

# Selberg integral involving the S generalized Gauss's hypergeometric function , a class of polynomials, the multivariable Aleph-function and the multivariable I-function

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**ABSTRACT**

In the present paper we evaluate the modified Selberg integral involving the product of a multivariable Aleph-function, a multivariable I-function defined by Prasad [2] , the S generalized Gauss hypergeometric function and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializing the parameters their in. We will study two particular cases.

**Keywords:** Multivariable Aleph-function, general class of polynomials, modified Selberg integral, S generalized Gauss hypergeometric function , multivariable I-function, multivariable H-function.

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## 1.Introduction and preliminaries.

In this paper, we evaluate the modified Selberg integral involving the product of a multivariable Aleph-function, a multivariable I-function defined by Prasad [2], The S generalized Gauss hypergeometric function and general class of polynomials of several variables.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables defined by Srivastava et al [7]. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

$$\text{We define : } \mathfrak{N}(z_1, \dots, z_r) = \mathfrak{N}_{p_i, q_i, \tau_i; R: p_i(1), q_i(1), \tau_i(1); R^{(1)}; \dots; p_i(r), q_i(r), \tau_i(r); R^{(r)}}^{0, n: m_1, n_1, \dots, m_r, n_r} \left( \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right)$$

$$[(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1, n}] \quad , [\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)})_{n+1, p_i}] :$$

$$\dots \dots \dots \quad , [\tau_i(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)})_{m+1, q_i}] :$$

$$[(c_j^{(1)}; \gamma_j^{(1)})_{1, n_1}], [\tau_{i(1)}(c_{ji(1)}^{(1)}; \gamma_{ji(1)}^{(1)})_{n_1+1, p_i^{(1)}}]; \dots ; [(c_j^{(r)}; \gamma_j^{(r)})_{1, n_r}], [\tau_{i(r)}(c_{ji(r)}^{(r)}; \gamma_{ji(r)}^{(r)})_{n_r+1, p_i^{(r)}}]$$

$$[(d_j^{(1)}; \delta_j^{(1)})_{1, m_1}], [\tau_{i(1)}(d_{ji(1)}^{(1)}; \delta_{ji(1)}^{(1)})_{m_1+1, q_i^{(1)}}]; \dots ; [(d_j^{(r)}; \delta_j^{(r)})_{1, m_r}], [\tau_{i(r)}(d_{ji(r)}^{(r)}; \delta_{ji(r)}^{(r)})_{m_r+1, q_i^{(r)}}]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) y_k^{s_k} ds_1 \dots ds_r \tag{1.1}$$

with  $\omega = \sqrt{-1}$

$$\psi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k)}{\sum_{i=1}^R [\tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^r \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^r \beta_{ji}^{(k)} s_k)]} \tag{1.2}$$

$$\text{and } \theta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_i^{(k)}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n_k+1}^{p_i^{(k)}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]} \quad (1.3)$$

Suppose , as usual , that the parameters

$$a_j, j = 1, \dots, p; b_j, j = 1, \dots, q;$$

$$c_j^{(k)}, j = 1, \dots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \dots, p_{i^{(k)}};$$

$$d_j^{(k)}, j = 1, \dots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \dots, q_{i^{(k)}};$$

$$\text{with } k = 1 \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)}$$

are complex numbers , and the  $\alpha' s, \beta' s, \gamma' s$  and  $\delta' s$  are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^n \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leq 0 \quad (1.4)$$

The reals numbers  $\tau_i$  are positives for  $i = 1$  to  $R$ ,  $\tau_{i^{(k)}}$  are positives for  $i^{(k)} = 1$  to  $R^{(k)}$

The contour  $L_k$  is in the  $s_k$ -p lane and run from  $\sigma - i\infty$  to  $\sigma + i\infty$  where  $\sigma$  is a real number with loop , if necessary ,ensure that the poles of  $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$  with  $j = 1$  to  $m_k$  are separated from those of  $\Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(k)} s_k)$  with  $j = 1$  to  $n$  and  $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$  with  $j = 1$  to  $n_k$  to the left of the contour  $L_k$  . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^n \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1, \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)} \quad (1.5)$$

The complex numbers  $z_i$  are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\alpha_1}, \dots, |z_r|^{\alpha_r}), \max(|z_1|, \dots, |z_r|) \rightarrow 0$$

