I_{gm} - Closed Sets

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Abstract

We define $I_{gm} - closed$ sets in (X, M, I) and discuss their properties.

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1 Introduction and preliminaries

An ideal I on a topological space (X, τ) is a non empty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A, B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : P(X) \to P(X)$ called a local function [5] of A with respect to τ and I is defined as follows: for $A \subset X, A^*(X, \tau) = \{x \in X / U \cap A \notin I,$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau / x \in U\}$.

A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(I,\tau)$ called the $*-topology$, finer than τ, is defined by $cl^*(A) = A \cup A^*(I,\tau)$ [8]. When there is no confusion we will simply write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal space. A subset A of an ideal space (X,τ,I) is said to be $*-closed$ [4] if $A^* \subset A$ and $*-dense$ in itself it $A \subset A^*$ [3]. A subset A of a an ideal space (X,τ,I) is said to be $I_g-closed$ [2] if $A^* \subset U$ whenever $A \subset U$ and U is open. A Subset A of an ideal space (X,τ,I) is said to be I_g-open if $X-A$ is $I_g-closed$. A subset A of an ideal space (X,τ,I) is said to be $I-locally*-closed$ [7] if there exists an open set U and a $*-closed$ set F such that $A = U \cap F$

By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X, cl(A)$ and $int(A)$ will respectively, denote the closure and interior of A in (X,τ) and $int^*(A)$ will denote the interior of A in (X,τ^*). A subset A of a topological space (X,τ) is said to be a $g-closed$ set [6] if $cl(A) \subset U$ whenever $A \subset U$ and U is open. A subset A of a topological space (X,τ) is said to be a $g-open$ set if $X-A$ is a $g-closed$ set. A sub collection M of $P(X)$ is called a minimal structure [1] on X, if (i) $\phi, X \in M$ and (ii) M is closed under finite intersection.

(X,M) is called a minimal space. If I is an ideal on X, (X,M,I) is called an ideal minimal space. If $U \in M$, U is said to be a $m-open$ set. The complement of a $m-open$ set is called $m-closed$ set. We set $mint(A) = \cup\{U \in M/U \subset A\}$ and $mcl(A) = \cap\{F/A \subset F$ and $X-F \in M\}$. A subset A of (X,M) is said to be $mg-closed$ [1] if $mvl(A) \subset U$ whenever $A \subset U$ and $U \in M$. The complement of $mg-closed$ set is called $mg-open$ set.

2 I_{gm} - closed sets

If (X, M) is a $m-space$, we denote the topology generated by M by τ_m. If (X, M, I) is an ideal $m-space$, then (X, τ_m, I) is an ideal topological space. We denote the $\star-topology$ generated by I and τ_m on X by τ_m^*.

For a subset A of X, we denote the local function of A with respect to I and τ_m by A^* and closure of A in τ_m and τ_m^* by $cl(A)$ and $cl^*(A)$ respectively.

A subset A of an ideal $m-space(X, M, I)$ is said to be $I_{gm}-closed$ if $A^* \subset U$ whenever A $\subset U$ and $U \in M$. The complement of an $I_{gm}-closed$ set is called an $I_{gm}-open$ set.

A subset A of (X, τ_m) is said to be $g_m-closed$ if $cl(A) \subset U$ whenever $A \subset U$ and $U \in M$. The complement of a $g_m-closed$ set is called a g_m-open set.

Since $cl^*(A) \subset cl(A) \subset mcl(A)$ and $M \subset \tau_m$ we have the following diagram.

$$\begin{array}{ccccc} m-closed & \rightarrow & closed & \rightarrow & \star-closed \\ \downarrow & & \downarrow & & \downarrow \\ mg-closed & \rightarrow & g-closed & \rightarrow & I_g-closed \\ & & \downarrow & & \downarrow \\ & & g_m-closed & \rightarrow & I_{gm}-closed \end{array}$$

If $M = \tau$, a topology on X, then $\tau_m = \tau$, $cl(A) = mcl(A)$ and hence the concepts $mg-closed$, $g-closed$ and $g_m-closed$ are coincide and the concepts $I_g-closed$ and $I_{gm}-closed$ are coincide.

The Theorems 2.1 and 2.2 gives characterizations for $I_{gm}-closed$ sets.

