Congruences on La-Semirings and Variant of Semigroup

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Abstract – *This paper deals with the properties of some congruence relations on variant of semigroup and LA-Semirings. The motivation to prove theorems in this paper is due to results of J.M.Howie[1] and congruences on regular semigroups[5].*

Keywords - *LA-semiring ,antiinverse semigroup ,variant of semi group.*

Introduction- The concept of congruence on semigroup was introduced by pondelick. In this paper we determine group congruence on semirings and also congruence relation in anti inverse semigroup becomes maximum 5-potent congruence. On other hand if (S, •) satisfies some identies then (S, a) is an idempotent separate congruence on S.

1.1. Definition: A semiring (S, +, •) is said to be LA-Semiring if

1. (S, +) is a LA-Semigroup
2. (S, •) is a LA-Semigroup
3. Distributive Laws hold in R.

Example: Let S = {a, b, c} is a mono semiring with the following tables 1, 2 which is a LA-Semiring

•	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

(1)

+	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

(2)

1.2. Definition : A semi group S is called anti inverse if every element of S is anti inverse element.

Example: Let S = {a, b} then (S, •) with following table 1 or 2 or 3 is an inverse semigroup

•	a	b
a	a	a
b	b	b

(1)

•	a	b
a	a	b
b	b	b

(1)

(2)

•	a	b
a	a	b
b	b	a

(3)

1.3. Definition: Let (S, •) be a semi group and for any a in S. we define a binary operation (sandwich operation) '**o**' on the set S by x**o**y = xay where x, y $\in$ S. Then S becomes a semigroup with respect to this operation. We denote it by (S, a) and we refer to (S, a) as a variant of (S, •) (or) a-connected semigroup.

1.4. Theorem: Let (S, +, •) be a semiring. Let ρ be a relation define on S. If S contains multiplicative identity which is also an additive identity then S/ ρ is a group congruence.

Proof: Given that (S, +, •) be a semiring.

Define a relation ρ on S by a ρ b $\Rightarrow$ ae = eb

For all a,b in S and

e $\in$ E(S)

Now we prove that ρ is an equivalence relation on S.

Clearly S is commutative ae = ea = a $\Rightarrow$ a ρ a

$\because$ ρ is reflexive

Again a ρ b $\Rightarrow$ ae = eb

$\Rightarrow$ be = ea

$\Rightarrow$ b ρ a

$\because$ ρ is symmetric

Now a ρ b, b ρ c $\Rightarrow$ ae = eb and be = ec

$\Rightarrow$ ae = be = ec

$\Rightarrow$ ae = ec

$\Rightarrow$ a ρ c

$\therefore$ ρ is transitive

$\Rightarrow$ ρ is an equivalence relation.

To prove that a ρ b $\Rightarrow$ ac ρ bc

If a ρ b = $\Rightarrow$ ae = eb

$\Rightarrow$ aec = ebc

$\Rightarrow$ ace = ebc $\Rightarrow$ ac ρ bc

Similarly a ρ b $\Rightarrow$ ca ρ cb

$\because$ ρ is an congruence relation.

To show S/ ρ is group congruence

Define S/ ρ = {a ρ : a$\in$S} where a ρ = {b$\in$ S/b ρ a}

Define 'o' on S/ ρ in the following way

a ρ , b ρ $\in$ S/ ρ s.t (a ρ) o (b ρ) = (ab) ρ

then (a ρ) (b ρ) = (ab) ρ

$\Rightarrow$ (a^1 ρ) (b^1 ρ)= (ab) ρ

$\Rightarrow (a^1b^1)\,\rho = (ab)\,\rho$

Hence S/ ρ is well defined

Clearly S/ ρ is a group

Hence S/ ρ is a group congruence.

1.5. Theorem: Let (S, •) be an anti-inverse semigroup. If η be a relation defined on S by a η b $\Leftrightarrow e_a a = e_b b$ where e_a, e_b are unit elements of a, b in S respectively in S then η is a maximum 5-potent congruence on S.

Proof: Let (S, •) be an anti-inverse semigroup. Define η is a on S by aη b $\Leftrightarrow e_a a = e_b b$ where e_a, e_b are unit elements of a, b in S respectively in S.

First we show that η is an equivalence relation on S.

