Forcing Edge Detour Domination Number of Graphs

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Abstract. *In this paper, we introduce the new concept, forcing edge detour domination number and obtain the forcing edge detour domination number for some well known graphs.*

Keywords — *Domination, forcing edge detour and forcing edge detour domination number.*

I. Introduction

The concept of domination was introduced by Ore and Berge [7]. Let G be a finite, undirected connected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number $\gamma(G)$ of G. For basic definitions and terminologies, we refer Harary [1]. (G, D)-number of a graph was introduced by Palani.K and Nagarajan.A [8]. Let G = (V, E) be any connected graph with at least two vertices. A subset S of V (G) which is both dominating and geodetic set of G is called a (G, D)-set of G. For vertices u and v in a connected graph G, the detour distance D(u,v) is the length of longest u-v path in G. A u-v path of length D(u,v) is called a u-v detour.

A subset S of V is called a detour set if every vertex in G lies on a detour joining a pair of vertices of S. The detour number dn(G) of G is the minimum order of a detour set and any detour set of order dn(G) is called a detour basis of G. These concepts were studied by Chartrand [3]. A subset S of V is called an edge detour set of G if every edge in G lies on a detour joining a pair of vertices of S. The edge detour number $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order dn_1 is an edge detour basis. A graph G is called an edge detour graph if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan [10]. Forcing (G,D)-number of a graph was introduced and studied by Palani.K and Nagarajan.A [9]. Let G be a connected graph and S be a $\gamma(G)$-set of G. A subset T of S is called a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S with minimum cardinality is called a minimum forcing subset for S. The forcing (G, D)-number of S, denoted by $f_{G,D}(S)$, is the cardinality of a minimum forcing subset of S. The forcing(G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken is over all γ_G - sets S of G and it is denoted by $f_{G,D}(G)$. That is, $f_{G,D}(G)$ = min $\{f_{G,D)}(S) : S$ is any γ_G - set of G$\}$.

An edge detour dominating set is a subset S of V (G) which is both a dominating and an edge detour set of G. An edge detour dominating set is said to be a minimal edge detour dominating set of G if no proper subset of S is an edge detour dominating set of G. An edge detour dominating set S is said to be minimum edge detour dominating set of G if there exists no edge detour dominating set S' such that $|S'| < |S|$.The smallest cardinality of an edge detour dominating set of G is called the edge detour domination number of G. It is denoted by $\gamma_{e,D}(G)$. Any edge detour dominating set S of G of minimum cardinality $\gamma_{e,D}$ is called a γ, eD - set of G. Edge detour domination number of a graph were introduced by A.Mahalakshmi, K.Palani and S.Somasundaram [6].

The following results are by A.Mahalakshmi, Palani.K and S.Somasundaram[6].

Theorem1.1. K_P is an edge detour dominating graph and for $p \geq 3$, $\gamma_{e,D}(K_p)$ = 3.

Theorem 1.2. $\gamma_{e,D}(K_{1,n}) = n$.

Theorem 1.3. $$\gamma_{e,D}(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 \text{ if } n \geq 5 \\ 2 \text{ if } n = 2, 3 \text{ or } 4. \end{cases}$$

Theorem1.4. For n > 5,
$$\gamma_{e,D}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Remark 1.5. If the set of all pendant vertices of a graph G forms an edge detour dominating G, then S is the unique minimum edge detour dominating set of G.

II. FORCING EDGE DETOUR DOMINATION NUMBER OF A GRAPH

Definition 2.1. Let G be an edge detour dominating graph and S be an edge detour dominating basis of G. A subset $T \subseteq S$ is called a forcing (γ, eD) subset for S if S is the unique edge detour dominating basis containing T. The forcing (γ, eD) - number of S denoted by $f\gamma_{eD}(S)$, is the cardinality of a minimum forcing subset for S. The forcing edge detour domination number of G is denoted by $f\gamma_{eD}(G)$ is $f\gamma_{eD}(G)$ = min { $f\gamma_{eD}(S)$ }, where the minimum is taken over all edge detour dominating bases S in G.

Observation 2.2. For every edge detour dominating graph, $0 \le f\gamma_{eD}(G) \le \gamma_{eD}(G)$.

