Catenation of Permutations on Caterpillars

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Abstract: ***Pnueli Lempel and Even introduced the concept of Permutation Graphs in 1971.*** *If i, j belong to a permutation on p symbols {1, 2, …, p} and i is less than j then there is an edge between i and j in the permutation graph if i appears after j in the sequence of permutation. The permutations realizing path, complete graph, bipartite graph, tripartite graph, double star, wind mill and elongated happy man had been characterized in the first phase of our research. The notion of extending a permutation by ConCatenating with other permutations was introduced by us earlier. This has persuaded to see the permutation of certain caterpillars as Catenation of permutations which is described in this paper.*

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I. INTRODUCTION

The permutation graph is introduced by Pnueli, Lempel and Even in 1971[1]. The idea of extending a permutation by finite number of elements is conceived by us which further led us to extend a permutation by ConCatenating permutations at either or both sides[7]. In the course of ConCatenating a permutation of a path with other permutations remarkably we came across some special types of caterpillars. The corresponding permutation is further decomposed into a product of cyclic permutations. We have characterised permutations realising some standard graphs earlier. In this paper we have proved that Catenating the permutation of a path with permutations of a star realise special types of caterpillars.

Section I contains an introduction. Section II includes the preliminary definitions required for proving the results in ConCatenating the permutations. Section III comprises of the results on permutations realising a path with even and odd number of elements Catenated with permutations realising stars.

II. PRELIMINARIES

Definition:2.1: Let π be a permutation on p symbols V={$a_1,a_2,\dots,a_p$} where $\pi(a_i) = a_i'$, $1\le i \le p$ and $|a_{i+1} - a_i| = c$, $c > 0$, $1\le i < p$. Then the sequence of permutation s(π) is {$a_1', a_2',\dots,a_p'$}. When the elements are ordered in a line L_1 and the elements of s(π) are ordered in a lineL_2 parallel to L_1, then a line joining a_i in L_1 and a_i in L_2 is known as line representation of a_i in π and is denoted by l_i [2]. The **Residue of a_i and a_j** is denoted by Res(a_i,a_j) and is given by $Res(a_i,a_j)=(a_i - a_j)(\pi^{-1}(a_i) - \pi^{-1}(a_j))$. Then l_i and l_j cross each other if Res(a_i,a_j) < 0. The neighbourhood of a_i in π is $N_\pi(a_i)$ = {$a_j \in V$/ Res(a_i,a_j) < 0} and $d_\pi(a_i) = |N_\pi(a_i)|$ is the number of lines that cross l_i in π.[3]

Definition: 2.2: Let π be a permutation on a finite set V = {$a_1,a_2,\dots,a_p$} given by $\pi = \begin{pmatrix} a_1 & a_2 \dots a_p \\ a_1' & a_2' \dots a_p' \end{pmatrix}$ where $|a_{i+1} - a_i| = c,\ c>0,$ $1 \le i < p-1$. Then the **permutation graph** $G_\pi = (V_\pi, E_\pi)$ where V_π = V and $a_i a_j \in E_\pi$ if Res (a_i ,a_j) < 0. A graph G is a permutation graph if there exists π such that $G_\pi \cong G$. That is a graph G is a permutation graph if it is realisable by a permutation π. Otherwise it is not a permutation graph [3]

Note: C_n, n ≥ 5, are not permutation graphs. [3]

Definition: 2.3: A **Caterpillar** is a tree T such that the removal of all pendant vertices leaves a path, P_k:($u_1, u_2, \dots, u_k$), which is called the spine of T. For each i, 1 ≤ i ≤ k, if v_i is the finite number of pendant vertices of T that are adjacent to the vertex u_i, then the caterpillar T can be represented by the finite sequence ($v_1, v_2, \dots, v_k$). Each v_i is non-negative and $v_1 > 0$, $v_k > 0$.[4]

Definition:2.4: The **double star** S(n,m), n ≥ 0, m ≥ 0, is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with the line joining their centres. The double star has a path P_2 joining the centres of the n-star and m-star. The Double Star is a Caterpillar T: (n,m). The generalised form S(n,m:k) has a path P_k joining the n-star and m-star.[6] S(n,m:k) is also known as **Elongated Double Star (EDS)** which is a Caterpillar (n,0,0,…,0,m)**.** When n = m = k = 2, then S(2,2:2) is the **Happy Man** and when n = m = 2, k ≥ 3 then S(2,2:k) is known as **Elongated Happy Man** [8]. S(n,0:k) is known as a **Coconut Tree**[7].

