Some New Families of Sum Divisor Cordial Graphs

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Abstract - *A sum divisor cordial labeling of a graph G with vertex set V is a bijection f from V(G) to {1,2,…, |V(G)|} such that an edge uv is assigned the label 1 if 2 divides f(u)+f(v) and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we investigate the sum divisor cordial labeling of switching of a pendent vertex in path P_n, switching of any vertex in cycle C_n, $DS(B_{n,n})$, $G*K_{2,n}$ and $G*K_{3,n}$.*

Keywords - *Divisor cordial labeling, sum divisor cordial labeling, sum divisor cordial graph.*

I. INTRODUCTION

We begin with simple, finite, undirected graph G = (V(G),E(G)) with p vertices and q edges. For standard terminology and notations related to graph theory we refer to Chartrand [2] while for number theory we refer to Burton [3]. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [4]. The concept of cordial labeling was introduced by Cahit [1]. Varatharajan et al.[6] introduced the concept of divisor cordial labeling of graphs. The concept of sum divisor cordial labeling was introduced by Lourdusamy et al.[5]. The present work is focused on some new families of sum divisor cordial graphs. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition :1.1

A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.

Definition :1.2

A mapping f :V(G) → {0,1} is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling f* : E(G) → {0,1} is given by f*(e) = |f(u) − f(v)|. Then $v_f(i)$ = number of vertices of having label i under f and $e_f(i)$ = number of edges of having label i under f*.

Definition :1.3

A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. A graph G is cordial if it admits cordial labeling.

Definition :1.4

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ka. It is denoted by a | b. If a does not divide b, then we denote $a \nmid b$.

Definition :1.5

Let G = (V(G), E(G)) be a simple graph and f : V(G)→{1,2,...,|V(G)|} be a bijection. For each edge uv, assign the label 1 if f(u)|f(v) or f(v)|f(u) and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \le 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition :1.6

Let G = (V (G), E(G)) be a simple graph and f : V(G)→{1,2,…, |V(G)|} be a bijection. For each edge uv, assign the label 1 if 2|(f(u)+f(v)) and the label 0 otherwise. The function f is called a sum divisor cordial labeling if $|e_f(0) - e_f(1)| \le 1$. A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.

Definition :1.7

A vertex switching G_v of a graph G is obtained by taking a vertex v of G, removing the entire edges incident with v and adding edges joining v to every vertex which are not adjacent to v in G.

Definition :1.8

Let G = (V(G), E(G)) be a graph with vertex set $V = S_1 \cup S_2 \cup \ldots \cup S_i \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of S_i for $1 \le i \le t$.

II. MAIN THEOREMS

Theorem : 2.1

Given a positive integer n, there is a sum divisor cordial graph G which has n vertices.

Proof.

Case 1 : Suppose n is even.

Construct a graph G of n vertices $v_1, v_2, \ldots, v_n$ in the following manner. Form a path containing $\frac{n}{2}$ vertices $v_1, v_2, \ldots, v_{\frac{n}{2}}$ and attach $\frac{n}{2}$ vertices $v_{\frac{n}{2}+1}$, $v_{\frac{n}{2}+2}$, ..., v_n to the vertex v_1.

Define $f : V(G) \to \{1,2,\ldots,n\}$ as follows

$$f(v_i) = 2i - 1 \quad \text{for } 1 \le i \le \frac{n}{2}$$

$$f(v_{\frac{n}{2}+i}) = 2i \quad \text{for } 1 \le i \le \frac{n}{2}$$

Then, the induced edge labeling f* is as follows.

$$f^*(v_i v_{i+1}) = 1 \quad \text{for } 1 \le i \le \frac{n}{2} - 1$$

$$f^*(v_1 v_i) = 0 \quad \text{for } \frac{n}{2}+1 \le i \le n$$

Thus, $e_f(0) = \frac{n}{2}$ and $e_f(1) = \frac{n}{2} - 1$.

Hence $| e_f(0) - e_f(1) | \le 1$.

Therefore, the resultant graph G is sum divisor cordial for n is even.

Case 2 : Suppose n is odd.