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\beta_1}, \dots, |z_r|^{\beta_r}), \min(|z_1|, \dots, |z_r|) \rightarrow \infty$$

where, with  $k = 1, \dots, r : \alpha_k = \min[\operatorname{Re}(d_j^{(k)} / \delta_j^{(k)})], j = 1, \dots, m_k$  and

$$\beta_k = \max[\operatorname{Re}((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \dots, n_k$$

Series representation of Aleph-function of several variables is given by

$$\begin{aligned} \aleph(y_1, \dots, y_r) &= \sum_{G_1, \dots, G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \dots \sum_{g_r=0}^{m_r} \frac{(-)^{G_1+\dots+G_r}}{\delta_{g_1}^{G_1} G_1! \dots \delta_{g_r}^{G_r} G_r!} \psi(\eta_{G_1, g_1}, \dots, \eta_{G_r, g_r}) \\ &\times \theta_1(\eta_{G_1, g_1}) \dots \theta_r(\eta_{G_r, g_r}) y_1^{-\eta_{G_1, g_1}} \dots y_r^{-\eta_{G_r, g_r}} \end{aligned} \tag{1.6}$$

Where  $\psi(\dots), \theta_i(\dots), i = 1, \dots, r$  are given respectively in (1.2), (1.3) and

$$\eta_{G_1, g_1} = \frac{d_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \dots, \eta_{G_r, g_r} = \frac{d_{g_r}^{(r)} + G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions  $\delta_{g_i}^{(i)} [d_j^i + p_i] \neq \delta_j^{(i)} [d_{g_i}^i + G_i]$  (1.7)

for  $j \neq m_i, m_i = 1, \dots, \eta_{G_i, g_i}; p_i, n_i = 0, 1, 2, \dots; y_i \neq 0, i = 1, \dots, r$  (1.8)

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, \dots, z_s) = I_{\substack{0, n_2; 0, n_3; \dots; 0, n_r; m', n'; \dots; m^{(s)}, n^{(s)} \\ p_2, q_2, p_3, q_3; \dots; p_s, q_s; p', q'; \dots; p^{(s)}, q^{(s)}}} \left( \begin{array}{c|c} z_1 & (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; \dots; \\ \cdot & \\ \cdot & \\ \cdot & (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; \dots; \\ z_s & \end{array} \right)$$

$$\left( \begin{array}{c} (a_{sj}; \alpha'_{sj}, \dots, \alpha_{sj}^{(s)})_{1, p_s} : (a'_j, \alpha'_j)_{1, p'}; \dots; (a_j^{(s)}, \alpha_j^{(s)})_{1, p^{(s)}} \\ (b_{sj}; \beta'_{sj}, \dots, \beta_{sj}^{(s)})_{1, q_s} : (b'_j, \beta'_j)_{1, q'}; \dots; (b_j^{(s)}, \beta_j^{(s)})_{1, q^{(s)}} \end{array} \right) \tag{1.9}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \dots \int_{L'_s} \xi(t_1, \dots, t_s) \prod_{i=1}^s \phi_i(t_i) z_i^{t_i} dt_1 \dots dt_s \tag{1.10}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}\Omega_i^{(k)}\pi$ , where

$$\Omega_i^{(k)} = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left( \sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \dots + \left( \sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left( \sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \dots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \tag{1.11}$$

where  $i = 1, \dots, s$

The complex numbers  $z_i$  are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form :

$$I(z_1, \dots, z_s) = O(|z_1|^{\alpha'_1}, \dots, |z_s|^{\alpha'_s}), \max(|z_1|, \dots, |z_s|) \rightarrow 0$$

$$I(z_1, \dots, z_s) = O(|z_1|^{\beta'_1}, \dots, |z_s|^{\beta'_s}), \min(|z_1|, \dots, |z_s|) \rightarrow \infty$$

where, with  $k = 1, \dots, s$  :  $\alpha'_k = \min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \dots, m_k$  and

$$\beta'_k = \max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \dots, n_k$$

