Solution of Reducible Quintic Equation by Properties of Continuous Function

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Abstract : *the general quintic equation can be divided in reducible and irreducible quantities. This paper presents a technique to solve several type of reducible quintic equation. This technique is based on property of continuous function and the fact that if a polynomial equation with rational coefficient has rational solution then its improved equation with coefficient of highest degree term unity and integer coefficient has an integer solution. The integer solution of the improved equation can easily be find out by the property of continuous function i.e if $f(\alpha)<0$ and $f(\beta)>0$, then there exist an integer such that $f(m)=0$.*

Keywords: *continuous function, irreducible, reducible, quintic equation*.

I. INTRODUCTION

In mathematics a quintic function is a function of the form $f(x)=A_0x^5+A_1x^4+A_2x^3+A_3x^2+A_4x+A_5$ where($A_0\neq0$) or in others words a function defined by a polynomial of degree 5, getting f(x)=0 produce a quintic equation. Where A_i's are rational. There are two type of quintic reducible and irreducible quantities[4,5,6] over main concern is about reducible quantities. A quintic is reducible in x if it is either divisible by ax+b or ax^2+bx+c. where a,b,c are rational. Otherwise it is said to be irreducible. Every reducible quintic is solvable as the quadratic, cubic and biquadratic were solved[1,3] but the general quentic equation is not solveable in redicals[2,6]. First the prove of the fact that "if a polynomial equation with rational coefficient has a rational solution, then its improved equation with coefficient of highest term unity and integer coefficients has an integer solution will be given. After that explanation of the method will be discussed.

II. PROOF OF THE FACT

Let us suppose that the polynomial equation of degree n with rational co-efficient has a rational solution

$A_0x^n+A_1x^{n-1}+\ldots \qquad \ldots+A_{n-1}x+A_n=0 \qquad \ldots$(A) (has a rational solution)

Equation (A) can be improved in the form

$x^n+B_0x^{n-1}+B_1x^{n-2}+\ldots \qquad \ldots+B_{n-2}x+B_{n-1}=0 \qquad \ldots$(B)

by multiplying the roots of (A) by a suitable factor

here $B_i \in Z$

as the equation (A) has a rational root. So equation (B) has a rational root.

Now we shall prove that the rational root of (B) is an integer.

Let if possible equation (B) has a rational root (x=p/q), where $p,q \in Z$, g.c.d.(p,q)=1

Putting in (B)

$(p/q)^n+B_0(p/q)^{n-1}+B_1(p/q)^{n-2}+\ldots \qquad \ldots+B_{n-2}(p/q)+B_{n-1}=0$

$\Rightarrow p^n+B_0p^{n-1}q+B_1p^{n-2}q^2+\ldots \qquad \ldots+B_{n-2}pq^{n-1}+B_{n-1}q^n=0$

$\Rightarrow q(B_0p^{n-1}+B_1p^{n-2}q^1+\ldots \qquad \ldots+B_{n-2}pq^{n-2}+B_{n-1}q^{n-1})=-p^n$

$\Rightarrow q/p^{n-1}$ but g.c.d.(p,q)=1

➔q=1

➔x=(p/q) =p (an integer)

Hence proved.

III. EXPLANATION OF METHOD

Let us suppose that the quintic equation

$A_0x^5+A_1x^4+A_2x^3+A_3x^2+A_4x+A_5=0$ …(1).

Where ($A_0\neq0$) and A_i's are rational for all "i" is reducible over rational . then equation (1) is reducible in the form either (Linear x Biquadratic) or (Quadratic x cubic)

Case I :

if equation (1) is reducible in the form (Linear x Biquadratic) then equation (1) has a linear factor.

➔equation (1) has a rational root.

But equation (1) can be reduce to an equation with coefficient of highest degree term unity and integer coefficient by multiplying the root of the equation (1) by some suitable factor "c" The improved equation will be of the form

$f(y)=y^5+B_1y^4+B_2y^3+B_3y^2+B_4y+B_5=0$ …(2)

where B_i's $\in Z$,y=cx

by the above proof of the fact.

