}A_j^{(i)} + \sum_{j=1}^{m}B_j^{(i)} - \sum_{j=m+1}^{q}B_j^{(i)} + \sum_{j=1}^{m_i}D_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)} + \sum_{j=1}^{n_i}C_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)$$

$$i=1,\cdots,r \tag{1.8}$$

Consider the second multivariable A-function.

$$A(z_1',\cdots,z_s') = A_{p',q':p_1',q_1';\cdots;p_r',q_r'}^{m',n':m_1',n_1';\cdots;m_r',n_r'}\left(\begin{array}{c|c} z_1 & (a'_j;A_j'^{(1)},\cdots,A_j'^{(s)})_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b'_j;B_j'^{(1)},\cdots,B_j'^{(s)})_{1,q'} : \end{array}\right.$$

$$\left.\begin{array}{l} (c'^{(1)}_j,C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(s)},C_j'^{(s)})_{1,p_s'} \\ \\ (d'^{(1)}_j,D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},D_j'^{(s)})_{1,q_s'} \end{array}\right) \tag{1.9}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\phi'(t_1,\cdots,t_s)\prod_{i=1}^{s}\theta_i'(t_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.10}$$

where $\phi'(t_1,\cdots,t_s), \theta_i'(t_i), i=1,\cdots,s$ are given by :

$$\phi'(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{m'}\Gamma(b_j' - \sum_{i=1}^{s}B_j'^{(i)}t_i)\prod_{j=1}^{n'}\Gamma(1-a_j' + \sum_{i=1}^{s}A_j'^{(i)}t_j)}{\prod_{j=n'+1}^{p'}\Gamma(a_j' - \sum_{i=1}^{s}A_j'^{(i)}t_j)\prod_{j=m'+1}^{q'}\Gamma\left(1-b_j' + \sum_{i=1}^{s}B_j'^{(i)}t_j\right)} \tag{1.11}$$

$$\theta_i'(t_i) = \frac{\prod_{j=1}^{n_i'}\Gamma(1-c_j'^{(i)} + C_j'^{(i)}t_i)\prod_{j=1}^{m_i'}\Gamma(d_j'^{(i)} - D_j'^{(i)}t_i)}{\prod_{j=n_i'+1}^{p_i'}\Gamma(c_j'^{(i)} - C_j'^{(i)}t_i)\prod_{j=m_i'+1}^{q_i'}\Gamma(1-d_j'^{(i)} - D_j'^{(i)}t_i)} \tag{1.12}$$

Here $m',n',p',m_i',n_i',p_i',c_i' \in \mathbb{N}^*; i=1,\cdots,r ; a_j',b_j',c_j'^{(i)},d_j'^{(i)},A_j'^{(i)},B_j'^{(i)},C_j'^{(i)},D_j'^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i')z_k'| < \frac{1}{2}\eta_k'\pi, \xi'^* = 0, \eta_i' > 0 \tag{1.13}$$

$$\Omega_i' = \prod_{j=1}^{p'}\{A_j'^{(i)}\}^{A_j'^{(i)}}\prod_{j=1}^{q'}\{B_j'^{(i)}\}^{-B_j'^{(i)}}\prod_{j=1}^{q_i'}\{D_j'^{(i)}\}^{D_j'^{(i)}}\prod_{j=1}^{p_i'}\{C_j'^{(i)}\}^{-C_j'^{(i)}}; i = 1, \cdots, s \tag{1.14}$$

$$\xi_i'^* = Im(\sum_{j=1}^{p'} A_j'^{(i)} - \sum_{j=1}^{q'} B_j'^{(i)} + \sum_{j=1}^{q_i'} D_j'^{(i)} - \sum_{j=1}^{p_i'} C_j'^{(i)}); i = 1, \cdots, s \tag{1.15}$$