We will use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \dots; p_{s-1}, q_{s-1}; V = 0, n_2; 0, n_3; \dots; 0, n_{s-1} \tag{1.12}$$

$$W = (p', q'); \dots; (p^{(s)}, q^{(s)}); X = (m', n'); \dots; (m^{(s)}, n^{(s)}) \tag{1.13}$$

$$A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k}); \dots; (a_{(s-1)k}, \alpha'_{(s-1)k}, \alpha''_{(s-1)k}, \dots, \alpha^{(s-1)}_{(s-1)k}) \tag{1.14}$$

$$B = (b_{2k}, \beta'_{2k}, \beta''_{2k}); \dots; (b_{(s-1)k}, \beta'_{(s-1)k}, \beta''_{(s-1)k}, \dots, \beta^{(s-1)}_{(s-1)k}) \tag{1.15}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{sk}, \alpha''_{sk}, \dots, \alpha^s_{sk}) : \mathfrak{B} = (b_{sk}; \beta'_{sk}, \beta''_{sk}, \dots, \beta^s_{sk}) \tag{1.16}$$

$$A' = (a'_k, \alpha'_k)_{1,p'}; \dots; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; B' = (b'_k, \beta'_k)_{1,q'}; \dots; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}} \tag{1.17}$$

The contracted form is :

$$I(z_1, \dots, z_s) = I_{U:p_s,q_s;W}^{V;0,n_s;X} \left( \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_s \end{matrix} \middle| \begin{matrix} A; \mathfrak{A}; A' \\ \\ \\ B; \mathfrak{B}; B' \end{matrix} \right) \tag{1.18}$$

Srivastava and Garg [6] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \dots, h_t} [z_1, \dots, z_t] = \sum_{R_1, \dots, R_t=0}^{h_1 R_1 + \dots + h_t R_t \leq L} (-L)_{h_1 R_1 + \dots + h_t R_t} B(E; R_1, \dots, R_t) \frac{z_1^{R_1} \dots z_t^{R_t}}{R_1! \dots R_t!} \tag{1.19}$$

the coefficients  $B(E; R_1, \dots, R_t)$  are arbitrary constants, real or complex.

### 2. S Generalized Gauss's hypergeometric function

The S generalized Gauss hypergeometric function  $F_p^{(\alpha, \beta; \tau, \mu)}(a, b, c; z)$  introduced and defined by Srivastava et al [4, page 350, Eq.(1.12)] is represented in the following manner :

$$F_p^{(\alpha, \beta; \tau, \mu)}(a, b, c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha, \beta; \tau, \mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \quad (|z| < 1) \tag{2.1}$$

provided that  $Re(p) \geq 0, \min Re(\alpha, \beta, \tau, \mu) > 0; Re(c) > Re(b) > 0$

where the S generalized Beta function  $B_p^{(\alpha, \beta; \tau, \mu)}(x, y)$  was introduced and defined by Srivastava et al [4, page 350, Eq( 1.13)]

$$B_p^{(\alpha, \beta; \tau, \mu)}(x, y) = \int_1^{x+y} t^{x-1} (1-t)^{y-1} {}_1F_1 \left( \alpha; \beta; \frac{-p}{t^\tau (1-t)^\mu} \right) dt \tag{2.2}$$

provided that  $Re(p) \geq 0, \min Re(x, y, \alpha, \beta) > 0; \min\{Re(\tau), Re(\mu)\} > 0$

### 3. Required integral

We note  $S(a, b, c)$ , the Selberg integral, see Askey et al ([1], page 402) by :

$$S(a, b, c) = \int_0^1 \dots \int_0^1 \prod_{i=1}^n x_i^{a-1} (1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} dx_1 \dots dx_n = \prod_{j=0}^{n-1} \frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{3.1}$$

with  $Re(a) > 0, Re(b) > 0, Re(c) > \max \left\{ -\frac{1}{n}, -\frac{Re(a)}{n-1}, -\frac{Re(b)}{n-1} \right\}$

We consider the new integral, see Askey et al ([1], page 402) defined by :

**Lemme**

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1} (1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} dx_1 \cdots dx_n = \prod_{i=1}^k \frac{(a + (n-i)c)}{(a + b + (2n-i-1)c)} S(a, b, c) \tag{3.2}$$

with  $Re(a) > 0, Re(b) > 0, Re(c) > Max \left\{ -\frac{1}{n}, -\frac{Re(a)}{n-1}, -\frac{Re(b)}{n-1} \right\}$  and  $k \leq n$

where  $S(a, b, c)$  is defined by (3.1). In this paper, we will denote the modified Selberg integral

