$\delta -\ I_g$ - Closed Sets

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Abstract

We define $\delta - I_g - closed$ sets and discuss their properties. Using these sets we characterize. $T_{1/2} - spaces$ and $T_I - spaces$.

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1 Introduction and preliminaries

An ideal I on a topological space (X, τ) is a non empty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A, B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : P(X) \to P(X)$ called a local function [8] of A with respect to τ and I is defined as follows: for $A \subset X, A^*(X, \tau) = \{x \in X | U \cap A \notin I,$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(I, \tau)$ called the

$\star-topology$, finer than τ, is defined by $cl^*(A) = A \cup A^*(I, \tau)$ [13]. When there is no confusion we will simply write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an *ideal space*. A subset A of an ideal space $(X\tau, I)$ is said to be $\star-closed$ [7] if $A^* \subset A$. A subset A of an ideal space (X, τ, I) is said to be $I_g-closed$ [2] if $A^* \subset U$ whenever $A \subset U$ and U is open. A subset A of an ideal space (X, τ, I) is said to be I_g-open if $(X-A)$ is $I_g-closed$. An ideal space (X, τ, I) is said to be a $T_I-space$ [2] if every $I_g-closed$ set is $\star-closed$. A subset A of an ideal space (X, τ, I) is said to be $I-locally\star-closed$ [12] if there exist an open set U and a $\star-closed$ set F such that $A = U \cap F$. If $I = \{\phi\}$, then $I-locally\star-closed$ sets coincide with locally $closed$ sets.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X, cl(A)$ and $int(A)$ will respectively, denote the closure and interior of A in (X, τ) and $int^*(A)$ will denote the interior of A in (X, τ^*). A subset A of a topological space (X, τ) is said to be a $g-closed$ set [9] if $cl(A) \subset U$ whenever $A \subset U$ and U is open. A subset A of a topological space (X, τ) is said to be a $g-open$ set if $X-A$ is a $g-closed$ set. A space (X, τ) is said to be a $T_{1/2}-space$ [9] if every $g-closed$ set is a closed set.

For a subset A of a space $(X, \tau), the\ \ \theta-interior$ [14] of A is the union of all open sets of X whose closures contained in A and is denoted by $int_\theta(A)$. The subset A is called $\theta-open$ if $A = int_\theta(A)$. The complement of a $\theta-open$ set is called a $\theta-closed$ set. Equivalently, $A \subset X$ is called $\theta-closed$ [14] if $A = cl_\theta(A) = \{x \in X | cl(U) \cap A \neq \phi$ for all $U \in \tau(x)\}$. The family of all $\theta-open$ sets of X forms a topology [14] on X, which is coarser than τ and is denoted by τ_θ. A subset A of a topological space (X, τ) is said to be a $\theta-g-closed$ set [3] if $cl_\theta(A) \subset U$ whenever $A \subset U$ and U is open. A subset A of a space (X, τ) is said to be a $\theta-g-open$ set [3] if $X-A$ is a $\theta-g-closed$ set. A subset A of a space (X, τ) is said to be a $\Lambda-set$ [10,11] if $A = A^\Lambda$, where $A^\Lambda = \cap\{U \in \tau | A \subset U\}$.

A subset A of an ideal space (X, τ, I) is said to be $\theta - I - closed$ [1] if $cl^*_\theta(A) = A$, where $cl^*_\theta(A) = \{x \in X | A \cap cl^*(U) \neq \phi$ for all $U \in \tau(x)\}$. A is said to be $\theta - I - open$ if $X - A$ is $\theta - I - closed$. If $I = \{\phi\}, cl^*_\theta(A) = cl_\theta(A)$. If $I = P(X), cl^*_\theta(A) = cl(A)$. For a subset A of X, $int_\theta I(A) = \cup\{U \in \tau | cl^*((U) \subset A\}$ [1]. We denote this $int_\theta I(A)$ by $int^*_\theta(A)$. The family of all $\theta - I - open$ sets of (X, τ, I) is a topology and it is denoted by $\tau_{\theta - I}$ (see [1, Theorem 1]).