$$\eta_i' = Re\left(\sum_{j=1}^{n'} A_j'^{(i)} - \sum_{j=n'+1}^{p'} A_j'^{(i)} + \sum_{j=1}^{m'} B_j'^{(i)} - \sum_{j=m'+1}^{q'} B_j'^{(i)} + \sum_{j=1}^{m_i'} D_j'^{(i)} - \sum_{j=m_i'+1}^{q_i'} D_j'^{(i)} + \sum_{j=1}^{n_i'} C_j'^{(i)} - \sum_{j=n_i'+1}^{p_i'} C_j'^{(i)}\right)$$

$$i = 1, \cdots, s \tag{1.16}$$

Srivastava and Garg [4] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_v}[z_1,\cdots,z_v] = \sum_{R_1,\cdots,R_v=0}^{h_1R_1+\cdots h_vR_v\leqslant L} (-L)_{h_1R_1+\cdots+h_vR_v} B(E;R_1,\cdots,R_v)\frac{z_1^{R_1}\cdots z_v^{R_v}}{R_1!\cdots R_v!} \tag{1.17}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j = 1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1,\cdots,r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\Bigg) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[3,page 454] given by :

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\Bigg)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+s_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+s_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-s_j)z_1^{s_1}\cdots z_l^{s_l}z_{l+1}^{s_{l+1}}\cdots,z_{l+k}^{s_{l+k}}\,\mathrm{d}s_1\cdots\mathrm{d}s_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [5, page 454].

3. Eulerian integral

In this section , we note :

$$\theta_i=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}},\zeta_j^{(i)}>0(i=1,\cdots,r)\,;\,\theta_i'=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}},\zeta_j'^{(i)}>0(i=1,\cdots,s)$$

$$\theta_i'' = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0 (i=1,\cdots,u) \tag{3.1}$$

$$X = m_1,n_1;\cdots;m_r,n_r;m_1',n_1';\cdots;m_s',n_s';1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.2}$$

$$Y = p_1,q_1;\cdots;p_r,q_r;p_1',q_1';\cdots;p_s',q_s';0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.3}$$

$$A = (a_j;A_j^{(1)},\cdots,A_j^{(r)},0,\cdots,0,0,\cdots,0,0,\cdots,0)_{1,p} \tag{3.4}$$

$$B = (b_j;B_j^{(1)},\cdots,B_j^{(r)},0\cdots,0,0\cdots,0,0\cdots,0)_{1,q} \tag{3.5}$$

$$A' = (a_j';0,\cdots,0,A_j'^{(1)},\cdots,A_j'^{(s)},0,\cdots,0,0,\cdots,0)_{1,p'} \tag{3.6}$$

$$B' = (b_j';0,\cdots,0,B_j'^{(1)},\cdots,B_j'^{(s)},0,\cdots,0,0,\cdots,0)_{1,q'} \tag{3.7}$$

$$\mathrm{C} = (\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r};(c_j'^{(1)},C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(r)},C_j'^{(s)})_{1,p_s'}$$

$$(1,0);\cdots;(1,0);(1,0);\cdots;(1,0) \tag{3.8}$$

$$D = (\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r};(\mathrm{d}'^{(1)}_j,D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},D_j'^{(s)})_{1,q_s'};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{3.9}$$

$$K_1 = (1-\alpha-\sum_{i=1}^{u} R_i a_i;\mu_1,\cdots,\mu_r,\mu_1',\cdots,\mu_s',h_1,\cdots,h_l,1,\cdots,1) \tag{3.10}$$

$$K_2 = (1-\beta-\sum_{i=1}^{u} R_i b_i;\rho_1,\cdots,\rho_r,\rho_1',\cdots,\rho_s',0,\cdots,0,0\cdots,0) \tag{3.11}$$

$$K_j = [1-\lambda_j-\sum_{i=1}^{u} R_i\zeta_j''^{(i)};\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,\underset{j}{1},\cdots,0,0\cdots,0]_{1,l} \tag{3.12}$$

$$K_j' = [1+\sigma_j-\sum_{i=1}^{u} R_i\lambda_j''^{(i)};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0\cdots,\underset{j}{1},\cdots,0]_{1,k} \tag{3.13}$$