**4. Main integral**

Let  $X_{u,v,w} = \prod_{i=1}^n x_i^u (1-x_i)^v \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2w}$  and

$$B_t = \frac{(-L)_{h_1 R_1 + \dots + h_t R_t} B(E; R_1, \dots, R_t)}{R_1! \cdots R_t!}$$

we have the following formula

**Theorem**

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1} (1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} F_p^{(\alpha, \beta; \tau, \mu)}(a, b, c; y X_{\alpha, \beta, \gamma}) S_L^{h_1, \dots, h_t} \left( \begin{matrix} y_1 X_{\alpha_1, \beta_1, \gamma_1} \\ \dots \\ y_t X_{\alpha_t, \beta_t, \gamma_t} \end{matrix} \right) N_{u:w}^{0, n; v} \left( \begin{matrix} z_1 X_{\delta_1, \psi_1, \phi_1} \\ \dots \\ z_r X_{\delta_r, \psi_r, \phi_r} \end{matrix} \right) I_{U:p_s, q_s; W}^{V; 0, n_s; X} \left( \begin{matrix} Z_1 X_{\epsilon_1, \eta_1, \zeta_1} \\ \dots \\ Z_s X_{\epsilon_s, \eta_s, \zeta_s} \end{matrix} \right) dx_1 \cdots dx_n = \sum_{R_1, \dots, R_t=0}^{h_1 R_1 + \dots + h_t R_t \leq L} \sum_{n'=0}^{\infty} \sum_{G_1, \dots, G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} (a)_{n'} \frac{B_p^{(\alpha, \beta; \tau, \mu)}(b+n', c-b)}{B(b, c-b)} \frac{(-)^{G_1 + \dots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} B_t$$

$$G(\eta_{G_1, g_1}, \dots, \eta_{G_r, g_r}) \frac{y^{n'}}{n'!} z_1^{\eta_{G_1, g_1}} \dots z_r^{\eta_{G_r, g_r}} y_1^{R_1} \dots y_t^{R_t} I_{U: p_s+3n+2k, q_s+2n+2k; W}^{V; 0, n_s+3n+2k; X} \left( \begin{array}{c|c} Z_1 & A; \\ \dots & \\ \dots & \\ Z_s & B; \end{array} \right)$$

$$[1-a-n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - j(c + \gamma'R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \epsilon_1 + j\zeta_1, \dots, \epsilon_s + j\zeta_s]_{0, n-1}$$

$$(-c-n'\gamma - \sum_{i=1}^t R_i \gamma_i - \sum_{i=1}^r \phi_i \eta_{G_i, g_i}; \zeta_1, \dots, \zeta_s), \dots,$$

$$[1-b-n'\beta - \sum_{i=1}^t R_i \beta_i - \sum_{i=1}^r \eta_{G_i, g_i} \psi_i - j(c + \gamma'R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \eta_1 + j\zeta_1, \dots, \eta_s + j\zeta_s]_{0, n-1}$$

$$(-c-n'\gamma - \sum_{i=1}^t \gamma_i R_i - \sum_{i=1}^r \phi_i \eta_{G_i, g_i}; \zeta_1, \dots, \zeta_s), B_1, B_2, B_3,$$

$$\left. \begin{array}{c} [-(j+1)(c+n'\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); (j+1)\zeta_1, \dots, (j+1)\zeta_s]_{0, n-1}, A_2, A_3, \mathfrak{A} : A' \\ \mathfrak{B} : B' \end{array} \right) \quad (4.1)$$

where  $B_1 = [1 - a - b - (\alpha + \beta)n' - \sum_{i=1}^t R_i(\alpha_i + \beta_i) - \sum_{i=1}^r (\delta_i + \phi_i)\eta_{G_i, g_i} - (n - 1 + j) \times$

$$(c + n'\gamma + \sum_{i=1}^t R_i \gamma_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \epsilon_1 + \eta_1 + j\zeta_1, \dots, \epsilon_s + \eta_s + j\zeta_s]_{0, n-1} \quad (4.2)$$

$$A_2 = [-a - n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - (n - j)(c + m\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i});$$

$$\epsilon_1 + (n - j)\zeta_1, \dots, \epsilon_s + (n - j)\zeta_s]_{1, k} \quad (4.3)$$

$$B_2 = [1 - a - n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - (n - j)(c + m\gamma + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i});$$

$$\epsilon_1 + (n - j)\zeta_1, \dots, \epsilon_s + (n - j)\zeta_s]_{1, k} \quad (4.4)$$