Theorem 2.1. *A subset A of an ideal $m-space$ (X, M, I) is $I_{gm}-closed$ if and only it $cl^*(A) \subset U$ whenever $A \subset U$ and $U \in M$.*

Proof. Suppose that A is $I_{gm}-closed$. Then $A^* \subset U$ whenever $A \subset U$ and $U \in M$. Therefore, $A \cup A^* \subset U$ whenever $A \subset U$ and U $\in M$. (ie) $cl^*(A) \subset U$

whenever $A \subset U$ and U $\in M$. Converse follows from the fact that $A^* \subset Cl^*(A)$.

For a subset A of an ideal $m - space$ (X, M, I), define $\Lambda_m(A) = \cap\{U \in M/ A \subset U\}$. A is said to be a $\Lambda_m - set$ if $\Lambda_m(A) = A$.

Theorem 2.2. *A subset A of an ideal* $m - space$ (X, M, I) *is* $I_{gm} - closed$ *if and only it* $cl^*(A) \subset \Lambda_m(A)$.

Proof. Suppose A is $I_{gm} - closed$. Let $U \in M$ be such that $A \subset U$. Then $cl^*(A) \subset U$. Therefore $cl^*(A) \subset \cap\{U \in M/A \subset U\}$. (ie) $cl^*(A) \subset \Lambda_m(A)$.

Conversely, suppose $cl^*(A) \subset \Lambda m(A)$. If $A \subset U$ and U $\in M$ then $\Lambda_m(A) \subset U$ and hence $cl^*(A) \subset U$. Therefore A is $I_{gm} - closed$.

The Theorem 2.3 gives some properties of $I_{gm} - closed$ sets and Example 2.4 shows that the converse need not be true.

Theorem 2.3. *Let* (X, M, I) *be an ideal* $m - space$ *If* $A \subset X$ *is* $I_{gm} - closed$, *then the following properties hold.*

(a) $cl^*(A) - A$ *contains no non empty* $m - closed$ *set.*

(b) $A^* - A$ *contains no non empty* $m - closed$ *set.*

Proof. Let A be $I_{gm} - closed$ set.

(a). Suppose $V \subset cl^*(A) - A$ and V is $m - closed$. since A is $I_{gm} - closed$ and $X - V$ is a $m - open$, set containing A, $cl^*(A) \subset X - V$. Hence $V \subset X - Cl^*(A)$. Since $V \subset Cl^*(A)$ and $V \subset cl^*(A) - A$, we get $V = \phi$

(b) If A is $I_{gm} - closed$, then by (a) $cl^*(A) - A$ contains no non empty closed set. But $cl^*(A) - A = A^* - A$. Therefore (b) follows.

Example 2.4. *Let* $X = \{a, b, c\}, M = \{\phi, \{a\}, \{b\}, \{b, c\}, X\}$ *and* $I = \{\phi, \{a\}\}$. *Then* $\tau_m = \{\phi, \{a\}, \{b\}\{a, b\}, \{b, c\}, X\}$

If $A = \{b\}$, *then* $A^* - A = \{c\}$, *which contains no non empty* $m - closed$ *sets. But* A *is not* $I_{gm} - closed$.

Theorem 2.5. *Suppose a subset A of an ideal $m-space$ is both $I_{gm}-closed$ and $m-open$. Then it is $\star-closed$.*

Proof. Since A is $I_{gm}-closed$. $A \subset A$ and $A \in M$ implies that $cl^*(A) \subset A$. Hence A is $\star-closed$.

Theorem 2.6. *Let (X, M, I) be an ideal $m-space$. Then every subset of X is $I_{gm}-closed$ if and only if every $m-open$ set is $\star-closed$.*

Proof. Suppose every subset of X is $I_{gm}-closed$. Let U be an $m-open$ set. Since $U \subset U$, from the definition of $I_{gm}-closed$ sets, $U^* \subset U$ and hence $cl^*(U) \subset U$. Therefore U is $\star-closed$.

Conversely, suppose that every $m-open$ set is $\star-closed$. If A is any subset of X and $A \subset U$, U is $m-open$, then $A^* \subset U^* \subset cl^*(U) = U$ and hence A is $I_{gm}-closed$.

Theorem 2.7. *If A is an $I_{gm}-closed$ subset of an ideal $m-space$ (X, M, I), then the following properties are equivalent.*

(a)A is a $\star-closed$ set

(b) $cl^*(A) - A$ is a $m-closed$ set

(c) $A^* - A$ is a $m-closed$ set.

Proof. $(a) \Rightarrow (b)$. If A is $\star-closed$ then $cl^*(A) = A$ and hence $cl^*(A) - A = \phi$, which is $m-closed$.

$(b) \Rightarrow (c)$. Since $cl^*(A) - A = A^* - A, A^* - A$ is $m-closed$.