$$L_1 = (1-\alpha-\beta-\sum_{i=1}^{u} R_i(a_i+b_i);\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu_1'+\rho_1',\cdots,\mu_r'+\rho_r',$$

$$h_1,\cdots,h_l,1,\cdots,1) \tag{3.14}$$

$$L_j = [1-\lambda_j - \sum_{i=1}^{u} R_i \zeta_j''^{(i)}; \zeta_j^{(1)}, \cdots, \zeta_j^{(r)}, \zeta_j'^{(1)} \cdots, \zeta_j'^{(s)}, 0, \cdots, 0, 0 \cdots, 0]_{1,l} \tag{3.15}$$

$$L_j' = [1+\sigma_j - \sum_{i=1}^{u} R_i \lambda_j''^{(i)}; \lambda_j^{(1)}, \cdots, \lambda_j^{(r)}, \lambda_j'^{(1)} \cdots, \lambda_j'^{(s)}, 0, \cdots, 0, 0, \cdots, 0]_{1,k} \tag{3.16}$$

$$P_1 = (b-a)^{\alpha+\beta-1} \left\{ \prod_{j=1}^{k} (af_j+g_j)^{\sigma_j} \right\} \tag{3.17}$$

$$P_u = (b-a)^{\sum_{i=1}^{u}(a_i+b_i)R_i} \left\{ \prod_{j=1}^{h} (af_j+g_j)^{-\sum_{l=1}^{u} \lambda_j''^{(i)} R_i} \right\} \tag{3.18}$$

$$B_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots, R_u)}{R_1! \cdots R_u!} \tag{3.19}$$

Let $\mathfrak{A} = A, A; \mathfrak{B} = B, B'; A, B, A'$ and B' are defined by (3.4), (3.5), (3.6) and (3.7), respectively

We the following generalized Eulerian integral :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j} \prod_{j=1}^{k} (f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u} \left(\begin{array}{c} \mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array} \right)$$

$$A \left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

$$A\begin{pmatrix} z'_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{pmatrix}\mathrm{d}t =$$

$$= P_1 \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} \prod_{k=1}^{u} z_k''^{R_k} P_u B_u$$

$$A^{m+m',n+n'+l+k+2;X}_{p+p'+l+k+2,q+q'+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ , \mathrm{K}_1, K_2, K_j, K'_j : C \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \\ \cdot\cdot\cdot & \\ \cdot\cdot\cdot & \\ \tau_l(b-a)^{h_l} & \\ \frac{(b-a)f_1}{af_1+g_1} & \mathfrak{B}\ , \mathrm{L}_1, L_j, L'_j : D \\ \cdot\cdot\cdot & \\ \cdot\cdot\cdot & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{3.20}$$

where $\mathfrak{A}, \mathfrak{B}, C, D, X, K_1, K_2, K_j, K'_j, L_1, L_j, L'_j, P_1, P_u, B_u$ and $\mathfrak{B}_1$ are defined above.

Provided that

(A) $a, b \in \mathbb{R}(a<b); \mu_i, \mu'_u, \rho_i, \rho'_u, \lambda_j^{(i)}, \lambda_j'^{(u)}, h_v \in \mathbb{R}^+, f_i, g_j, \tau_v, \sigma_j, \lambda_v \in \mathbb{C}\ \ (i=1,\cdots,r; j=1,\cdots;k;$

$u=1,\cdots,s; v=1,\cdots,l), a_i, b_i, \zeta_j''^{(i)} \in \mathbb{R}^+, (i=1,\cdots,u; j=1,\cdots,k)$

(B) $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*; i=1,\cdots,r$ **;** $a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

$m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i=1,\cdots,r\ ; a'_j, b'_j, c_j'^{(i)}, d_j'^{(i)}, A_j'^{(i)}, B_j'^{(i)}, C_j'^{(i)}, D_j'^{(i)} \in \mathbb{C}$