$$B_3 = [-a - n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - b - m\beta - \sum_{i=1}^t R_i \beta_i - \sum_{i=1}^r \eta_{G_i, g_i} \psi_i$$

$$\epsilon_1 + \eta_1 + (2n - j - 1)\zeta_1, \dots, (2n - j - 1)(c + m\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i});$$

$$\epsilon_s + \eta_s + (2n - j - 1)\zeta_s]_{1, k} \quad (4.5)$$

$$\begin{aligned}
 A_3 = & \left[ 1 - a - n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - b - m\beta - \sum_{i=1}^t R_i \beta_i - \sum_{i=1}^r \eta_{G_i, g_i} \psi_i \right. \\
 & \left. \epsilon_1 + \eta_1 + (2n - j - 1)\zeta_1, \dots, (2n - j - 1)(c + m\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \right. \\
 & \left. \epsilon_s + \eta_s + (2n - j - 1)\zeta_s \right]_{1,k} \tag{4.6}
 \end{aligned}$$

Provided that

a)  $\min\{\alpha, \beta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_j, \psi_j, \phi_j, \epsilon_l, \eta_l, \zeta_l\} > 0, i = 1, \dots, t, j = 1, \dots, r, l = 1, \dots, s,$

b)  $A = \operatorname{Re}\left[a + n'\alpha + \sum_{i=1}^r \delta_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^s \epsilon_i \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0$

c)  $B = \operatorname{Re}\left[b + n'\beta + \sum_{i=1}^r \psi_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^s \eta_i \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0$

d)  $C = \operatorname{Re}\left[c + n'\gamma + \sum_{i=1}^r \phi_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^s \zeta_i \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > \operatorname{Max}\left\{-\frac{1}{n}, -\frac{A}{n-1}, -\frac{B}{n-1}\right\}$

e)  $|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi$ , where  $A_i^{(k)}$  is defined by (1.5);  $i = 1, \dots, r$

f) The conditions (f) are satisfied and  $k \leq n$

g)  $|\arg Z_k| < \frac{1}{2} \Omega_i^{(k)} \pi$ , where  $\Omega_i^{(k)}$  is defined by (1.11);  $i = 1, \dots, s$

h) The series occurring on the right-hand side of (3.1) are absolutely and uniformly convergent.

i)  $(\operatorname{Re}(p) \geq 0, \min \operatorname{Re}(\alpha, \beta, \tau, \mu) > 0; \operatorname{Re}(c) > \operatorname{Re}(b) > 0; |z| < 1)$

The quantities  $A, B, \mathfrak{A}, \mathfrak{B}, A'$  and  $B'$  are defined by (1.16), (1.17), (1.18) and (1.19).

**Proof**

First, expressing the generalized the S generalized Gauss hypergeometric function  $F_p^{(\alpha, \beta; \tau, \mu)}(a, b, c; z)$  in serie with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables  $S_L^{h_1, \dots, h_t}[\cdot]$  with the help of equation (1.19) and the I-function of s variables in defined by Prasad [2] in Mellin-Barnes contour integral with the help of equation (1.10), changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now evaluating the resulting modified Selberg integral with the help of equation (3.2). Use the following relations  $\Gamma(a)(a)_n = \Gamma(a+n)$  and  $a = \frac{\Gamma(a+1)}{\Gamma(a)}$  several times with  $\operatorname{Re}(a) > 0$ . Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

5. Particular cases

1) If  $U = V = A = B = 0$ , the multivariable I-function defined by Prasad [2] reduces to multivariable H-function



defined by Srivastava et al [7]. We have the following result.

**Corollary 1**

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1} (1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} F_p^{(\alpha, \beta; \tau, \mu)}(a, b, c; y X_{\alpha, \beta, \gamma})$$

$$S_L^{h_1, \dots, h_t} \left( \begin{matrix} y_1 X_{\alpha_1, \beta_1, \gamma_1} \\ \dots \\ y_t X_{\alpha_t, \beta_t, \gamma_t} \end{matrix} \right) X_{u:v}^{0, n; w} \left( \begin{matrix} z_1 X_{\delta_1, \psi_1, \phi_1} \\ \dots \\ z_r X_{\delta_r, \psi_r, \phi_r} \end{matrix} \right) H_{p_s, q_s; W}^{0, n_s; X} \left( \begin{matrix} Z_1 X_{\epsilon_1, \eta_1, \zeta_1} \\ \dots \\ Z_s X_{\epsilon_s, \eta_s, \zeta_s} \end{matrix} \right) dx_1 \cdots dx_n$$