Example. 2.3.
i) For a graph 2.1 (a), $\{v_1, v_4\}$ is a unique edge detour dominating basis and so $\gamma_{eD}(G) = 2 = dn_1(G)$. Therefore, $f\gamma_{eD}(G)$ = 0.
ii) For the graph G in the given figure 2.1 (b), S_1 = {u, x, y}, S_2 = {u, x, z} are the edge detour dominating bases of G {x} and {z} are the forcing subsets of S_1 and S_2 respectively. Hence, $f\gamma_{eD}(G)$ = 1.
iii) For the graph G in Figure 2.1(c), S_1 = {u, y, z}, S_2 = {u, v, x}, S_3 = {u, w, z}, S_4 = {v, x, z}, S_5 = {x, y, z}, S_6 = {w, x, z} are the six edge detour dominating bases of G. And every single element appears in atleast two of the edge detour dominating sets. And also, {u, v}, {u, y}, {u, w}, {u, x}, {u, z}, {v, w}, {v, z}, contained in only one of the six edge detour dominating bases. Therefore, $f\gamma_{eD}(G)$ = 2.

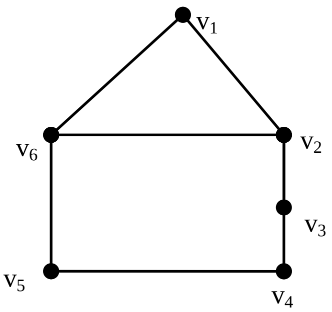

Figure 2.1 (a)

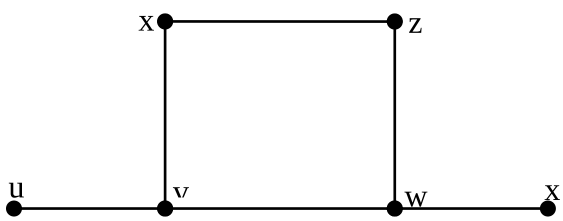

Figure 2.1(b)

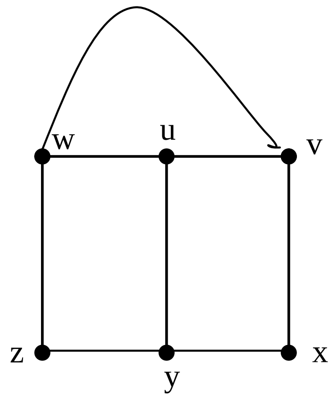

Figure 2.1(c)

Remark 2.4. The bounds in the above observation are sharp.
i) $f\gamma_{eD}(G)$ = 0 if and only if G has a unique edge detour dominating basis.
ii) $f\gamma_{eD}(G)$ = 1 if and only if G has atleast two edge detour dominating bases, one of which is a unique detour dominating bases containing one of its elements.
ii) $f\gamma_{eD}(G) = \gamma_{eD}(G)$ if and only if no edge detour dominating bases of G is the unique edge detour dominating basis containing any of its proper subsets.

Definition 2.6. A vertex v of G is said to be an edge detour dominating vertex of G if v belong to every γ_{eD} - set of G.

Theorem 2.7. *Let G be an edge detour dominating graph and W be the set of all edge detour dominating vertices of G. Then,* $f\gamma_{eD}(G) \le \gamma_{eD}(G) - |W|$.

Proof. Let W = $\{v_1, v_2, \ldots v_n\}$. By the definition of W, $W \subseteq$ S, for all edge detour dominating basis of G. Therefore, no minimum forcing subset of S contains the vertices of W for all S. Hence, $f\gamma_{eD}(G) \le \gamma_{eD}(G) - |W|$.

Corollary. 2.8. *Let G be any graph and S be the set of all end vertices of G. Since, end vertices lie in every* γ_{eD} *-set,* $f\gamma_{eD}(G) \le \gamma_{eD}(G) - |S|$.

Remark.2.9. The bound in the above corollary is sharp. For the graph given in Figure 2.1(b), $\gamma_{eD}(G)$ = 3, $|W|$ = 2 and $f\gamma_{eD}(G)$ = 1. Also the inequality can be strict. For the graph in Figure 2.1(a), γ_{eD} = 2, $|W|$ = 0. Thus $f\gamma_{eD}(G) < \gamma_{eD}(G) - |W|$.

Proposition.2.10.
$f\gamma_{eD}(P_n) = 0$ if $n \equiv 1 (\mathrm{mod}\ 3)$.

Proof. Let $P_n = \{v_1, v_2, \ldots v_{3k+1}\}$, $k > 0$. $S = \{v_1, v_4, \ldots, v_{3k+1}\}$ is the unique edge detour dominating set of P_n. Therefore, by observation 2.5(1), $f\gamma_{eD}(P_n) = 0$.

Proposition.2.11. *Every three element subsets of $V(K_p)$ are the edge detour dominating bases of K_p, for $p \geq 3$.*

Proof. Let S = {u, v, w} be a three element subset of $V(K_p)$. Every edge other than uv lie in some edge detour joining u and v. And uv is in some edge detour joining v and w. Also, S dominates all the vertices of K_p. Therefore, S is an edge detour dominating set of K_p.