Theorem 2.5: Let π be a permutation on p symbols V = {$a_1,a_2,\dots,a_p$} given by $\pi(a_i)=a_i'$,$1\le i \le p$, where $|a_{i+1} - a_i| = c$, c > 0. (i.e) s(π) = {$a'_1,a'_2,\dots, a'_p$}. If π realises a path then π^R (Restructured Permutation), where $s(\pi^R)$ = is {$a'_2, a'_3, a'_1,a'_4,a'_5,\dots, a'_{p-4},a'_{p-1},a'_{p-}$

$_2$,a'$_p$,a'$_{p-3}$} for odd p and s(π^R) = {a'$_2$, a'$_3$, a'$_1$, a'$_4$,a'$_5$,…,a'$_{p-3}$, a'$_p$, a'$_{p-2}$, a'$_{p-1}$} for even p, realise the Elongated Happy Man S(2,2:p-4).[8]

Theorem 2.6: Let π be a permutation on p symbols V = {$a_1,a_2,\ldots,a_p$} given by $\pi(a_i)=a_i'$, $1\le i\le p$, where $|a_{i+1} - a_i| = c$, c > 0. (i.e) s(π) = {a'$_1$,a'$_2$,…, a'$_p$}. If π realises a path then $\pi^R = \pi_2\pi\,\pi_1$, where π_1 = (a_{p-3} a_p a_{p-1}), even p (or) π_1 = (a_{p-2} a_{p-1} a_p), odd p and π_2 =($a_1a_2a_3$).[8]

III. PERMUTATIONS EXTENDED BY FINITE NUMBER OF ELEMENTS

Definition 3.1: Let π be a connected permutation on a finite set V = {$a_1,a_2,\ldots,a_p$} such that $|a_{i+1}- a_i|$ = c, c > 0, 1 ≤ i < p. **Permutation extended by r elements** a_{p+1}, a_{p+2}, …a_{p+r} where r is finite, denoted by π_{r^+} is given by (i) $\pi_{r^+}(a_{p+r})=a_m$ (ii) $\pi_{r^+}(a_n)=a_{p+r}$ (iii) $\pi_{r^+}(a_i)=\pi_{(r-1)^+}(a_i)$, 1 ≤ m,n < p, provided $G_{\pi_{r^+}}$ is connected[6].

Definition 3.2: An extension of a permutation π by r elements π_{r^+} is said to be ***similar*** if the extension preserves the structure of $G_{\pi_{r^+}}$ containing G_π[6].

We have stretched this idea of extending a permutation by finite number of elements to extending a permutation by another permutation either or both sides of a permutation. The operation is named as ***ConCatenation of Permuations.*** The definitions are as follows:

Definition 3.3: Let π be a connected permutation on a finite set V ={$a_1,a_2,\ldots,a_p$} such that $|a_{i+1}- a_i|$ = k, k > 0, 1 ≤ i < p and $\pi(a_i) = a'_i$, 1 ≤ i ≤ p; π_1 be a connected permutation on a finite set V_1 = {$b_1, b_2, \ldots, b_m$} such that $|b_{i+1}- b_i|$ = k, k > 0, 1 ≤ i < m and $\pi_1(b_i) = b'_i$, 1 ≤ i ≤ m and π_2 be a connected permutation on a finite set V_2 = {$c_1, c_2, \ldots, c_n$} such that $|c_{i+1}- c_i|$ = k, k> 0, 1 ≤ i < n and $\pi_2(c_i) = c'_i$, 1 ≤ i ≤ n. Then the (i) π ConCatenated with π_1 at left is known as **Left ConCatenation** of π, denoted by **ConCatπ(m[p])**. This is obtained by interchanging b'$_m$ under π_1 with a'$_1$ under π in s(ConCat π(m[p])), where s(ConCat π(m[p])) is the image sequence of ConCat π(m[p]) given by {b'$_1$, b'$_2$, … ,b'$_{m-1}$, a'$_1$, b'$_m$, a'$_2$, …, a'$_p$}.(ii))π ConCatenated with π_2 at right is known as **Right ConCatenation** of π, denoted by **ConCatπ([p]n)**. This is obtained by interchanging a'$_p$ under π with c'$_1$ under π_2 in s(ConCat π([p]n)), where s(ConCat π([p]n)) is the image sequence of ConCat π([p]n) given by {a'$_1$, a'$_2$, …, a'$_{p-1}$, c'$_1$, a'$_p$, c'$_2$, …, c'$_n$ }.
(iii) The connected permutation π can be extended at both sides by ConCatenating π_1 at left and π_2 at right as mentioned above, will be denoted by **ConCat π(m[p]n)**. If the π realises P_p then the ConCatenation of permutations π_1 and π_2 either / both side of π is known as **Catenation** and denoted by **Catπ(m[p])**, **Catπ([p]n)** and **Catπ(m[p]n)** respectively.[7]