$(c) \Rightarrow (a)$. Suppose $A^* - A$ is $m-closed$. Since A is $I_{gm}-closed$, by Theorem 2.3, $A^* - A$ contains no non empty $m-closed$ set. Therefore, $A^* - A = \phi$ and hence $A^* \subset A$. So A is $\star-closed$.

Theorem 2.8. *Let (X, M, I) be an ideal $m-space$. Then a subset A of X is $\star-closed$ if and only if $A^* - A$ is $m-closed$ and A is $I_{gm}-closed$.*

Proof. Suppose A is $\star - closed$. Then $A^* - A = cl^*(A) - A = \phi$, which is $m - closed$. Also every $\star - closed$ set is $I_{gm} - closed$. Hence A is $I_{gm} - closed$.

Conversely, suppose $A^* - A$ is $m - closed$ and A is $I_{gm} - closed$. Then by Theorem 2.7, A is $\star - closed$.

Theorem 2.9. *Let (X, M, I) be an ideal $m - space$. If A is $\star - dense$ in itself, then $A^* = mcl(A^*) = mcl(A)$*

Proof. Clearly , $A^* \subset mcl(A^*)$. It $x \notin A^*$, then there exist $U \in \tau_m$ such that $x \in U$ and $U \cap A \in I$. Since τ_m is generated by M , there exist $V \in M$ such that $x \in V \subset U$. Since $V \cap A \subset U \cap A \in I$, we have $V \cap A \in I$. If $x \in V$, then $x \in U$ and $U \cap A \in I$ and hence $x \notin A^*$. Therefore $V \cap A^* = \phi$. So $x \notin mcl(A^*)$. This proves that $mcl(A^*) \subset A^*$. Therefore $A^* = mcl(A^*)$. Since A is $\star - dense$ in itself, $A \subset A^*$ and hence $mcl(A) \subset mcl(A^*)$.

On the other hand, $A^* \subset cl^*(A) \subset cl(A) \subset mcl(A)$ and hence $mcl(A^*) \subset mcl(A)$. Therefore $mcl(A^*) = mcl(A)$.

In general $I_{gm} - closed$ sets need not be $mg - closed$. The following Theorem 2.10 gives a condition where it is $mg - closed$.

Theorem 2.10. *Let (X, M, I) be an ideal $m - space$ and A is a subset of X. If A is $\star - dense$ in itself and $I_{gm} - closed$, then A is $mg - closed$.*

Proof. A is $\star - dense$ in itself. So $A \subset A^*$. Therefore, $cl^*(A) = A \cup A^* = A^*$. Suppose $U \in M$ and $A \subset U$. Since A is $I_{gm} - closed$, by Theorem 2.1, $cl^*(A) \subset U$. Therefore $A^* \subset U$. Since A is $\star - dense$ in itself, by Theorem 2.9, $A^* = mcl(A)$. Therefore $mcl(A) \subset U$ and hence A is $mg - closed$.

Theorem 2.11. *Let (X, M, I) be an ideal $m - space$ and A, B be subsets of X. If A is $I_{gm} - closed$ and $A \subset B \subset cl^*(A)$, then B is also $I_{gm} - closed$.*

Proof. Suppose $B \subset U$ and U is $m-open$. Then $A \subset U$. Since A is $I_{gm}-closed, cl^*(A) \subset U$. Since $A \subset B \subset cl^*(A), cl^*(A) \subset cl^*(B) \subset cl^*(cl^*(A)) = cl^*(A)$.

Hence $cl^*(A) = cl^*(B)$. Therefore, $cl^*(B) \subset U$ and hence B is $I_{gm}-closed$.

Theorem 2.12. *Union of two $I_{gm}-closed$ sets is an $I_{gm}-closed$ set.*

Proof. Let (X, M, I) be an ideal $m-space$ and let A and B be $I_{gm}-closed$ sets in X. Suppose $A \cup B \subset U$ and U is $m-open$. Since A is $I_{gm}-closed$ and $A \subset U$, we have $cl^*(A) \subset U$. Similarly $cl^*(B) \subset U$. Therefore $cl^*(A \cup B) = cl^*(A) \cup cl^*(B) \subset U$. Hence $A \cup B$ is $I_{gm}-closed$.

The following Example 2.13 shows that intersection of two $I_{gm}-closed$ sets need not be $I_{gm}-closed$.