Claim: No two element subset of $V(K_p)$ is an edge detour dominating set of K_p. Suppose, let S' ={u', v'} be an edge detour dominating set of K_p. Clearly, the edge u'v' lie in no edge detour joining u' and v'. Therefore, no two element subset of $V(K_p)$ is an edge detour dominating set of K_p. Hence, S is an edge detour dominating basis of K_p.

Theorem 2.12. *Let G be a complete graph K_p of order $p \geq 4$. Then, $\gamma_{eD}(K_p) = 3$ and $f\gamma_{eD}(K_p) = 3$.*

Proof. Let G be a complete graph K_p. Let $p \geq 4$. By Proposition 2.11, $\gamma_{eD}(K_p) = 3$. Since every three element subset of $V(K_p)$ is an edge detour dominating basis and no edge detour dominating basis of G is the unique edge detour dominating basis containing any of its proper subsets. Therefore, by observation 2.5 (3), $f\gamma_{eD}(K_p) = 3$.

Remark 2.13. When p = 3, {u, v, w} is a unique edge detour dominating basis. Therefore, $\gamma_{eD}(K_3) = 3$. And so by observation 2.5 (1), $f\gamma_{eD}(K_p) = 0$.

Theorem 2.14. *Let G be a cycle C_n of order n = 3k, k>1. Then, $f\gamma_{eD}(C_n) = 1$.*

Proof. Let n = 3k, k>1. Let $V(C_n) = \{v_1, v_2, \ldots, v_{3k}\}$. Clearly, $S_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\}$, $S_2 = \{v_2, v_5, \ldots, v_{3(k-1)+2}\}$ and $S_3 = \{v_3, v_6, \ldots, v_{3k}\}$ are the only three edge detour dominating bases of $V(C_n)$. Clearly, for i = 1 to n, $\{v_i\} \subseteq S_j$ for exactly one j such that $1 \leq j \leq 3$. Hence, $f\gamma_{eD}(C_n) = 1$.

Corollary 2.15. *Let $G \cong K_{m,n}$ $(2 \leq m \leq n)$. Suppose $S \subseteq V$ is an edge detour dominating basis G. Then,*

i)If m = 1 and n > 1 (or n = 1 and m > 1) then, $\gamma_{eD}(K_{m,n}) = n$ (or m) and $f\gamma_{eD}(K_{m,n}) = 0$.

ii) If m = n =1 and m = 2 and $n \geq 2$ then, $\gamma_{eD}(K_{m,n}) = 2$ and $f\gamma_{eD}(K_{m,n}) = 0$.

iii) If m, $n \geq 3$ then, $\gamma_{eD}(K_{m,n}) = 3$ and $f\gamma_{eD}(K_{m,n}) = 3$.

Proof. Let G = $K_{m,n}$ with bipartition $V_1=\{a_1, a_2, .., a_m\}$ and $V_2=\{b_1, b_2, \ldots, b_n\}$.

i) For, m = 1 and n > 1 (or n = 1 and m > 1) then $K_{m,n} = K_{1,n}$ (or $K_{m,1}$) correspondingly V_2 (or V_1) is the unique minimum edge detour dominating basis of G and $\gamma_{eD}(K_{m,n}) = 2$. Therefore, by Observation 2.5 (1), $f\gamma_{eD}(K_{m,n}) = 0$.

ii) If m = n = 1, then G $\cong$ K_2, and has a unique edge detour dominating basis. Therefore, $f\gamma_{eD}(K_{m,n}) = 0$.

If m = 2 and $n \geq 2$ (or n = 2 and $m \geq 2$) then, V_1, (or V_2) is the unique minimum edge detour dominating basis of G and $\gamma_{eD}(K_{m,n}) = 2$. Hence, $f\gamma_{eD}(K_{m,n}) = 0$.

iii) For, m,n$\geq$. Consider, any three element set of the form $\{a_i, ,a_j ,b_k / 1 \leq i, j \leq m; 1 \leq k \leq n\}$. Here, every edge of G lie on a detour joining a_i and a_j. Also, a_i, ,a_j, b_k dominates all the vertices of G. Therefore,$\{a_i, ,a_j ,b_k\}$ is an edge detour dominating basis of G. Further, no two element subset of V is an edge detour dominating basis of $K_{m,n}$. Hence, $\{a_i, a_j, b_k\}$ is a minimum edge detour dominating basis of $K_{m,n}$ and $\gamma_{eD}(K_{m,n}) = 3$ Also, each vertex belongs to more than one edge detour dominating basis and since $\gamma_{eD}(K_{m,n}) = 3$, $f\gamma_{eD}(K_{m,n}) = 3$.