IV. CATERPILLARS AS CATENATION OF PERMUTATIONS

Theorem:4.1 *Let π be a connected permutation on a finite set V ={$a_1,a_2,\ldots,a_p$} such that $|a_{i+1}- a_i|$ = k, k > 0, 1 ≤ i < p and $\pi(a_i) = a'_i$, 1 ≤ i ≤ p realising a path. Let π_1 be a permutation on {$b_1,b_2,b_3,\ldots,b_m$} such that $|b_{i+1}- b_i|$ = k, k > 0, 1 ≤ i < m realising a star $K_{1,m-1}$and π_3 on {$c_1,c_2,c_3, \ldots, c_n$} such that $|c_{i+1}- c_i|$ = k, k > 0, 1 ≤ i < n realising a star $K_{1,n-1}$. Then π Catenated with π_1 or π_3 realsie a Caterpillar.*

Proof: Given π realise a path with p elements. Then

$$\pi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{p-1} & a_p \\ a_2 & a_4 & a_1 & a_6 & \cdots & a_p & a_{p-2} \end{pmatrix},$$

odd p and,

$$\pi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{p-1} & a_p \\ a_2 & a_4 & a_1 & a_6 & \cdots & a_{p-3} & a_{p-1} \end{pmatrix}$$

even p.

Given π_1 on {$b_1,b_2,b_3,\ldots,b_m$} realise a star $K_{1,p-1}$. Then

$$\pi_1 = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \cdots & b_{m-1} & b_m \\ b_2 & b_3 & b_4 & b_5 & \cdots & b_m & b_1 \end{pmatrix}$$

= ($b_1b_2b_3 \ldots b_m$)

$$\text{(or)} \quad = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \cdots & b_{m-1} & b_m \\ b_m & b_1 & b_2 & b_3 & \cdots & b_{m-2} & b_{m-1} \end{pmatrix}$$

= ($b_1b_mb_{m-1} \ldots b_2$).

Let us denote the former permutation on m elements realising a star $K_{1,m-1}$ as π_1 and the later as π_2. Similarly let π_3 and π_4 be permutations on n elements realising a star $K_{1,n-1}$ given by

$$\pi_3 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_{n-1} & c_n \\ c_2 & c_3 & c_4 & c_5 & \cdots & c_n & c_1 \end{pmatrix}$$

= ($c_1c_2c_3 \ldots c_n$) (or)

$$\pi_4 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_{n-1} & c_n \\ c_n & c_1 & c_2 & c_3 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}$$

= ($c_1c_nc_{n-1} \ldots c_2$).

(A) Let π be Catenated with π_1 or π_2 at left or at right.

Case(i): Let p be even

(a) Let π be catenated with π_1 at left. π realises a path P_p = (2,1,4,3,6,5,...,p,p-1) and π_1 realises $K_{1,m-1}$. Then Catπ(m[p]) has p+m elements. Here $d_\pi(a_i)$ = 1, i = 2,p-1; $d_\pi(a_i)$ = 2, i = 1,3,4, ..., p-2,p; $d_{\pi_1}(b_i) = 1$, i = 2,3,...,m and $d_{\pi_1}(b_1) = m-1.$

π is Catenated with π_1 at left.

$$Cat\,\pi(m[p]) =$$

Then $\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \dots & b_{m-1} & b_m & a_1 & a_2 & a_3 & a_4 & \dots & a_{p-1} & a_p \\ b_2 & b_3 & b_4 & b_5 & \dots & b_m & a_2 & b_1 & a_4 & a_1 & a_6 & \dots & a_{p-3} & a_{p-1} \end{pmatrix}$