$$= \sum_{R_1, \dots, R_t=0}^{h_1 R_1 + \dots + h_t R_t \leq L} \sum_{n'=0}^{\infty} \sum_{G_1, \dots, G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} (a)_{n'} \frac{B_p^{(\alpha, \beta; \tau, \mu)}(b+n', c-b)}{B(b, c-b)} \frac{(-)^{G_1 + \dots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} B_t$$

$$G(\eta_{G_1, g_1}, \dots, \eta_{G_r, g_r}) \frac{y^{n'}}{n'!} z_1^{\eta_{G_1, g_1}} \cdots z_r^{\eta_{G_r, g_r}} y_1^{R_1} \cdots y_t^{R_t} H_{p_s+3n+2k, q_s+2n+2k; W}^{0, n_s+3n+2k; X} \left( \begin{matrix} Z_1 \\ \dots \\ Z_s \end{matrix} \right)$$

$$[1-a-n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - j(c + \gamma' R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \epsilon_1 + j\zeta_1, \dots, \epsilon_s + j\zeta_s]_{0, n-1}$$

$$\begin{matrix} \dots \\ (-c-n'\gamma - \sum_{i=1}^t R_i \gamma_i - \sum_{i=1}^r \phi_i \eta_{G_i, g_i}; \zeta_1, \dots, \zeta_s), \dots, \end{matrix}$$

$$[1-b-n'\beta - \sum_{i=1}^t R_i \beta_i - \sum_{i=1}^r \eta_{G_i, g_i} \psi_i - j(c + \gamma' R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \eta_1 + j\zeta_1, \dots, \eta_s + j\zeta_s]_{0, n-1}$$

$$\begin{matrix} \dots \\ (-c-n'\gamma - \sum_{i=1}^t \gamma_i R_i - \sum_{i=1}^r \phi_i \eta_{G_i, g_i}; \zeta_1, \dots, \zeta_s), B_1, B_2, B_3, \end{matrix}$$

$$\left. \begin{matrix} [-(j+1)(c+n'\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); (j+1)\zeta_1, \dots, (j+1)\zeta_s]_{0, n-1}, A_2, A_3, \mathfrak{A} : A' \\ \dots \\ \mathfrak{B} : B' \end{matrix} \right) \quad (5.1)$$

under the same notations and conditions that (4.1) with  $U = V = A = B = 0$

$$2) \text{ If } B(L; R_1, \dots, R_t) = \frac{\prod_{j=1}^{\bar{A}} (a_j)_{R_1 \theta'_j + \dots + R_t \theta_j^{(t)}} \prod_{j=1}^{B'} (b'_j)_{R_1 \phi'_j} \cdots \prod_{j=1}^{B^{(t)}} (b_j^{(t)})_{R_t \phi_j^{(t)}}}{\prod_{j=1}^{\bar{C}} (c_j)_{m_1 \psi'_j + \dots + m_t \psi_j^{(t)}} \prod_{j=1}^{D'} (d'_j)_{R_1 \delta'_j} \cdots \prod_{j=1}^{D^{(t)}} (d_j^{(t)})_{R_t \delta_j^{(t)}}} \quad (5.2)$$

then the general class of multivariable polynomial  $S_L^{h_1, \dots, h_t}[z_1, \dots, z_t]$  reduces to generalized Lauricella function defined by Srivastava et al [5].

$$F_{\bar{C}:D'; \dots; D^{(t)}}^{1+\bar{A}:B'; \dots; B^{(t)}} \left( \begin{matrix} z_1 \\ \dots \\ z_t \end{matrix} \middle| \begin{matrix} [(-L); R_1, \dots, R_t] [(a); \theta', \dots, \theta^{(t)}] : [(b'); \phi']; \dots; [(b^{(t)}); \phi^{(t)}] \\ [(c); \psi', \dots, \psi^{(t)}] : [(d'); \delta']; \dots; [(d^{(t)}); \delta^{(t)}] \end{matrix} \right) \quad (5.3)$$