For a subset A of $(X, \tau, I), [A]_{\delta - I} = \{x \in X/A \cap int(cl^*(U)) \neq \phi$ for all $U \in \tau(x)\}$ [15], is called $\delta - I - closure$ of A. We denote $[A]_{\delta - I}$ by $cl^*_\delta(A)$. The set A is said to be $\delta - I - closed$ if $cl^*_\delta(A) = A$. The complement of $\delta - I - closed$ set is said to be $\delta - I - open$. For a subset A of a space (X, τ), $cl_\delta(A) = \{x \in X/A \cap int(cl(U)) \neq \phi$ for all $U \in \tau(x)$[14]. If $cl_\delta(A) = A$, then A is said to be $\delta - closed$. The complement of a $\delta - closed$ set is said to be a $\delta - open$ set. The family of all $\delta - open$ sets of X form a topology τ_δ. The family of all $\delta - I - open$ sets form a topology $\tau_{\delta - I}$ on X.

Lemma 1.1. *[15, Theorem 2.3] Let (X, τ, I) be an ideal space and $\tau_{\delta - I} = \{A \subset X/A$ is a $\delta - I - open$ set of $(X, \tau, I)\}$. Then $\tau_{\delta - I}$ is a topology such that $\tau_\delta \subset \tau_{\delta - I} \subset \tau$.*

Lemma 1.2. *[15, proposition 2.1] Let (X, τ, I) be an ideal space. (1). If $I = \{\phi\}$ or the ideal N of nowhere dense sets of (X, τ), then $\tau_{\delta - I} = \tau_\delta$. (2). If $I = P(X)$, then $\tau_{\delta - I} = \tau$.*

2 $\delta - I_g$ - closed sets

A subset A of an ideal space (X, τ, I) is said to be $\delta - I_g - closed$ if $cl^*_\delta(A) \subset U$, whenever $A \subset U$ and U is open. The complement of $\delta - I_g - closed$ set is called $\delta - I_g - open$ set. The set A is said to be $\delta - g - closed$ [4], if $cl_\delta(A) \subset U$, whenever $A \subset U$ and U is open. The complement of $\delta - g - closed$ set is $\delta - g - open$.

Every $\delta - I - closed$ set is $\delta - I_g - closed$. If $I = \{\phi\}$ or the ideal N of nowhere dense subsets of (X, τ), then $\delta - I_g - closed$ sets coincide with $\delta - g - closed$ sets. If $I = P(X)$, then $\delta - I_g - closed$ sets coincide with $g - closed$ sets. Since $cl^*(A) \subset cl(A) \subset cl^*_\delta(A) \subset cl^*_\theta(A) \subset cl_\theta(A)$, we have the following inclusion diagram.

$$\theta - g - closed \;\rightarrow\; \theta - I_g - closed \;\rightarrow\; \delta - I_g - closed$$
$$\rightarrow\; g - closed \;\rightarrow\; I_g - closed.$$

Since $cl(A) \subset cl^*_\delta(A) \subset cl_\delta(A)$ we have the following inclusion diagram.

$$\delta - g - closed \;\rightarrow\; \delta - I_g - closed \;\rightarrow\; g - closed$$

The following Example 2.1 shows that a $g - closed$ set need not be $\delta - I_g - closed$.

Example 2.1. *Let* $X = \{a, b, c, d\}, \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$ *and* $I = \{\phi, \{a\}, \{c\}, \{a, c\}\}$.

Then $int(cl^*(U)) = X$, *for all* $U \in \tau$. *Therefore,* $cl^*_\delta(A) = X$, *for all subsets* A *of* X. *Let* $A = \{d\}$. *Then* A *is closed and hence* $g - closed$. *But* A *is not* $\delta - I_g - closed$, *because,* $A \subset \{a, b, d\}, \{a, b, d\}$ *is open and* $cl^*_\delta(A) = X \not\subset \{a, b, d\}$

The following Example 2.2 shows that $\delta - I_g - closed$ set need not be $\theta - I_g - closed$ set.