Here the lines of a_2 and b_1 cross. Here $Res(b_1,a_2) < 0$ and all other crossing of lines remain unaltered. Hence $d_{Cat\pi(m[p])}(a_2) = 2$ and $d_{Cat\pi(m[p])}(b_1) = m$. Therefore $d_{Cat\pi(m[p])}(a_{p-1}) = 1$; $d_{Cat\pi(m[p])}(a_i) = 2$, $i = 1,2,3,4, \dots, a_{p-2}, a_p$; $d_{Cat\pi(m[p])}(b_i) = 1$, $i = 2,3,\dots,m$. Here b_1 is a support of m-1 elements.

$$\sum_{i=1}^{p} d_{Cat\pi(m[p])}(a_i) + \sum_{j=1}^{m} d_{Cat\pi(m[p])}(b_j)$$

= m+m+2(p-1) = 2(p+m-1). Therefore $Cat\pi(m[p])$ is a tree. Hence $Cat\pi(m[p])$ realise a caterpillar of the form (m-1,0,0,0, ..., 1) whose spine is P_p. -----(4.1.1)

(b) Let π be catenated with π_2 at left. π realises a path P_p = (2,1,4,3,6,5,...,p,p-1) and π_2 realises $K_{1,m-1}$. Then $Cat\pi(m[p])$ has p+m elements. Here $d_\pi(a_i) = 1$, i = 2,p-1; $d_\pi(a_i) = 2$, i = 1,3,4, ..., p-2,p;

$d_{\pi_2}(b_i) = 1$, i = 1,2,3,...,m-1 and

$d_{\pi_2}(b_m) = m-1$. π is Catenated with π_2 at left.

Then

$$Cat\,\pi(m[p]) =$$

$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \dots & b_{m-1} & b_m & a_1 & a_2 & a_3 & a_4 & \dots & a_{p-1} & a_p \\ b_m & b_1 & b_2 & b_3 & \dots & b_{m-2} & a_2 & b_{m-1} & a_4 & a_1 & a_6 & \dots & a_{p-3} & a_{p-1} \end{pmatrix}$

Here the lines of a_2 and b_{m-1} cross. Here $Res(a_2,b_{m-1}) < 0$ and all other crossing of lines remain unaltered. Hence $d_{Cat\pi(m[p])}(a_2) = 2$ and $d_{Cat\pi(m[p])}(b_{m-1}) = 2$. Therefore $d_{Cat\pi(m[p])}(a_{p-1}) = 1$; $d_{Cat\pi(m[p])}(a_i) = 2$, i = 1,2,3,4, ..., p-2,p; $d_{Cat\pi(m[p])}(b_i) = 1$, i =1, 2,3,...,m-2. Here b_m is a support of m-1 elements.

$$\sum_{i=1}^{p} d_{Cat\pi(m[p])}(a_i) + \sum_{j=1}^{m} d_{Cat\pi(m[p])}(b_j) =$$

m-1+m-1+2p = 2(p+m-1). Therefore $Cat\pi(m[p])$ is a tree. Hence $Cat\pi(m[p])$ realise a caterpillar of the form (m-2,0,0,0, ...,1) whose spine is P_{p+1}. ----(4.1.2)

(c) Let π be catenated with π_1 at right. π realises a path P_p = (2,1,4,3,6,5,...,p,p-1) and π_1 realises $K_{1,m-1}$. Then $Cat\pi([p]m)$ has p+m elements. Here $d_\pi(a_i) = 1$, i = 2,p-1; $d_\pi(a_i) = 2$, i = 1,3,4, ..., p-2,p;

$d_{\pi_1}(b_i) = 1$, i = 2,3,...,m and $d_{\pi_1}(b_1) = m-1$.

π is Catenated with π_1 at right. Then

$$Cat\,\pi([p]m) =$$

$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_{p-1} & a_p & b_1 & b_2 & b_3 & b_4 & \dots & b_{m-1} & b_m \\ a_2 & a_4 & a_1 & a_6 & \dots & a_{p-3} & b_2 & a_{p-1} & b_3 & b_4 & b_5 & \dots & b_m & b_1 \end{pmatrix}$

Here the lines of a_{p-1} and b_2 cross. Here $Res(b_2,a_{p-1}) < 0$ and all other crossing of lines remain unaltered. Hence $d_{Cat\pi([p]m)}(a_2) = 1$ and $d_{Cat\pi([p]m)}(a_i) = 2$, i =1,3,4,…, p-2,p-1,p. Therefore $d_{Cat\pi([p]m)}(b_2) = 2$; $d_{Cat\pi([p]m)}(b_i) = 1$, i = 3,4, ..., m and $d_{Cat\pi([p]m)}(b_1) = m$. Here b_1 is a support of m-1 elements.