Equation (2) has an integer solution this integer solution can easily be find out by using the properties of continuous function that if $f(\alpha)<0$and $f(\beta)>0$ then there exist an integer "d" such that f(d)=0. Then y=d is a solution of (2). Other four roots of the equation (2)can be find out by solving the depressed biquadratic equation. As the roots of equation (2) are known it means the roots of equation (1) are known by x=y/c

Case II :

If quintic equation (1) is not reducible in the form (Linear x Biquadratic) it means it is reducible in the form (quadratic x cubic). The quintic equation (1) can be reduced to the form $y^5+a_1y^3+a_2y^2+a_3y+a_4=0$ …(3)

Where $a_i \in Z$

By multiplying and decreasing the root of equation (1) by some suitable factor.

Now let us put

$$y^5+a_1y^3+a_2y^2+a_3y+a_4 = (y^3+b_1y^2+b_2y+b_3)(y^2-b_1y+c_1) \quad \text{…(4)}$$

comparing the coefficient of like power of y in (4) we get

$c_1-b_1^2+b_2=a_1$ …(5)

$b_1c_1-b_1b_2+b_3=a_2$ …(6)

$b_2c_1-b_1b_3=a_3$ …(7)

$b_3c_1=a_4$ …(8)

putting the value of c_1 from (8) in (5),(6),(7) we have

$(-b_1^2+b_2-a_1)b_3+a_4=0$ …(9)

$b_3^2+(-b_1b_2-a_2)b_3+a_4b_1=0$ …(10)

$-b_1b_2^2-a_3b_3+a_4b_2=0$ …(11)

Now $b_1(10) + (11)$

$\Rightarrow (-b_1^2b_2-b_1a_2-a_3)b_3+(a_4b_1^2+a_4b_2)=0$ …(12)

From (9) and (10)

$b_3(9) – (-b_1^2+b_2-a_1)(10)$

$\Rightarrow [a_4-b_2b_1^3-a_2b_1^2+(b_2^2-a_1b_2)b_1+a_2b_2-a_1a_2]b_3$

$+ [a_4b_1^3+(-a_4b_2+a_1a_4)b_1]=0$ ...(13)

From (9) and (12)

From (9) $b_3=a_4/(b_1^2-b_2+a_1)$ putting in (12) we have

$b_1^4+(a_1-a_2)b_1^2-a_2b_1-b_2^2+a_1b_2-a_3=0$ …(14)

From (9) $b_3=a_4/(b_1^2-b_2+a_1)$ putting in (13) we have

$b_1^5+(2a_1-3b_2)b_1^3-a_2b_1^2+(2b_2^2+a_1^2-3a_1b_2)b_1+a_4+a_2b_2-a_1a_2=0$ …(15)

now $(15) – b_1(14)$

$\Rightarrow (a_1-2b_2)b_1^3+(3b_2^2-4a_1b_2+a_1^2+a_3)b_1+a_4+a_2b_2-a_1a_2=0$ …(16)

$b_1(16) – (a_1-2b_2)(14)$

$\Rightarrow (b_2^2-a_1b_2+a_3)b_1^2+(a_4-a_2b_2)b_1$

$-[2b_2^3-3a_1b_2^2+(a_1^2+2a_3)b_2-a_1a_3]=0$ …(17)

$(a_1-2b_2)b_1(17) – (b_2^2-a_1b_2+a_3)(16)$

$\Rightarrow (a_1a_4-a_1a_2b_2-2a_4b_2+2a_2b_2^2)b_1^2-[a_1b_2^3-a_1a_3b_2-b_2^4+a_3^2]b_1$

$-[a_4b_2^2+a_2b_2^3-2a_1a_2b_2^2-a_1a_4b_2+a_1^2a_2b_2+a_3a_4+a_2a_3b_2-a_1a_2a_3] = 0$ …(18)

Equation (17) and (18) are of the type

$Ab_1^2+Bb_1+C=0$ …(19)

$A'b_1^2+B'b_1+C'=0$ …(20)

Eliminating b_1 from (19) and (20) we get

$\Rightarrow A(A'C-AC')^2-B(A'B-AB')(A'C-AC')+C(A'B-AB')^2=0$ …(21)

Where A,B,C,A',B' and C' are given by

$A=b_2^2-a_1b_2+a_3$, $B=a_4-a_2b_2$, $C=-[2b_2^3-3a_1b_2^2+(a_1^2+2a_3)b_2-a_1a_3]$,

$A'=(a_1a_4-a_1a_2b_2-2a_4b_2+2a_2b_2^2)$, $B'=(-a_1b_2^3+a_1a_3b_2+b_2^4-a_3^2)$,

$C'=-[a_2b_2^3+(a_4-2a_1a_2)b_2^2+(a_2a_3+a_1^2a_2-a_1a_4)b_2-a_1a_2a_3+a_3a_4]$

equation (21) gives an equation of atmost fifteen degree in b_2 which has a rational solution. This rational solution can be find out by using Case I: select the value of b_2 such that equation (19) and (20) have a unique solution in b_1. Now b_1 and b_2 are known. Then b_3 and c_1 can be find out using equation (5), (6), (7) and (8). Putting all these value in (4) we get reduced form of quintic (4).