Example 2.2. *Let* $X = \{a, b, c, d, e\}, \tau = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{c, d, e\}, \{b, c, d, e\}, X\}$ *and* $I = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. *Let* $A = \{a, b\}$. *Then* $cl^*_\delta(A) = A$. *Therefore,* A *is* $\delta - I - closed$ *and hence* $\delta - I_g - closed$. *But* A *is not* $\theta - I_g - closed$, *because,* $A \subset \{a, b, c\}, \{a, b, c\}$ *is open and* $cl^*_\theta(A) = X \not\subset \{a, b, c\}$.

The following Example 2.3 shows that every $\delta - I_g - closed$ set need not be $\delta - I - closed$ set.

Example 2.3. *Let* $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ *and* $I = \{\phi, \{a\}\}$. *Let* $A = \{d\}$. *Since* X *is the only open set containing* A, A *is* $\delta - I_g - closed$. *But* A *is not* $\delta - I - closed$, *because,* $cl^*_\delta(A) = \{b, d\} \neq A$.

The following Example 2.4 shows that $\delta - I_g - closed$ set need not be $\delta - g - closed$ set.

Example 2.4. *Let $X = R$, the real line, $\tau = \{\phi, \{x\}, (-\infty, x], [x, \infty), X\}$ where x is any element of R and $I = P((-\infty, x])$.*

Let $A = (x, \infty)$. Then $cl(U) = X$, for all nonempty $U \in \tau$, $int(cl(U)) = X$ and hence $cl_\delta(A) = X$. Therefore, A is not $\delta - g - closed$. Let $U = (-\infty, x]$. Then $U^* = \phi, cl^*(U) = U$, $int(cl^*(U)) = U$, $A \cap int(cl^*(U)) = A \cap U = \phi$. Therefore, $y \in U$ implies $y \notin cl^*_\delta(A)$. Therefore, $cl^*_\delta(A) = A$. Therefore, A is $\delta - I - closed$ and hence $\delta - I_g - closed$.

The following Example 2.5 shows that $\delta - I_g - closed$ set need not be $\theta - I_g - closed$ set.

Example 2.5. *Let $X = R$, the real line, $\tau = \{\phi, (0, 1), [1, 2), (0, 2), (-\infty, 2), [1, \infty), (0, \infty), X\}$ and $I = P((-\infty, 1))$. Let $A = (-\infty, 1)$. Then $cl^*_\delta(A) = A$. Therefore, A is $\delta - I - closed$ and hence $\delta - I_g - closed$. Now $A \subset (-\infty, 2)$. But $cl^*_\theta(A) = X \not\subset (-\infty, 2)$. Therefore, A is not $\theta - I_g - closed$.*

The following Theorem 2.6 gives characterization for $\delta - I_g - closed$ sets.

Theorem 2.6. *If A is a subset of an ideal space (X, τ, I), then the following are equivalent.*

(a) A is $\delta - I_g - closed$

(b) For all $x \in cl^*_\delta(A), cl(\{x\}) \cap A \neq \phi$

(c) $cl^*(A) - A$ contains no nonempty closed sets.

Proof. $(a) \Rightarrow (b)$. Suppose $x \in cl^*_\delta(A)$. If $cl(\{x\}) \cap A = \phi$, then $A \subset X - cl(\{x\})$. Since A is $\delta - I_g - closed$, $cl^*_\delta(A) \subset X - cl(\{x\})$, it is a contradiction to the fact that $x \in cl^*_\delta(A)$.

$(b) \Rightarrow (c)$. Suppose $F \subset cl_\delta^*(A) - A, F$ is closed and $x \in F$. Since $F \subset X - A$ and F is closed, $cl(\{x\}) \cap A \subset cl(F) \cap A = F \cap A = \phi$, which is a contradiction. This proves (c).

$(c) \Rightarrow (a)$. Let $A \subset U$, U is open. Since $cl_\delta^*(A)$ is closed, $cl_\delta^*(A) \cap (X - U)$ is closed and $cl_\delta^*(A) \cap (X - U) = cl_\delta^*(A) - U \subset cl_\delta^*(A) - A$. By hypothesis, $cl_\delta^*(A) \cap (X - U) = \phi$ and hence $cl_\delta^*(A) \subset U$. Therefore A is $\delta - I_g - closed$.