Example 2.13. *Let $X = \{a, b, c\}, M = \{\phi, \{a\}, \{b\}, \{b, c\}, X\}$ and $I = \{\phi, \{a\}\}$. Then $\tau_m = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$.*

Let $A = \{a, b\}$ and $B = \{b, c\}$. Since X is the only $m-open$ set containing A, A is $I_{gm}-closed$. Since, $B^* = \{b, c\}, B$ is $\star-closed$ and hence $I_{gm}-closed$. Now $A \cap B = \{b\}$, which is $m-open$ and $(A \cap B)^* = \{b, c\}$. Therefore $A \cap B$ is not $I_{gm}-closed$.

Theorem 2.14. *Let (X, M, I) be an ideal $m-space$ and A, B be subsets of X. If $A \subset B \subset A^*$ and A is $I_{gm}-closed$, then B is $mg-closed$.*

Proof. Since $A \subset B \subset A^*$, we have $A^* \subset B^* \subset (A^*)^* = A^*$. Therefore $A^* = B^*$ and hence A and B are $\star-dense$ in itself. Since $A \subset B \subset A^* \subset cl^*(A)$ and A is $I_{gm}-closed$, by Theorem 2.11, B is $I_{gm}-closed$. Since B is $\star-dense$ in itself and $I_{gm}-closed$, by Theorem 2.10, B is $mg-closed$.

Theorem 2.15. *Let(X, M, I) be an ideal $m-space$ and $I = \{\phi\}$. Then A is $I_{gm}-closed$ if and only if A is $mg-closed$.*

Proof. Since $I = \{\phi\}, cl(A) = cl^*(A)$, for every subset of X. Therefore $cl^*(A) \subset U$ if and only if $cl(A) \subset U$. Therefore A is $I_{gm} - closed$ if and only if A is $mg - closed$.

Theorem 2.16. *Let (X, M, I) be an ideal $m - space$. For every $x \in X$, the set $X - \{x\}$ is $I_{gm} - closed$ or $m - open$.*

Proof. Suppose $X - \{x\}$ is not $m - open$. Then X is the only $m - open$ set contains $X - \{x\}$ and $(X - \{x\})^* \subset X$. Hence $X - \{x\}$ is $I_{gm} - closed$.

The following theorem 2.17, gives a characterization for $I_{gm} - open$ sets.

Theorem 2.17. *Let (X, M, I) be an ideal $m - space$ and $A \subset X$. Then A is $I_{gm} - open$ if and only if $F \subset int^*(A)$ whenever F is $m - closed$ and $F \subset A$.*

Proof. Suppose A is $I_{gm} - open$ and $F \subset A$, F is $m - closed$. Then $X - A \subset X - F$, $X - F$ is $m - open$ and $X - A$ is $I_{gm} - closed$. Therefore $cl^*(X - A) \subset X - F$. Therefore $F \subset X - cl^*(X - A) = int^*(A)$.

Conversely, suppose $F \subset int^*(A)$ whenever $F \subset A$ and F is $m - closed$. If $X - A \subset U$ and U is $m - open$, then $X - U \subset A$ and $X - U$ is $m - closed$. Therefore, by hypothesis, $X - U \subset int^*(A)$ and hence $cl^*(X - A) = X - int^*(A) \subset U$. Therefore $X - A$ is $I_{gm} - closed$ and hence A is $I_{gm} - open$.

Since every $m - closed$ set is $I_{gm} - closed$, every $m - open$ set is $I_{gm} - open$.

Theorem 2.18. *Let (X, M, I) be an ideal $m - space$ and A, B be subsets of X. If A is $I_{gm} - open$ and $int^*(A) \subset B \subset A$, then B is $I_{gm} - open$.*

The proof follows from the Theorem 2.11.

Theorem 2.19. *Intersection of two $I_{gm} - open$ sets is an $I_{gm} - open$ set.*

The proof follows from the Theorem 2.12.

Theorem 2.20. *If a subset A of an ideal $m - space$ (X, M, I) is $I_{gm} - closed$ then $A \cup (X - A^*)$ is also $I_{gm} - closed$.*

Proof. Suppose A is $I_{gm}-closed$. If $A\cup(X-A^*)\subset U$ and U is $m-open$, then $X-U\subset X-[A\cup(X-A^*)]=(X-A)\cap A^*=A^*-A$ and $X-U$ is $m-closed$. Since A is $I_{gm}-closed$, by, Theorem 2.3, $X-U=\phi$ and hence $X=U$. Therefore X is the only $m-open$ set containing $A\cup(X-A^*)$ and hence $A\cup(X-A^*)$ is $I_{gm}-closed$.

Theorem 2.21. *Let (X,M,I) be an ideal $m-space$. Then the following are equivalent.*

(a) Every $I_{gm}-closed$ set $\star-closed$

(b) Every singleton of X is either $m-closed$ or $\star-open$.