$$\sum_{i=1}^{p} d_{Cat\pi([p]m)}(a_i) + \sum_{j=1}^{m} d_{Cat\pi([p]m)}(b_j) =$$

m-1+m-1+2p = 2(p+m-1). Therefore $Cat\pi([p]m)$ is a tree. Hence $Cat\pi([p]m)$ realise a caterpillar of the form (1,0,0,0, ..., m-2) whose spine is P_{p+1}-----(4.1.3)

(d) Let π be Catenated with π_2 at right. π realises a path P_p = (2,1,4,3,6,5,...,p,p-1) and π_2 realises $K_{1,m-1}$. Then $Cat\pi([p]m)$ has p+m elements.

$d_{\pi_2}(b_i) = 1$, i = 2,3,...,m and

$d_{\pi_2}(b_1) = m-1$. π is Catenated with π_2 at right.

$$Cat\,\pi([p]m) =$$

Then $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_{p-1} & a_p & b_1 & b_2 & b_3 & b_4 & \dots & b_{m-1} & b_m \\ a_2 & a_4 & a_1 & a_6 & \dots & a_{p-3} & b_m & a_{p-1} & b_1 & b_2 & b_3 & \dots & b_{m-2} & b_{m-1} \end{pmatrix}$

Here the lines of a_{p-1} and b_m cross. Here $Res(b_m,a_{p-1}) < 0$ and all other crossing of lines remain unaltered. Hence $d_{Cat\pi([p]m)}(a_2) = 1$; $d_{Cat\pi([p]m)}(a_i) = 2$, i =1,3,4,…, p-2,p-1,p; $d_{Cat\pi([p]m)}(b_i) = 1$, i = 1,2,3,4, ..., m-1 and $d_{Cat\pi(m[p])}(b_m) = m$. Here b_m is a support of m-1 elements.

$$\sum_{i=1}^{p} d_{Cat\pi([p]m)}(a_i) + \sum_{j=1}^{m} d_{Cat\pi([p]m)}(b_j) =$$

m+m+2(p-1) = 2(p+m-1). Therefore $Cat\pi([p]m)$ is a tree. Hence $Cat\pi([p]m)$ realise a caterpillar of the form (1,0,0,0, ..., m-1) whose spine is P_p. -----(4.1.4)

Case(ii) Let p be odd.

Let π be catenated with (a) π_1 and (b) π_2 at left.

The proof for (a) and (b) for odd p is the same as for even p. The $Cat\pi(m[p])$ and $Cat\pi([p]m)$ realise Caterpillars (m-1,0,0,0,...,1) whose spine is P_p(4.2.1) and (m-2,0,0, ...,1) and P_{p+1} ----- (4.2.2) respectively.

(c) Let π be catenated with π_1 at right. π realises a path P_p = (2,1,4,3,6,5,...,p-1,p-2,p) and π_1 realises $K_{1,m-1}$. Then $Cat\pi([p]m)$ has p+m elements. Here $d_\pi(a_i) = 1$, i=2,p; $d_\pi(a_i) = 2$, i = 1,3,4, ..., p-2, p-1;

$d_{\pi_1}(b_i) = 1$, i = 2,3,...,m and $d_{\pi_1}(b_1) = m-1$.

π is Catenated with π_1 at right. Then

$$Cat\pi([p]m) =$$

$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_p & b_1 & b_2 & b_3 & b_4 & \dots & b_{m-1} & b_m \\ a_2 & a_4 & a_1 & a_6 & \dots & a_p & b_2 & a_{p-2} & b_3 & b_4 & b_5 & \dots & b_m & b_1 \end{pmatrix}$

Here the lines of a_{p-2} and b_2 cross. Here $Res(b_2,a_{p-2}) < 0$ and all other crossing of lines remain unaltered. Hence $d_{Cat\pi([p]m)}(a_{p-2}) = 3$, $d_{Cat\pi([p]m)}(a_i) = 1$, i = 2,p; $d_{Cat\pi([p]m)}(a_i) = 2$, i =1,3,4,…,p-1; $d_{Cat\pi([p]m)}(b_2)=2$;

$d_{Cat\pi([p]m)}(b_i) = 1$, i =1,3,4, ..., m and $d_{Cat\pi([p]m)}(b_1) = m-1$. Here b_1 is a support of m-2 elements.