For any a in S a = a $\Rightarrow a^5 = a^5 \Rightarrow e_a a = e_a a \Rightarrow a\,\eta\,a$

Hence η is reflexive

Symmetric: $a\eta b \Leftrightarrow e_a a = e_b b$

$$\Leftrightarrow a^4 a = b^4 b$$
$$\Leftrightarrow a^5 = b^5$$
$$\Leftrightarrow a = b$$
$$\Leftrightarrow b = a$$
$$\Leftrightarrow b^5 = a^5$$
$$\Leftrightarrow b^4 b = a^4 a$$
$$\Leftrightarrow e_b b = e_a a$$
$$\Leftrightarrow b\eta a$$

$\therefore$ η is symmetric

Transitive: Let aη b and bη c $\Leftrightarrow e_a a = e_b b$ and $e_b b = e_c c$

So $e_a a = e_b b = e_c c$

$\Rightarrow e_a a = e_c c$

$\Rightarrow$ aη c

$\therefore$ aη b and bη c $\Rightarrow$ aη c

$\therefore$ η is transitive

Hence η is equivalence relation

Left compatibility: Let $a\eta b \Leftrightarrow e_a a = e_b b$

$$\Leftrightarrow z^5 e_a a = z^5 e_b b$$
$$\Leftrightarrow z^5 a^5 = z^5 b^5$$
$$\Leftrightarrow (za)^5 = (zb)^5$$

$$\Leftrightarrow (za)^4 za = (zb)^4 zb$$
$$\Leftrightarrow e_{za} za = e_{zb} zb$$
$$\Leftrightarrow za\,\eta\, zb$$

$\therefore$ η is left compatibility

Right compatibility

$$a\eta b \Leftrightarrow e_a a = e_b b$$
$$\Leftrightarrow e_a a z^5 = e_a b z^5$$
$$\Leftrightarrow a^5 z^5 = b^5 z^5$$
$$\Leftrightarrow (az)^5 = (bz)^5$$
$$\Leftrightarrow (az)^4 az = (bz)^4 bz$$
$$\Leftrightarrow e_{az} az = e_{bz} bz$$
$$\Leftrightarrow az\,\eta\, bz$$

$\therefore$ η is right compatibility

Hence η is compatibility

Therefore n is a congruence relation on S.

To show that n is 5-potent congruence

Let $a^5 \eta b^5 \Leftrightarrow e_a a^5 = e_b b^5$

$$\Leftrightarrow e_a a = e_b b$$
$$\Leftrightarrow a\eta b$$

$\therefore$ η is 5-potent congruence relation on S.

To show that η is maximum 5-potent congruence relation on S.

To prove that η is maximum, let μ be any 5-potent congruence relation on S.

Let $a, b \in \mu \Leftrightarrow (a^5, b^5) \in \mu \Leftrightarrow (e_a, e_b) \in \mu$

We know that for all $(e_a, e_b) \in \mu$ and $(a, b) \in \mu \Leftrightarrow (e_a a, e_b b) \in \mu$

Since $(e_a a, e_b b) \Leftrightarrow a\,\eta\,b \Leftrightarrow (a, b) \in \eta$ and $(a, b) \in \mu$

$\therefore \mu \subseteq n$

Hence η is maximum 5-potent congruence relation on S.

1.6. Theorem: Let (S, +, •) be a LA-semiring and (S, •) be an anti inverse semigroup and let η be a congruence relation on S then S/η is a anti inverse sub semigroup.

Proof: Let (S, +, •) be a LA-semiring and (S, •) be an anti inverse semigroup and η be a congruence relation on S.

Therefore we can construct the congruence class S/ η such that S/η = $\{a\eta : a \in (S, \bullet)\}$ where η is a congruence class on S.

For any $a\eta, b\eta \in S/\eta$

Define 'o' on S/η in the following way.

Such that $(a\eta) o (b\eta) = (ab)\eta$

Let $a\eta = a^1\eta$ and $b\eta = b^1\eta$ then $(a^1\eta) o (b^1\eta) = (ab)\eta$ & $(a^1b^1)\eta = (ab)\eta$

Hence 'o' is well defined it is also associative. Hence (S/η , •) is an anti-inverse semi group.

1.7. Theorem: Let η be a congruence relation on an anti-inverse semi group S then η^n is also a congruence relation on S.

Proof: Let η be a congruence relation on an anti-inverse semigroup S.

