Explanation of Method to Particular Quintic Equation $X^5+5Ax^3+5A^2x+B=0$

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Abstract: *A particular quintic is the quintic which satisfy some specific condition. In other words, a quintic with co-efficient satisfy some perticular condition. This paper gives a simple explanation of method to find the roots and to discuss the nature of roots of the particular quintic equation which is known as, "De moivers quintic" i.e* $x^5+5Ax^3+5A^2x+B=0$.

Keywords: *Quintic equation, reducible, irreducible, real root, complex root*

I. INTRODUCTION

In mathematics a quintic function is a function of the form $f(x)=a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5$ where $(a_0\neq0)$ or in other words a function defined by a polynomial of degree 5 getting f(x)=0 produce usually a quintic equation of the form $a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0$ where a_i's are rational. Solving quintic eqution[4] in term of redicals was a major problem in algebra from 16^{th} century , where cubic and Biquadretic equation[1,2,3] were solved until the half of the century, when the impossiblity of such a general solution was proved (Abel –Ruffini theorem). Some quintic equation[6,7] can be solved in terms of redicals these include the reducible quantities and solvable irreducible quantities. For characterizing solvable quantities "Everiste galois" developed technique which gave a rise to group theory and Galois theory[5]. But the particular quintics of the type $x^5+5Ax^3+5A^2x+B=0$ are always solvable in redicals[4,7] whether it is reducible or irreducible.

II. EXPLANATION OF METHOD

Let us try to solve the quintic equation $x^5+a_1x^3+a_2x+a_3=0$ … (1)

Put x=a+b

$\Rightarrow x^5=(a+b)^5$

$=(a^2+b^2+2ab)(a+b)^3$

$=(5ab-3ab+a^2+b^2)(a+b)^3$

$=5ab(a+b)^3+(a^2+b^2-3ab)(a+b)^3$

$=5abx^3+[(a^2+b^2-3ab)(a+b)^2](a+b)$

$=5abx^3+[(a^2+b^2-3ab)(a^2+b^2+2ab)](a+b)$

$=5abx^3+[a^4+a^2b^2+2a^3b+a^2b^2+b^4+ab^3-3a^3b-3ab^3-6a^2b^2](a+b)$

$=5abx^3+(a^4+b^4-4a^2b^2-a^3b-ab^3)(a+b)$

$=5abx^3-5a^2b^2(a+b)+(a^4+b^4+a^2b^2-a^3b-ab^3)(a+b)$

$=5abx^3-5a^2b^2x+(a^5+b^5)$

$\Rightarrow x^5-5abx^3+5a^2b^2x-(a^5+b^5)=0$

$\Rightarrow x^5+5(-ab)x^3+5(-ab)^2x-(a^5+b^5)=0$ …(2)

Comparing (1) and (2)

$5(-ab)=a_1 \Rightarrow -ab=a_1/5$ …(3a)

$5(-ab)^2=a_2$ …(3b)

$-(a^5+b^5)=a_3$ …(3c)

$\Rightarrow$x=a+b is a root of the equation (1) if a, b satisfies the three equation given by (3) there are three equation and two unknowns. So it becomes a particular quintic equation. The quintic equation (1) can be solved by this method if $a_2=5(a_1/5)^2= a_1^2/5$

Let us put $a_1=5A$,$a_3=B$

Then equation (1) become

$x^5+5Ax^3+5A^2x+B=0$ …(4)

comparing (2) and (4)

$-ab=A \Rightarrow (ab)^5=-A^5$ …(5a)

$-(a^5+b^5)=B \Rightarrow a^5+b^5=-B$ …(5b)

a^5, b^5 are the roots of

$y^2+By-A^5=0$ …..(6)

$y=[-B\pm\sqrt{(B^2+4A^5)}]/2$

$a=\{[-B+\sqrt{(B^2+4A^5)}]/2\}^{1/5}$, $b=\{[-B-\sqrt{(B^2+4A^5)}]/2\}^{1/5}$

Nature of the roots:

Case I : if $B^2+4A^5\geq 0$

Subcase (i):

$[-B+\sqrt{(B^2+4A^5)}]/2=\alpha^5$, $[-B-\sqrt{(B^2+4A^5)}]/2=\beta^5$

Where α,β are rational

$\Rightarrow x=a+b=\alpha+\beta$(rational)

the quintic equation $x^5+5Ax^3+5A^2x+B=0$ is reducible ,one root is rational and other four roots may be real or imaginary.