$$\sum_{i=1}^{p} d_{Cat\pi([p]m)}(a_i) + \sum_{j=1}^{m} d_{Cat\pi([p]m)}(b_j)$$

= m+m-1+3+2(p-2) = 2(p+m-1). Therefore Catπ([p]m) is a tree. Hence Catπ([p]m) realise a caterpillar of the form (1,0,0,0, ...,0,1,0, m-2) whose spine is P_p. ------------------------(4.2.3)

(d) Let π be Catenated with π_2 at right. π realises a path P_p = (2,1,4,3,6,5,...,p-1,p-2,p) and π_2 realises $K_{1,m-1}$. Then Catπ([p]m) has p+m elements.

$d_{\pi_2}(b_i) = 1$, i = 2,3,...,m and

$d_{\pi_2}(b_1) = m-1$. π is Catenated with π_2 at right.

Then
$$Cat\,\pi([p]m) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{p-1} & a_p & b_1 & b_2 & b_3 & b_4 & \cdots & b_{m-1} & b_m \\ a_2 & a_4 & a_1 & a_6 & \cdots & a_p & b_m & a_{p-2} & b_1 & b_2 & b_3 & \cdots & b_{m-2} & b_{m-1} \end{pmatrix}$$

Here the lines of a_{p-2} and b_m cross. Here $Res(b_m,a_{p-2}) < 0$ and all other crossing of lines remain unaltered. Hence $d_{Cat\pi([p]m)}(a_i) = 1$, i =2,p; $d_{Cat\pi([p]m)}(a_i) = 2$, i =1,3,4,…,p-3,p-1; $d_{Cat\pi([p]m)}(a_{p-2}) = 3$; $d_{Cat\pi([p]m)}(b_i) = 1$, i = 1,2,3,4, ..., m-1 and $d_{Cat\pi([p]m)}(b_m) = m$. Here b_m is a support of m-1 elements.

$$\sum_{i=1}^{p} d_{Cat\pi([p]m)}(a_i) + \sum_{j=1}^{m} d_{Cat\pi([p]m)}(b_j) =$$

m+1+m+3+2(p-3) = 2(p+m-1). Therefore Catπ([p]m) is a tree. Hence Catπ([p]m) realise a caterpillar of the form (1,0,0,0, ...,0,1, m-1) whose spine is P_{p-1}. --------------------(4.2.4)

(B) Let π be Catenated with π_1 or π_2 or π_3 or π_4 at left and π_1 or π_2 or π_3 or π_4 at right

(i) Let p be even..

(a) Let us consider that π is Catenated with π_1 at left and π_3 at right. The other cases can be similarly discussed.

Given π realise a path with p elements. Then

$$\pi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{p-1} & a_p \\ a_2 & a_4 & a_1 & a_6 & \cdots & a_p & a_{p-2} \end{pmatrix},$$

odd p

$$\pi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{p-1} & a_p \\ a_2 & a_4 & a_1 & a_6 & \cdots & a_{p-3} & a_{p-1} \end{pmatrix}$$

even p.

and $\pi_1 = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \cdots & b_{m-1} & b_m \\ b_2 & b_3 & b_4 & b_5 & \cdots & b_m & b_1 \end{pmatrix}$

$= (b_1b_2b_3 \dots b_m)$

$$\pi_3 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_{n-1} & c_n \\ c_2 & c_3 & c_4 & c_5 & \cdots & c_n & c_1 \end{pmatrix}$$

$= (c_1c_2c_3 \dots c_n)$

Then $Cat\pi(m[p]n) =$

$$\begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m & a_1 & a_2 & \cdots & a_{p-1} & a_p & c_1 & c_2 & \cdots & c_{n-1} & c_n \\ b_2 & b_3 & b_4 & \cdots & b_m & a_2 & b_1 & a_4 & \cdots & a_{p-3} & c_2 & a_{p-1} & c_3 & \cdots & c_n & c_1 \end{pmatrix}$$

Here $d_{Cat\pi(m[p]n)}(b_1) = m$; $d_{Cat\pi(m[p]n)}(c_1) = n-1$; $d_{Cat\pi(m[p]n)}(b_i) = 1$, i = 2,3,4, ..., m; $d_{Cat\pi(m[p]n)}(c_i) = 1$, i = 3,4, ..., n; $d_{Cat\pi(m[p]n)}(a_i) = 2$, i =1,2,3,4, ..., p and $d_{Cat\pi(m[p]n)}(c_2) = 2$.

$$\sum_{i=1}^{p} d_{Cat\pi(m[p]n)}(a_i) + \sum_{j=1}^{m} d_{Cat\pi(m[p]n)}(b_j) +$$

$$\sum_{k=1}^{n} d_{Cat\pi(m[p]n)}(c_k) = m+n-1+m-1+n-2+2(p+1)$$

= 2(p+m+n-1).