Construct a graph G of n vertices $v_1, v_2, \ldots, v_n$ in the following manner. Form a path containing $\frac{n+1}{2}$ vertices $v_1, v_2, \ldots, v_{\frac{n+1}{2}}$ and attach $\frac{n-1}{2}$ vertices $v_{\frac{n+3}{2}}$, $v_{\frac{n+3}{2}}$, ..., v_n to the vertex v_1.

Define $f : V(G) \to \{1,2,\ldots, n\}$ as follows

$$f(v_i) = 2i - 1 \quad \text{for } 1 \le i \le \frac{n+1}{2}$$

$$f(v_{(\frac{n+1}{2})+i}) = 2i \quad \text{for } 1 \le i \le \frac{n-1}{2}$$

Then, the induced edge labeling f* is as follows.

$$f^*(v_i v_{i+1}) = 1 \quad \text{for } 1 \le i \le \frac{n-1}{2}$$

$$f^*(v_1 v_i) = 0 \quad \text{for } \frac{n+3}{2} \le i \le n$$

Thus, $e_f(0) = \frac{n-1}{2}$ and $e_f(1) = \frac{n-1}{2}$.

Hence $|e_f(0) - e_f(1)| \le 1$. Therefore, the resultant graph G is sum divisor cordial for n is odd.

Therefore, given a positive integer n, there is a sum divisor cordial graph G which has n vertices.

Example : 2.1

Sum divisor cordial labeling of the graph G with 10 vertices is shown in figure 2.1.

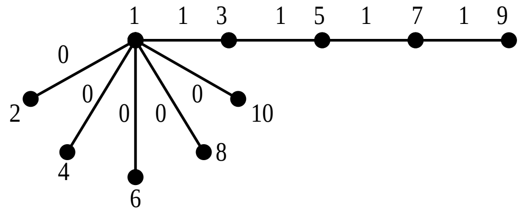

Figure 2.1

Theorem 2.2

Switching of a pendent vertex in path P_n is sum divisor cordial graph.

Proof.

Let $v_1, v_2, \ldots, v_n$, be the vertices of path P_n. The graph G is obtained by switching of a pendent vertex in path P_n. v_1 and v_n are pendent vertex of path P_n. Then $|V(G)| = n$ and $|E(G)| = 2n - 4$.

Define $f : V(G) \to \{1, 2, 3, \ldots, n\}$ as follows

Case 1 : n is odd

Without loss of generality, let the switched vertex be v_1.

Subcase 1(a) : $n \equiv 1 (\text{mod } 4)$

$f(v_1) = n$

$$f(v_i) = \begin{cases} i-1 & \text{if } i = 1,2 (\text{mod} 4) \\ i & \text{if } i = 3 (\text{mod} 4) \\ i-2 & \text{if } i = 0 (\text{mod} 4) \end{cases} \text{ for } 2 \le i \le n$$

Then $e_f(0) = e_f(1) = n - 2$.

Subcase 1(b) : $n \equiv 3 (\text{mod } 4)$

$f(v_1) = n - 1$

$$f(v_i) = \begin{cases} i-1 & \text{if } i = 1,2 (\text{mod} 4) \\ i & \text{if } i = 3 (\text{mod} 4) \\ i-2 & \text{if } i = 0 (\text{mod} 4) \end{cases} \text{ for } 2 \le i \le n$$

Then $e_f(0) = e_f(1) = n - 2$.

Therefore, in all sub cases, $|e_f(0) - e_f(1)| \le 1$.

Case 2 : n is even

Without loss of generality, let the switched vertex be v_n.

Subcase 2(a) : $n \equiv 0 (\text{mod } 4)$

$f(v_n) = n - 1$

$$f(v_i) = \begin{cases} i & \text{if } i = 1,2 (\text{mod} 4) \\ i+1 & \text{if } i = 3 (\text{mod} 4) \\ i-1 & \text{if } i = 0 (\text{mod} 4) \end{cases} \text{ for } 1 \le i \le n-1$$

Then $e_f(0) = e_f(1) = n - 2$.

Subcase 2(b) : $n \equiv 2 (\text{mod } 4)$

$f(v_1) = n$

$$f(v_i) = \begin{cases} i & \text{if } i = 1,2 (\text{mod} 4) \\ i+1 & \text{if } i = 3 (\text{mod} 4) \\ i-1 & \text{if } i = 0 (\text{mod} 4) \end{cases} \text{ for } 1 \le i \le n-1$$

Then $e_f(0) = e_f(1) = n - 2$.