Solution of equation (4) can be find out by solving the cubic and quadratic . roots of equation (4) are known. So roots of quintic (1) are known.

Examples :

1. The given equation is $2x^5+5x^4+5x^3+5x^2+5x+3=0$ …(a)
Multiplying the roots of the equation (a) by 2. To make the coefficient of higher term unity and integer coefficients
$2x^5+2.5x^4+2^2.5x3+2^3.5x^2+2^4.5x+2^5.3=0$
$x^5+5x^4+10x^3+20x^2+40x+48=0$ …(b)
if equation (a) has a rational solution then equation (b) will have an integer solution.
Here $f(0)=48 > 0$
$f(x) > 0$ for all $x \geq 0$
clearly $f(-3)=0$ ➔ $x=-3$ is a root of equation (b)
➔$x= -3/2$ is a root of equation (a)
Other four roots can be find out by solving residue biquadratic equation.

2. The given equation is $x^5+5x^2+x+3=0$ …(a)

Let us try that equation (a) has a rational root. This is an equation with coefficient of highest term unity and integer coefficient. So this rational root will be an integer.

Here $f(0)=3 > 0$, $f(x) > 0$ for all $x \geq 1$, $f(x) < 0$ for all $x \leq -2$,

$f(-1) =6$ so equation (a) has no integer solution. So quintic (a) is not be reducible in the form (Linear x Biquadratic)

Let us put

$x^5+5x^2+x+3=(x^3+b_1x^2+b_2x+b_3)(x^2-b_1x+c_1)$ …(b)

Here $a_1=0$, $a_2=5$, $a_3=1$, $a_4=3$

Now from equation (19) and (20) we have

$A=b_2^2+1$, $B=3-5b_2$, $C=-(2b_2^3+2b_2)$

$A'=-6b_2+10b_2^2$, $B'=b_2^4-1$, $C'=-(5b_2^3+3b_2^2+5b_2+3)$

Now from equation (21) we have

$A(A'C-AC')^2-B(A'B-AB')(A'C-AC')+C(A'B-AB')^2=0$

➔$-2b_2^{15}-6b_2^{13}+100b_2^{12}-322b_2^{11}+448b_2^{10}-2290b_2^9-580b_2^8+1170b_2^7+1732b_2^6-1510b_2^5+1096b_2^4+348b_2^3-364b_2^2+190b_2=0$

➔$-2b_2[b_2^{14}+3b_2^{12}-50b_2^{11}+161b_2^{10}-224b_2^9+1145b_2^8+290b_2^7-585b_2^6-866b_2^5+755b_2^4-548b_2^3-174b_2^2+182b_2-95]=0$ …(c)

➔either $b_2=0$ or $b_2=1$

{rejecting $b_2=0$ which not gives Common solution to (19) and (20) }

Putting the value of b_2 in (19) and (20) we get

$2b_1^2-2b_1-4=0 \rightarrow b_1^2-b_1-2=0$ …(d)

$4b_1^2+0b_1-16=0 \rightarrow b_1^2=4 \rightarrow b_1=\pm 2$ …(e)

$b_1=2$ also satisfied (d) so we have $b_1=2$, $b_2=1$

by equation (5), (6), (7) and (8) we have

$c_1-4+1=0 \rightarrow c_1=3$

$3b_3=3 \rightarrow b_3=1$

Putting all these value in (b) we get

$$x^5+5x^2+x+3=(x^3+2x^2+x+1)(x^2-2x+3) \quad \text{…(f)}$$

two roots of the equation (a) are $1\pm\sqrt{2}i$ other three roots can be find out by cardan's methods to solve the cubic equation

$x^3+2x^2+x+1=0$.

IV. CONCLUSION

In this study, a method to solve the reducible quintic equation has been given. Further it has been revealed that this method can be used to check the irreducibility of the general quintic. So, the proposed work of this paper has been done.

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