Let $a\eta b$ then therexist $t_1, t_2, t_3, \ldots\ldots, t_{n-1} \in S$ and by transitivity

We have

$$a\eta t_1, t_1\eta t_2, t_2\eta t_3, \ldots\ldots, t_{n-1}\eta b \Rightarrow a\eta^n b.$$

It is easy to see that η^n is an equivalence relation.

Let $c \in S$ then ca η cb ($\therefore$ since η is compatible)

$$ca\eta ct_1, ct_1\eta ct_2, ct_2\eta ct_3, \ldots\ldots, ct_{n-1}\eta cb \Rightarrow ca\eta^n cb.$$

Hence $a\eta^n b = ca\eta^n cb.$

Similarly we can prove that $a\eta^n b = ac\eta^n bc$

Hence η^n is compatible

Therefore η^n is a congruence relation on S.

1.8. Theorem: Let (S, •) be a semigroup. For any a $\in$ S, (S, a) is a variant of semigroup such that (S, •) is a rectangular band define a relation η on (S, a) such that x η y $\Leftrightarrow$ xoy = x then η is an idempotent separate congruence on S.

Proof: Let (S, •) be a rectangular band. Then for all a, b $\in$ S aba = a.

Let (S, a) be a variant of S. Define η on (S, a) by x η y $\Leftrightarrow$ xoy = x

$\Leftrightarrow$ xay = x

Now we show that η is a congruence on (S, a)

Since S is rectangular xax = x

$\Rightarrow$ xox = x

$\Rightarrow$ x η x

$\Rightarrow$ η is reflexive

Let x η y $\Rightarrow$ xoy = x

$\Rightarrow$ yoxoy = yox

$\Rightarrow$ yaxay = yax

$\Rightarrow$ y(axa)y = yax

$\Rightarrow$ yay = yax

$\Rightarrow$ y = yax

$\Rightarrow$ y = yox

$\Rightarrow$ y η x

$\Rightarrow$ η is symmentric

Let x η y and y η z $\Rightarrow$ xoy = x and yoz = y

Now yoz = y

$\Rightarrow$ xoyoz = xoy

$\Rightarrow$ xayaz = xay

$\Rightarrow$ xaz = x

$\Rightarrow$ x η z

$\Rightarrow$ η is transitive

$\therefore$ η is an equivalence relation on (S, a)

To prove compatibility, i.e., x η y $\Rightarrow$ xoz η yoz $\forall$ x, y, z in S

We have to show that (xoz) o (yoz) = xoz

We know that x η y $\Rightarrow$ xoy = x

$\Rightarrow$ xozoxoyoz = xozoxoz

$\Rightarrow$ xaz(axa)yaz = xazaxaz

$\Rightarrow$ xazayaz = xaz(axa)z

$\Rightarrow$ xazayaz = xa(zaz)

$\Rightarrow$ xazayaz = xaz

$\Rightarrow$ (xoz)o(yoz) = xoz

Similarly we show that x η y $\Leftrightarrow$ zox η zoy

Hence η is a congruence relation on (S, a)

Now we show that η is separative on (S, a)

i.e xox η xoy η yoy $\Rightarrow$ x = y

xox η xoy $\Rightarrow$ xox η xoy = xox

$\Rightarrow$ xaxaxay = xax

$\Rightarrow$ x(axa)xay = xax

$\Rightarrow$ xaxay = xax [axa = a]

$\Rightarrow$ xay = x [axa = a] -------------(1)

yoy η xoy $\Rightarrow$ yoy η xoy = yoy

$\Rightarrow$ xoy η yoy = yoy

$\Rightarrow$ xayayay = yay

$\Rightarrow$ x(aya)yay = yay

$\Rightarrow$ xa(yay) = yay (aya = a)

$\Rightarrow$ xay = y (yay = y) -------------(2)

From (1) and (2) x = y

Hence η is separative congruence on (S, a)

Let (x, y) $\in$ (S, a) $\Rightarrow$ xox = x, yoy = y

Consider (xox) η (yoy) $\Rightarrow$ (xox)o(yoy) = xox

Let x η y $\Rightarrow$ xay = x

$\Rightarrow$ xaxayay = xaxay = xax

$\Rightarrow$ (xox) o (yoy) = xox

Hence η is an idempotent separative congruence on (S, a).

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