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$

(D) $Re\big[\alpha+\sum_{j=1}^{r}\mu_j\min_{1\leqslant k\leqslant m_i}\frac{d_k^{(j)}}{D_k^{(j)}}+\sum_{j=1}^{s}\mu'_i\min_{1\leqslant k\leqslant m'_i}\frac{d_k'^{(j)}}{D_k'^{(j)}}\big]>0$

$$Re\big[\beta+\sum_{j=1}^{r}\rho_j\min_{1\leqslant k\leqslant m_i}\frac{d_k^{(j)}}{D_k^{(j)}}+\sum_{j=1}^{s}\rho'_i\min_{1\leqslant k\leqslant m'_i}\frac{d_k'^{(j)}}{D_k'^{(j)}}\big]>0$$

(E) $Re\left(\alpha+\sum_{i=1}^{u}R_ia_i+\sum_{i=1}^{r}\mu_is_i+\sum_{i=1}^{s}t_i\mu'_i\right)>0\,;Re\left(\beta+\sum_{i=1}^{u}R_ib_i+\sum_{i=1}^{r}v_is_i+\sum_{i=1}^{s}t_i\rho'_i\right)>0$

$$Re\left(\lambda_j+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\zeta_j^{(i)}+\sum_{i=1}^{s}t_i\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);$$

$$Re\left(-\sigma_j+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\lambda_j^{(i)}+\sum_{i=1}^{s}t_i\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(F) $|arg(\Omega_i)z_k|<\frac{1}{2}\eta_i\pi,\xi^*=0,\eta_i>0$

$$\Omega_i=\prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}};i=1,\cdots,r$$

$$\xi_i^*=Im\Big(\sum_{j=1}^{p}A_j^{(i)}-\sum_{j=1}^{q}B_j^{(i)}+\sum_{j=1}^{q_i}D_j^{(i)}-\sum_{j=1}^{p_i}C_j^{(i)}\Big);i=1,\cdots,r$$

$$\eta_i=Re\left(\sum_{j=1}^{n}A_j^{(i)}-\sum_{j=n+1}^{p}A_j^{(i)}+\sum_{j=1}^{m}B_j^{(i)}-\sum_{j=m+1}^{q}B_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)$$

$$-\mu_i-\rho_i-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}>0\,;\ i=1,\cdots,r$$

$$|arg(\Omega'_i)z'_k|<\frac{1}{2}\eta'_i\pi,\xi'^*=0,\eta'_i>0$$

$$\Omega'_i=\prod_{j=1}^{p'}\{A_j'^{(i)}\}^{A_j'^{(i)}}\prod_{j=1}^{q'}\{B_j'^{(i)}\}^{-B_j'^{(i)}}\prod_{j=1}^{q'_i}\{D_j'^{(i)}\}^{D_j'^{(i)}}\prod_{j=1}^{p'_i}\{C_j'^{(i)}\}^{-C_j'^{(i)}};i=1,\cdots,s$$

$$\xi_i'^*=Im\Big(\sum_{j=1}^{p'}A_j'^{(i)}-\sum_{j=1}^{q'}B_j'^{(i)}+\sum_{j=1}^{q'_i}D_j'^{(i)}-\sum_{j=1}^{p'_i}C_j'^{(i)}\Big);i=1,\cdots,s$$

$$\eta'_i=Re\left(\sum_{j=1}^{n'}A_j'^{(i)}-\sum_{j=n'+1}^{p'}A_j'^{(i)}+\sum_{j=1}^{m'}B_j'^{(i)}-\sum_{j=m'+1}^{q'}B_j'^{(i)}+\sum_{j=1}^{m'_i}D_j'^{(i)}-\sum_{j=m'_i+1}^{q'_i}D_j'^{(i)}+\sum_{j=1}^{n'_i}C_j'^{(i)}-\sum_{j=n'_i+1}^{p'_i}C_j'^{(i)}\right)$$

$$-\mu'_i-\rho'_i-\sum_{l=1}^{k}\lambda_j'^{(i)}-\sum_{l=1}^{l}\zeta_j'^{(i)}>0\,;i=1,\cdots,s$$