Theorem 2.16. *Let T be a tree. If the set of all end vertices of T forms an edge detour dominating set of G then, $f\gamma_{eD}(T) = 0$.*

Proof. By the remark 1.5, the set of all end vertices of a tree is the unique edge detour dominating basis of T. Therefore, $f\gamma_{eD}(T) = 0$.

Theorem 2.17. *For, each pair a, b of integers with $0 \leq a \leq b$, and a is even, there is an edge detour dominating graph with $f\gamma_{eD}(G) = a$ and $\gamma_{eD}(G) = b$.*

Proof. **Case 1:** a = 0 and $b \geq 2$. Let G be the graph P_4. Attach r and s end vertices to P_4 such that r + s = b. Then, the set of all end vertices of P_4 is the unique edge detour dominating set. Therefore, $\gamma_{eD}(G) = b$. By Observation 2.5(1), $f\gamma_{eD}(G) = 0$.

Case 2: $a \geq 1$. Consider H = $K_{1, b-a}$ be the star with end vertices $t_1, t_2, \ldots, t_{b-a}$. Suppose for i = 1 to a/2, $F_i = C_5$ (x_i, y_i, z_i, u_i, v_i, x_i) be the cycle. Let v be the central vertex of H. By an edge, attach a/2 cycles F_i of length 5 to v. The graph is as in Figure 2.2.

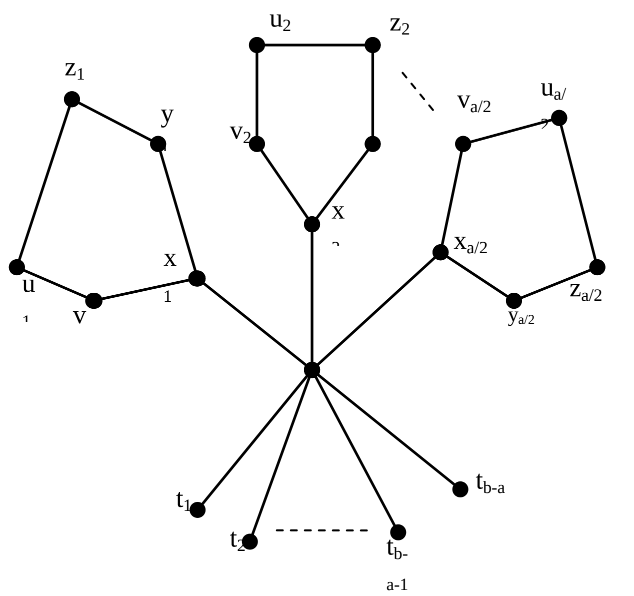

Figure 2.2

Claim1: $\gamma_{eD}(G) = b$. Obviously, $\{t_1, t_2, ..., t_{b-a}, u_i, x_i\}$ is an edge detour dominating set of G. Therefore, $\gamma_{eD}(G) \leq b - a + 2(a/2) = b$. --------------------(1)

To dominate the vertices of each of the a/2 cycles, we need at least two vertices from each cycle. Also, $\{t_1, t_2, ..., t_{b-a}\} \subseteq S$ for all γ_{eD} -set of G. Therefore, $\gamma_{eD}(G) \leq b - a + 2(a/2) = b$. --------------------(2)

By (1) and (2), $\gamma_{eD}(G) = b$.

Claim 2: $f\gamma_{eD}(G) = a$.

Since, $\{t_1, t_2, ..., t_{b-a}\}$ is the set of all edge detour dominating vertices, by Theorem 2.7, $f\gamma_{eD}(G) \leq b - (b - a)$. Clearly, a set S is an edge detour dominating basis of G if and only if S = $\{t_1, t_2, ..., t_{b-a}\} \cup \backslash\{s_{i1}, s_{i2}/1 \leq i \leq a/2\}$ where $\{s_{i1}, s_{i2}\}$ is a dominating set of F_i. It is obvious that any set T which is a proper subset of S- $\{t_1, t2, ..., t_{b-a}\}$ is contained in at least two edge detour dominating basis and $\{x_1, u_1, x_2, u_2, .., x_{a/2}, u_{a/2}\}$ is a subset of S - $\{t_1, t_2, ..., t_{b-a}\}$ such that S is edge detour dominating basis containing it. Therefore, $f\gamma_{eD}(G) = |x_1, u_1, x_2, u_2, \cdots, x_{a/2}, u_{a/2}|$ = 2(a/2) = a.

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