Therefore, in all sub cases, $|e_f(0) - e_f(1)| \le 1$.

Therefore, in both cases, $|e_f(0) - e_f(1)| \le 1$.

Hence, G is sum divisor cordial graph.

Example 2.2

The sum divisor cordial labeling of switching of a pendent vertex in path P_6 is given in figure 2.2.

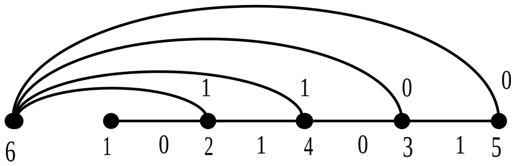

Figure 2.2

Theorem 2.3

Switching of a vertex in cycle C_n admits sum divisor cordial labeling for $n \geq 4$.

Proof.

Let $v_1, v_2, \ldots, v_n$ be the successive vertices of C_n. G_v denotes graph is obtained by switching of vertex v of G = C_n. Without loss of generality let the switched vertex be v_1.

Then $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 5$.

Define $f : V(G_{v_1}) \rightarrow \{1,2,3,\ldots,n\}$ as follows.

Case 1 : n is odd

$$f(v_i) = \left\{ \begin{array}{ll} i & \text{if } i = 1,2(\text{mod}\,4) \\ i+1 & \text{if } i = 3(\text{mod}\,4) \\ i-1 & \text{if } i = 0(\text{mod}\,4) \end{array} \right\} \text{for } 1 \leq i \leq n-1$$

$f(v_n) = n$

Then $e_f(0) = n - 3$ and $e_f(1) = n - 2$, if $n \equiv 1(\text{mod } 4)$

$e_f(0) = n - 2$ and $e_f(1) = n - 3$, if $n \equiv 3(\text{mod } 4)$

Case 2 : n is even

$$f(v_i) = \left\{ \begin{array}{ll} i & \text{if } i = 1,2(\text{mod}\,4) \\ i+1 & \text{if } i = 3(\text{mod}\,4) \\ i-1 & \text{if } i = 0(\text{mod}\,4) \end{array} \right\} \text{for } 1 \leq i \leq n$$

Then $e_f(0) = n - 3$ and $e_f(1) = n - 2$, if $n \equiv 0(\text{mod } 4)$

$e_f(0) = n - 2$ and $e_f(1) = n - 3$, if $n \equiv 2(\text{mod } 4)$

In both cases, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by switching of a vertex in cycle C_n is sum divisor cordial graph for $n \geq 4$.

Example 2.3

The sum divisor cordial labeling of switching of a vertex in cycle C_8 is shown in figure 2.3.

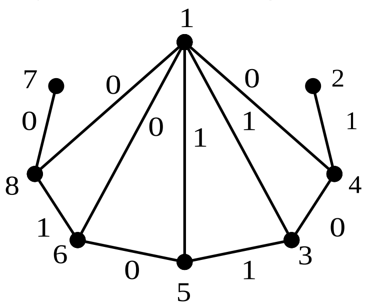

Figure 2.3

Theorem 2.4

$DS(B_{n,n})$ is sum divisor cordial graph.

Proof.

Let $B_{n,n}$ be a graph with vertex set $\{u,v,u_i,v_i,\ 1 \leq i \leq n\}$, where u_i, v_i are pendant vertices. Here, $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_i,v_i : 1 \leq i \leq n\}$ and $V_2 = \{u,v\}$. In order to obtain $DS(B_{n,n})$ from $B_{n,n}$, add w_1,w_2 corresponding to V_1 and V_2. Let G be a graph $DS(B_{n,n})$. Then, $|V(G)| = 2n+4$ and $|E(G)| = 4n+3$.

Define $f : V(G) \rightarrow \{1,2, \ldots, 2n+4\}$ as follows.

$f(u) = 1$,

$f(v) = 3$,

$f(w_1) = 2n+4$,

$f(w_2) = 2$,

$f(u_i) = 2i+2 \quad$ for $1 \leq i \leq n-1$

$f(v_i) = 2i+3 \quad$ for $1 \leq i \leq n-1$.