(H) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\eta_i\pi\ \ (a\leqslant t\leqslant b;i=1,\cdots,r)$$

$$\left|arg\left(z'_i\prod_{j=1}^{l}\left[1-\tau'_j(t-a)^{h'_i}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}\eta'_i\pi\ (a\leqslant t\leqslant b;i=1,\cdots,s)$$

Proof

To prove (3.20), first, we express in serie a class of multivariable polynomials defined by Srivastava et al [4], $S_L^{h_1,\cdots,h_u}[.]$ with the help of (1.7), espressing the A-function of r variables by the Mellin-Barnes contour integral with the help of the equation (1.2), the A-function of s variables by the Mellin-Barnes contour integral with the help of the equation (1.5). Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable A-function defined by Gautam et al [1], we obtain the equation (3.20).

Remarks

If a) $\rho_1=\cdots,\rho_r=\rho'_1=\cdots,\rho'_s=0$; b) $\mu_1=\cdots,\mu_r=\mu'_1=\cdots,\mu'_s=0$, we obtain the similar formulas that (3.20) with the corresponding simplifications.

4. Particular cases

a) If $A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)}\in\mathbb{R}$,$m=0$ and $A_j'^{(i)},B_j'^{(i)},C_j'^{(i)},D_j'^{(i)}\in\mathbb{R}$ and $m'=0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [6], we obtain the following result.

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\begin{pmatrix} z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}}\end{pmatrix}$$

$$H\begin{pmatrix} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}}\end{pmatrix}$$

$$H\left(\begin{array}{c} z_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ z_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t =$$

$$= P_1 \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} \prod_{k=1}^{u} z_k''^{R_k} P_u B_u$$

$$H^{0,n+n'+l+k+2;X}_{p+p'+l+k+2,q+q'+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_j,K_j':C \\ \cdots & \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & . \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \tau_1(b-a)^{h_1} & . \\ \cdots & . \\ \cdots & \\ \tau_l(b-a)^{h_l} & \\ \frac{(b-a)f_1}{af_1+g_1} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_j':D \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (4.1)$$

under the same notations and validity conditions that (3.24) with$A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)}\in\mathbb{R}$,$m=0$ and $A_j'^{(i)},B_j'^{(i)},C_j'^{(i)},D_j'^{(i)}\in\mathbb{R}$ and $m'=0$

b) If $B(L;R_1,\cdots,R_u) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta_j'+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b_j')_{R_1\phi_j'}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi_j'+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d_j')_{R_1\delta_j'}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [3]. We have

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F^{1+\bar{A}:B';\cdots;B^{(u)}}_{\bar{C}:D';\cdots;D^{(u)}}\left(\begin{array}{c} z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda''^{(1)}_j} \\ . \\ . \\ . \\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda''^{(u)}_j} \end{array}\right.$$

$$\left.\begin{array}{c} [(-L);R_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}] \\ [(c);\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}] \end{array}\right)$$

$$A\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(1)}_j} \\ . \\ . \\ . \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(r)}_j} \end{array}\right)$$

$$A\left(\begin{array}{c} z'_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(1)}_j} \\ . \\ . \\ . \\ z'_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(s)}_j} \end{array}\right)\mathrm{d}t = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{k=1}^{u} z''^{R_k}_k P_u B'_u$$

$$A^{m+m',n+n'+l+k+2;X}_{p+p'+l+k+2,q+q'+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda^{(1)}_j}} & \mathfrak{A}\ ,\ K_1,K_2,K_j,K'_j:C \\ \cdots & . \\ \cdots & . \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda^{(r)}_j}} & . \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & . \\ \tau_1(b-a)^{h_1} & \\ \cdots & \\ \cdots & \\ \tau_l(b-a)^{h_l} & \\ \frac{(b-a)f_1}{af_1+g_1} & \mathfrak{B}\ ,\ L_1,L_j,L'_j:D \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \qquad (4.4)$$

under the same conditions that (3.24)

and $B'_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$; $B(L;R_1,\cdots,R_u)$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable A-functions and a class of multivariable polynomials defined by Srivastava et al [4].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions defined by Gautam et [1] and a class of multivariable polynomials defined by Srivastava et al [4] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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