We have the following formula

**Corollary 2**

$$\int_0^1 \dots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1} (1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} F_p^{(\alpha, \beta; \tau, \mu)}(a, b, c; yX_{\alpha, \beta, \gamma})$$

$$F_{\bar{C}:D'; \dots; D^{(t)}}^{1+\bar{A}:B'; \dots; B^{(t)}} \left( \begin{matrix} y_1 X_{\alpha_1, \beta_1, \gamma_1} \\ \dots \\ y_t X_{\alpha_t, \beta_t, \gamma_t} \end{matrix} \middle| \begin{matrix} [(-L); R_1, \dots, R_t] [(a); \theta', \dots, \theta^{(t)}] : [(b'); \phi']; \dots; [(b^{(t)}); \phi^{(t)}] \\ [(c); \psi', \dots, \psi^{(t)}] : [(d'); \delta']; \dots; [(d^{(t)}); \delta^{(t)}] \end{matrix} \right)$$

$$N_{u:w}^{0, n; v} \left( \begin{matrix} z_1 X_{\delta_1, \psi_1, \phi_1} \\ \dots \\ z_r X_{\delta_r, \psi_r, \phi_r} \end{matrix} \right) I_{U:p_s, q_s; W}^{V; 0, n_s; X} \left( \begin{matrix} z_1 X_{\epsilon_1, \eta_1, \zeta_1} \\ \dots \\ z_s X_{\epsilon_s, \eta_s, \zeta_s} \end{matrix} \right) dx_1 \dots dx_n = \sum_{R_1, \dots, R_t=0}^{h_1 R_1 + \dots + h_t R_t \leq L}$$

$$\sum_{n'=0}^{\infty} \sum_{G_1, \dots, G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \dots \sum_{g_r=0}^{m_r} \frac{(-)^{G_1 + \dots + G_r}}{\delta_{g_1} G_1! \dots \delta_{g_r} G_r!} G(\eta_{G_1, g_1}, \dots, \eta_{G_r, g_r})(a)_{n'} \frac{B_p^{(\alpha, \beta; \tau, \mu)}(b+n', c-b)}{B(b, c-b)} B'_t$$

$$\frac{y^{n'}}{n'!} z_1^{\eta_{G_1, g_1}} \dots z_r^{\eta_{G_r, g_r}} y_1^{R_1} \dots y_t^{R_t} I_{U:p_s+3n+2k, q_s+2n+2k; W}^{V; 0, n_s+3n+2k; X} \left( \begin{matrix} z_1 & | & A; \\ \dots & & \\ \dots & & \\ z_s & | & B; \end{matrix} \right)$$

$$[1-a-n'\alpha - \sum_{i=1}^t R_i \alpha_i - \sum_{i=1}^r \eta_{G_i, g_i} \delta_i - j(c + \gamma' R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \epsilon_1 + j\zeta_1, \dots, \epsilon_s + j\zeta_s]_{0, n-1}$$

$$(-c-n'\gamma - \sum_{i=1}^t R_i \gamma_i - \sum_{i=1}^r \phi_i \eta_{G_i, g_i}; \zeta_1, \dots, \zeta_s), \dots,$$

$$[1-b-n'\beta - \sum_{i=1}^t R_i \beta_i - \sum_{i=1}^r \eta_{G_i, g_i} \psi_i - j(c + \gamma' R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); \eta_1 + j\zeta_1, \dots, \eta_s + j\zeta_s]_{0, n-1}$$

$$(-c-n'\gamma - \sum_{i=1}^t \gamma_i R_i - \sum_{i=1}^r \phi_i \eta_{G_i, g_i}; \zeta_1, \dots, \zeta_s), B_1, B_2, B_3,$$

$$\left[ \begin{array}{c} -(j+1)(c+n'\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i \eta_{G_i, g_i}); (j+1)\zeta_1, \dots, (j+1)\zeta_s \\ \vdots \\ \mathfrak{B} : B' \end{array} \right]_{0, n-1, A_2, A_3, \mathfrak{A} : A'} \quad (5.4)$$

under the same conditions and notations that (4.1)

$$\text{and } B'_t = \frac{(-L)_{h_1 R_1 + \dots + h_t R_t} B(E; R_1, \dots, R_t)}{R_1! \dots R_t!}; \quad B(L; R_1, \dots, R_t) \text{ is defined by (5.2)}$$

## 6. Conclusion

In this paper we have evaluated a modified Selberg integral involving the product of the multivariable  $I$ -function defined by Prasad [2], the multivariable Aleph-function, the S generalized Gauss's hypergeometric function and a general class of polynomials of several variables. The integral established in this paper is of very general nature as it contains multivariable I-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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