If we put $I = \{\phi\}$, in Theorem 2.6, we get Corollary 2.7, which gives characterizations for $\delta - g - closed$ sets. If we put $I = P(X)$, in Theorem 2.6, we get Corollary 2.8 which gives characterizations for $g - closed$ sets.

Corollary 2.7. *If A is a subset of a topological space (X, τ), then the following are equivalent*

(a) A is $\delta - g - closed$

(b)For all $x \in cl_\delta(A)$, $cl(\{x\}) \cap A \neq \phi$

(c)$cl_\delta(A) - A$ contains no nonempty closed set.

Corollary 2.8. *If A is a subset of a topological space(X, τ), then the following are equivalent.*

(a) A is $g - closed$

(b) For all $x \in cl(A)$, $cl(\{x\}) \cap A \neq \phi$

(c) $cl(A) - A$ contains no nonempty closed set.

(X, τ) is said to be a $T_1 - space$, if given any two different points a and b of X , each has a neighbourhood not containing the other.

The following Corollary 2.9 shows that in $T_1 - spaces$, $\delta - I_g - closed$ sets are $\delta - I - closed$, the proof of which follows from Theorem 2.6(c). Corollary 2.10 gives a relation between $\delta - I_g - closed$ and $\delta - I - closed$.

Corollary 2.9. *If (X, τ, I) is a $T_1 - space$ and A is $\delta - I_g - closed$ set, then A is a $\delta - I - closed$ set.*

Corollary 2.10. *If (X, τ, I) is an ideal space and A is $\delta - I_g - closed$ set, then the following are equivalent.*

(a) A is a $\delta - I - closed$ set

(b) $cl^*_\delta(A) - A$ is a closed set.

Proof. $(a) \Rightarrow (b)$. If a is $\delta - I - closed$, then $cl^*_\delta(A) - A = \phi$ and so $cl^*_\delta(A) - A$ is closed.

$(b) \Rightarrow (a)$. If $cl^*_\delta(A) - A$ is closed, since A is $\delta - I_g - closed$, by Theorem 2.6, $cl^*_\delta(A) - A = \phi$ and so $cl^*_\delta(A) = A$, which proves (a).

If we put $I = \{\phi\}$ in Corollary 2.10, we get Corollary 2.11. If we put $I = P(X)$ in Corollary 2.10, we set Corollary 2.12.

Corollary 2.11. *If (X, τ) is a topological space and A is a $\delta - g - closed$ set, then the following are equivalent.*

(a) A is a $\delta - closed$ set

(b) $cl_\delta(A) - A$ is a closed set.

Corollary 2.12. *If (X, τ) is a topological space and A is a $g - closed$ set, then the following are equivalent.*

(a) A is a closed set

(b) $cl(A) - A$ is a closed set.

Theorem 2.13. *If every open set of an ideal space (X, τ, I) is $\star - closed$, then every $g - closed$ set is $\delta - I_g - closed$.*

Proof. Since every open set is $\star - closed, cl^*(U) = U$ for every $U \in \tau$. Therefore, for every subset A of X, $cl^*_\delta(A) = \{x \in X / A \cap int(cl^*(U)) \neq \phi$ for all $U \in \tau(x)\}$ $= cl(A)$. This implies that every $g - closed$ set is $\delta - I_g - closed$.

Corollary 2.14. *If every subset of an ideal space* (X,τ,I) *is* $I_g-closed$*, then every* $g-closed$ *set is* $\delta-I_g-closed$*.*

The proof follows from the fact that every subset of X is $I_g-closed$ if and only if every open set is $\star-closed$ and Theorem 2.13.

Theorem 2.15. *Let* (X,τ,I) *be an ideal space. Then every subset of* X *is* $\delta-I_\delta-closed$ *if and only if every open set is* $\delta-I-closed$*.*

Proof. Suppose every subset of X is $\delta-I_g-closed$. If U is open, then U is $\delta-I_\delta-closed$, and so $cl^*_\delta(U)\subset U$. Hence U is $\delta-I-closed$.