Then, $e_f(0) = 2n+2$ and $e_f(1) = 2n+1$.

In both cases, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DS(B_{n,n})$ is sum divisor cordial graph.

Example 2.4

The sum divisor cordial labeling of $DS(B_{3,3})$ is shown in figure 2.4.

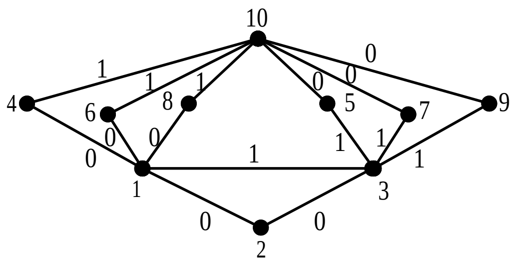

Figure 2.4

Theorem : 2.5

Let G be any sum divisor cordial graph of order p and $K_{2,n}$ be a bipartite graph with the bipartition $V = V_1 \cup V_2$ with $V_1 = \{v_1,v_2\}$ and $V_2 = \{u_1,u_2,\ldots,u_n\}$. Then the graph $G*K_{2,n}$ obtained by identifying the vertices v_1 and v_2 of $K_{2,n}$ with that labeled any one odd number and any one even number respectively in G is also sum divisor cordial graph.

Proof.

Let G be any sum divisor cordial graph of order p. Then in G, $|e_f(0) - e_f(1)| \leq 1$.

Consider any two vertices of G having one odd label and one even label. Without loss of generality, let w_i and w_j be the consider vertices with the labels 1 and 2. Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{v_1,v_2\}$ and $V_2 = \{u_1,u_2,\ldots,u_n\}$.

Now assign the labels p+1, p+2, ..., p+n to the vertices u_1, u_2, ..., u_n respectively and identify the vertices v_1 and v_2 of $K_{2,n}$ with that labeled 1 and 2 respectively in G.

The edges of $K_{2,n}$ incident with the vertex w_i have the label 1 and with the vertex w_j have the label 2. Then, the edges of $K_{2,n}$ contribute equal numbers, namely n to both $e_f(1)$ and $e_f(0)$ in $G*K_{2,n}$.

Thus, $|e_f(0) - e_f(1)| \leq 1$ in $G*K_{2,n}$.

Hence $G*K_{2,n}$ is sum divisor cordial.

Example : 2.5

Sum divisor cordial labeling of the graph G and $G^*K_{2,5}$ are shown in figure 2.5.

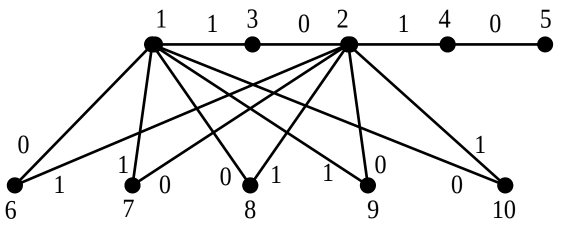

Figure 2.5

Theorem : 2.6

Let G be any sum divisor cordial graph of order p and $K_{3,n}$ be a bipartite graph with the bipartition $V = V_1 \cup V_2$ with $V_1 = \{x_1,x_2,x_3\}$ and $V_2 = \{y_1,y_2, ..., y_n\}$, where n is even. Then the graph $G^*K_{3,n}$ obtained by identifying the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled any two odd number and any one even number or any one odd number and any two even number respectively in G is also sum divisor cordial graph.

Proof.

Let G be any sum divisor cordial graph of order p. Then in G, $|e_f(0) - e_f(1)| \leq 1$.

Let $V = V_1 \cup V_2$ be the bipartition of $K_{3,n}$ such that $V_1 = \{x_1,x_2,x_3\}$ and $V_2 = \{y_1,y_2,...,y_n\}$.

Case 1 : In G, $e_f(0) = e_f(1)$.

Consider any three vertices of G having one odd label and two even labels.

Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.

Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 4 respectively in G.

Subcase 1(a) : p is odd or even and n is even

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute equal numbers of 0's and 1's in $G^*K_{3,n}$.