Proof. $(a)\Rightarrow(b)$. Let $x\in X$. If $\{x\}$ is not $m-closed$, then $X-\{x\}$ is not $m-open$. Therefore X is the only $m-open$ set containing $X-\{x\}$ and $X-\{x\}$ is $I_{gm}-closed$ set. By hypothesis, $X-\{x\}$ is $\star-closed$ and hence $\{x\}is\ \star-open$.

$(b)\Rightarrow(a)$. Let A be an $I_{gm}-closed$ set and $x\in A^*$. We have to prove that $x\in A$.

Case (i). If $\{x\}$ is $m-closed$ and $x\notin A$, then $A\subset X-\{x\}$ and $X-\{x\}$ is $m-open$. Since A is $I_{gm}-closed$, A $^*\subset X-\{x\}$ and hence $x\notin A^*$, which is a contradiction.

Case (ii). If {x} is $\star-open$, since $x\in A^*$ we have $x\in cl^*(A)$ and hence $\{x\}\cap A\neq\phi$. (ie) $x\in A$. Therefore $A^*\subset A$ and hence A is $\star-closed$.

A subset A of an ideal $m-space$ (X,M,I) is said to be $m-locally\ \star-closed$ if there exist a $m-open$ set U and a $\star-closed$ set F of (X,τ_m^*) such that $A=U\cap F$. The set A is said to be $m-locally$ closed if there exist a $m-open$ set U and a closed set F of (X,τ_m) such that $A=U\cap F$.

If $I=\{\phi\}$, then the concept $m-locally\ *-closed$ sets coincide with $m-locally$ closed set.

Theorem 2.22. *Let* (X, M, I) *be an ideal* $m-space$ *and* A *be a subset of* X*. Then the following statements are equivalent.*

(a) A *is* $m-locally \star -closed$

(b) $A = U \cap cl^*(A)$*, for some* $m-open$ *set* U*.*

Proof. $(a) \Rightarrow (b)$. If A is $m-locally \star -closed$ Then there exist a $m-open$ set U and a $\star - closed$ set F such that $A = U \cap F$. Clearly $A \subset U \cap cl^*(A)$. On the other hand, since F is $\star - closed$, $A \subset F$ implies that $cl^*(A) \subset cl^*(F) = F$ and so $U \cap cl^*(A) \subset U \cap F = A$. Therefore $A = U \cap cl^*(A)$

$(b) \Rightarrow (a)$ is clear.

Theorem 2.23. *Let* (X, M, I) *be an ideal* $m-space$ *and* A *be a* $m-locally \star - closed$ *subset of* X*. Then the following properties hold.*

(a) $A^* - A$ *is closed*

(b) $(X - A^*) \cup A = A \cup (X - cl^*(A))$ *is open*

(c) $A \subset int(A \cup (X - A^*))$

(d) A *is* $I-locally \star -closed$ *in* (X, τ_m^*).

Proof. (a) Since A is $m-locally \star -closed$, by Theorem 2.21, $A = U \cap cl^*(A)$, for some $m-open$ set U. Then, $A^* - A = A^* \cap (X - A) = A^* \cap [X - (U \cap cl^*(A))]$ $A^* \cap [(X - U) \cup (X - cl^*(A))] = (A^* \cap (X - U)) \cup (A^* \cap (X - cl^*(A)) = A^* \cap (X - U)$, which is closed (b). $X - (A^* - A) = X - [A^* \cap (X - A)] = (X - A^*) \cup A$

By (a), $(X - A^*) \cup A$ is open. Also $(X - A^*) \cup A = A \cup (X - cl^*(A))$.

(c). Since A is $m-locally \star -closed$, by (b) , $A \cup (X - A^*)$ is open.

Therefore $A \cup (X - A^*) \subset int[A \cup (X - A^*)]$ and hence $A \subset int[A \cup (X - A^*)]$.

(d). The proof follows from the fact that every $m-open$ set is open.

Theorem 2.24. *Let* (X, M, I) *be an ideal* $m-space$ *and* A *is a* $m-locally\star -closed$ *and* $I-dense$ *subset of* X*. Then* A *is open.*

Proof. If A is $m-locally\star -closed$, then by Theorem 2.23(d), A is $I-locally\star -closed$. Therefore, by Theorem 3.1 (e)[7], $A \subset int[A \cup (X-A^*)]$. Since A is $I-dense$, $A^* = X$ and so $A \subset int(A)$. Therefore A is open.

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