Catπ(m[p]n) has p+m+n elements. Therefore Catπ(m[p]n) is a tree and realises a Caterpillar of the form (m-1,0,0,…,0,n-2) whose Spine is P_{p+3}.

(b) Let π be Catenated with π_1 at left and π_4 at right. Then Catπ(m[p]n) realises a Caterpillar of the form (m-1,0,0,…,0,n-1) whose Spine is P_{p+2}.

(c) If π is Catenated with π_2 at left and π_3 at right, then Catπ(m[p]n) realises a Caterpillar of the form (m-2,0,…,0,n-2) whose Spine is P_{p+4}.

(d) If π is Catenated with π_2 at left and π_4 at right then Catπ(m[p]n) realises a Caterpillar of the form (m-2,0,…,0,n-1) whose Spine is P_{p+3}.

By Definition **Catπ(m[p]n) is a EDS** when p is even.

(ii) Let p be odd.

Analogous proof can be given for odd p as in the previous case. The results are as follows:

(a) Let π be Catenated with π_1 at left and π_3 at right. Then Catπ(m[p]n) realises a Caterpillar of the form (m-1,0,0,…,0,1,0,n-2) whose Spine is P_{p+2}.

(b) Let π be Catenated with π_1 at left and π_4 at right. Then Catπ(m[p]n) realises a Caterpillar of the form (m-1,0,0,…,0,1,n-1) whose Spine is P_{p+1}.

(c) If π is Catenated with π_2 at left and π_3 at right, then Catπ(m[p]n) realises a Caterpillar of the form (m-2,0,…,0,1,0,n-2) whose Spine is P_{p+3}.

(d) If π is Catenated with π_2 at left and π_4 at right then Catπ(m[p]n) realises a Caterpillar of the form (m-2,0,…,0,1,n-1) whose Spine is P_{p+2}.

Hence the theorem.

Theorem 4.2: Let π be a connected permutation on a finite set, **p even,** V ={$v_1,v_2,...,v_p$} such that $|v_{i+1}- v_i| = k$, k > 0, 1 ≤ i < p realising a path. Let π_1 be a permutation on {b_1,b_2,b_3}such that $|b_{i+1}- b_i| = k > 0$, 1 ≤ i < 3 given by $(b_1b_2b_3)$ and π_2 be a permutation on {c_1,c_2,c_3} such that $|c_{i+1}- c_i| = k > 0$, 1 ≤ i < 3 given by $(c_1c_3c_2)$. Let π_3,π_4 and π_5 be permutations on {$a_1,a_2,…,a_{p+6}$}, such that $|a_{i+1} – a_i| = k > 0$, 1 ≤ i < p, where π_4 realises a path $\pi_3 = (a_{p+3}\ a_{p+6}\ a_{p+5})$ and $\pi_5 = (a_1a_2a_3)$. If π *Catenated with π_1 at* *left and π_2 at right,* then Catπ(3[p]3) $\cong_s$ π_4^R $\cong_s$ $\pi_5\pi_4\pi_3$.

Proof: *The permutation of a Path with p+6 elements when restructured realises a S(2,2:p+2), by Theorem 2.5. By Theorem 2.6,* π_4^R *is expressed as a product of cyclic permutations as* $\pi_5\pi_4\pi_3$. *When π Catenated with π_1 at left and π_2 at right, m=n=3 Catπ(m[p]n) realises S(2,2:p+2) when p is even by Theorem 4.1.* Hence *Catπ(m[p]n)* $\cong_s$ π_4^R $\cong_s$ $\pi_5\pi_4\pi_3$.

Result : Since $\pi \cong_s \pi^{-1}$, $(\pi_4^R)^{-1} \cong_s$ *Catπ*(m[p]n).

V. Conclusions

The curiosity in applying the properties of permutations with graph theoretic perspective drives us to various avenues in Permutation Graphs. Many more results are in progress.

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