Thus, $|e_f(0) - e_f(1)| = 0$ in $G^*K_{3,n}$.

Subcase 1(b) : p is even and n is odd.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute $\frac{3n+1}{2}$ numbers of 0's and $\frac{3n-1}{2}$ numbers of 1's in $G^*K_{3,n}$.

Thus, $|e_f(0)-e_f(1)| \leq 1$ in $G^*K_{3,n}$.

Subcase 1(c) : p is odd and n is odd.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute $\frac{3n-1}{2}$ numbers of 0's and $\frac{3n+1}{2}$ numbers of 1's in $G^*K_{3,n}$.

Thus, $|e_f(0)-e_f(1)| \leq 1$ in $G^*K_{3,n}$.

Case 2 : In G, $e_f(0) = e_f(1) + 1$.

Subcase 2(a) : p is odd or even and n is even

Consider any three vertices of G having one odd label and two even labels.

Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.

Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute equal numbers of 0's and 1's in $G^*K_{3,n}$.

Thus, $|e_f(0) - e_f(1)| \leq 1$ in $G^*K_{3,n}$.

Subcase 2(b) : p is even and n is odd.

Consider any three vertices of G having two odd labels and one even label.

Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 3.

Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 3 respectively in G.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute $\frac{3n-1}{2}$ numbers of 0's and $\frac{3n+1}{2}$ numbers of 1's in $G^*K_{3,n}$.

Thus, $|e_f(0)-e_f(1)| = 0$ in $G^*K_{3,n}$.

Subcase 2(c) : p is odd and n is odd.

Consider any three vertices of G having one odd label and two even labels. Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4. Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute $\frac{3n-1}{2}$ numbers of 0's and $\frac{3n+1}{2}$ numbers of 1's in $G^*K_{3,n}$.

Thus, $|e_f(0)-e_f(1)| = 0$ in $G^*K_{3,n}$.

Case 3 : In G, $e_f(0) + 1 = e_f(1)$.

Subcase 3(a) : p is odd or even and n is even

Consider any three vertices of G having one odd label and two even labels.

Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.

Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute equal numbers of 0's and 1's in $G^*K_{3,n}$.

Thus, $|e_f(0) - e_f(1)| \leq 1$ in $G^*K_{3,n}$.

Subcase 3(b) : p is even and n is odd.

Consider any three vertices of G having one odd label and two even labels.

Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.

Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute $\frac{3n+1}{2}$ numbers of 0's and $\frac{3n-1}{2}$ numbers of 1's in $G*K_{3,n}$.

Thus, $|e_f(0) - e_f(1)| = 0$ in $G*K_{3,n}$.

Subcase 3(c) : p is odd and n is odd.

Consider any three vertices of G having two odd labels and one even label.

Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 3.

Now assign the labels p+1,p+2,...,p+n to the vertices $y_1,y_2,...,y_n$ respectively and identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and labeled 3 respectively in G.

Then, the edges of $K_{3,n}$ incident with the vertices v_i, v_j and v_k contribute $\frac{3n+1}{2}$ numbers of 0's and $\frac{3n-1}{2}$ numbers of 1's in $G*K_{3,n}$.

Thus, $|e_f(0) - e_f(1)| = 0$ in $G*K_{3,n}$.

Hence $G*K_{3,n}$ is sum divisor cordial.

Example : 2.6

Sum divisor cordial labeling of the graph G and $G*K_{3,4}$ are shown in figure 2.6.

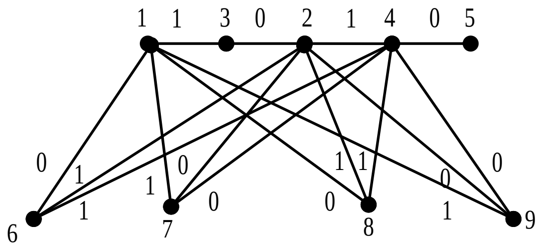

Figure 2.6

III. CONCLUSIONS

In this paper, we prove that the switching of a pendent vertex in path P_n, switching of any vertex in cycle C_n, $DS(B_{n,n})$, $G*K_{2,n}$ and $G*K_{3,n}$ are sum divisor cordial graph.

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