Subcase (ii):

$[-B+\sqrt{(B^2+4A^5)}]/2\neq\alpha^5$ and $[-B-\sqrt{(B^2+4A^5)}]/2\neq\beta^5$

where α, β are rational then the quintic equation $x^5+5Ax^3+5A^2x+B=0$ is irreducible . then one rot is real and other four roots are imaginary given by

$x_1=a+b$

$x_i=a\omega^i+b\omega^{5-i}$, $i=1,2,3,4$

where ω^i are complex fifth root of unity.

So one root of quintic $x^5+5Ax^3+5A^2x+B=0$ is real and other four root are complex.

Case II : if $B^2+4A^5<0$

Then both a^5 and b^5 are complex conjugate of the type

$a^5=r(\cos\theta+i\sin\theta)$, $b^5= r(\cos\theta-i\sin\theta)$

$a=r^{1/5}\{\cos[(2r\pi+\theta)/5]+i\sin[(2r\pi+\theta)/5]\}$

$b=r^{1/5}\{\cos[(2r\pi+\theta)/5]-i\sin[(2r\pi+\theta)/5]\}$

$x=a+b=2r^{1/5}\cos[(2r\pi+\theta)/5]$, $r=0,1,2,3,4$.

In this case all the roots are real.

Examples :

1. the given equation is
   $x^5-10x^3+20x-33=0$ …(a)
   comparing (a) with equation (2)
   $-ab=-2=A$
   $\Rightarrow (ab)^5=32$
   $-(a^5+b^5)=-33$
   $\Rightarrow a^5+b^5=33$
   $\Rightarrow$ A=-2 and B=33

   a^5, b^5 are the roots of

   $y^2-33y+32=0$

$a^5=1$, $b^5=32 \Rightarrow a=1$ and $b=2$

$\Rightarrow x=a+b = 1+2 = 3$

Other four root are given by

$x=\{[(-3-\sqrt{5})/2]\pm\sqrt{(30+6\sqrt{5})}\}/4$

$x=\{[(-3+\sqrt{5})/2]\pm\sqrt{(30+6\sqrt{5})}\}/4$.

2. The given equation is
   $x^5+15x^3+45x+18=0$ …(a)
   comparing equation (a) with equation (2)
   $-ab=3=A$
   $\Rightarrow (ab)^5=-243$
   $-(a^5+b^5)=18$
   $\Rightarrow a^5+b^5=-18$
   $\Rightarrow$ A=3 and B=18
   $y^2+18y-243=0$
   $\Rightarrow (y+27)(y-9)=0$
   $\Rightarrow y=-27,9$
   $\Rightarrow x=(9)^{1/5}+(-27)^{1/5}$
   Other four root are given by sub-case –II

3. The given equation is
   $x^5-15x^3+45x-31=0$ …(a)
   Comparing equation (a) with equation (2)
   $-ab=-3=A \Rightarrow (ab)^5=243$
   $-(a^5+b^5)=-31 \Rightarrow a^5+b^5=31$
   $\Rightarrow$ A=-3 and B=-31

$y^2-31y+243=0$
$y=(31\pm i\sqrt{11})/2$
$a^5=(31+i\sqrt{11})/2,\ b^5=(31-i\sqrt{11})/2$
$a=r^{1/5}\{\cos[(2k\pi+\theta)/5]+i\sin[(2k\pi+\theta)/5]\}$
$b=r^{1/5}\{\cos[(2k\pi+\theta)/5]-i\sin[(2k\pi+\theta)/5]\}$
$x=2r^{1/5}\cos[(2k\pi+\theta)/5]$, $k=1,2,3,4$
where
$r=\sqrt{[(31/2)^2+(\sqrt{11}/2)^2]}=\sqrt{[(961+11)/4]}=\sqrt{(972/4)}$
and $\theta=\tan^{-1}(\sqrt{11}/31)$.

III. CONCLUSION

In this study, it has been given that every "De moivers quintic" is solvable, whether it is reducible or irreducible. The nature of the roots also can be determined. So the proposed work of the paper has been done.

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