Conversely, suppose $A\subset U$ and U is open. Since every open set is $\delta-I-closed$, $cl^*_\delta(A)\subset U$ and so A is $\delta-I_g-closed$.

If we put $I=\{\phi\}$ in Theorem 2.15, we set Corollary 2.16. If we put $I=P(X)$ in Theorem 2.15, we set Corollary 2.17.

Corollary 2.16. *Let* (X,τ) *be a topological space. Then every subset of* X *is* $\delta-g-closed$ *if and only if every open set is* $\delta-closed$*.*

Corollary 2.17. *Let* (X,τ) *be a topological space. Then every subset of* X *is* $g-closed$ *if and only if every open set is* $closed$*.*

Theorem 2.18. *Intersection of a* $\delta-I_g-closed$ *set and a* $\delta-I-closed$ *set is* $\delta-I_g-closed$*.*

Proof. Let A be a $\delta-I_g-closed$ set and F a $\delta-I-closed$ set of an ideal space (X,τ,I). Suppose $A\cap F\subset U$ and U is open in X. Then $A\subset U\cup(X-F)$. Now $X-F$ is an open set containing A. Since A is $\delta-I_g-closed$, $cl^*_\delta(A)\subset U\cap(X-F)$. Therefore $cl^*_\delta(A)\cap F\subset U$, which implies that $cl^*_\delta(A\cap F)\subset U$. So $A\cap F$ is $\delta-I_g-closed$.

If we put $I=\{\phi\}$ in Theorem 2.18, we get Corollary 2.19. If we put $I=P(X)$ in Theorem 2.18, we get Corollary 2.20.

Corollary 2.19. *Intersection of a* $\delta - g - closed$ *set and a* $\delta - closed$ *set is always* $\delta - g - closed$.

Corollary 2.20. *Intersection of a* $g - closed$ *set and a closed set is always a* $g - closed$ *set.*

Theorem 2.21. *A subset A of an ideal space* (X, τ, I) *is* $\delta - I_g - closed$ *if and only if* $cl^*_\delta(A) \subset A^\Lambda$

Proof. Suppose A is $\delta - I_g - closed$ and $x \in cl^*_\delta(A)$. If $x \notin A^\Lambda$, then there exists an open set U such that $A \subset U$, but $x \notin U$. Since A is $\delta - I_g - closed$, $cl^*_\delta(A) \subset U$ and $x \notin cl^*_\delta(A)$, a contradiction. Therefore, $cl^*_\delta(A) \subset A^\Lambda$.

Conversely, suppose that $cl^*_\delta(A) \subset A^\Lambda$. If $A \subset U$ and U is open, then $A^\Lambda \subset U$ and so $cl^*_\delta(A) \subset U$. Therefore, A is $\delta - I_g - closed$.

If we put $I = \{\phi\}$ in Theorem 2.21, we get Corollary 2.22. If we put $I = P(X)$ in Theorem 2.21, we get Corollary 2.23.

Corollary 2.22. *A subset A of a space* (X, τ) *is* $\delta - g - closed$ *if and only if* $cl_\delta(A) \subset A^\Lambda$.

Corollary 2.23. *A subset A of a space* (X, τ) *is* $g - closed$ *if and only if* $cl(A) \subset A^\Lambda$.

Theorem 2.24. *Let A be a* $\Lambda - set$ *of an ideal space* (X, τ, I). *Then A is* $\delta - I_g - closed$ *if and only if A is* $\delta - I - Closed$.

Proof. Suppose A is $\delta - I_g - closed$. By Theorem 2.21, $cl^*_\delta(A) \subset A^\Lambda = A$, since A is a $\Lambda - set$. Therefore, A is $\delta - I - closed$. Converse follows from the fact every $\delta - I - closed$ is $\delta - I_g - closed$.

If we put $I = \{\phi\}$ in Theorem 2.24, we get Corollary 2.25. If we put $I = P(X)$ in Theorem 2.24, we get Corollary 2.26.

Corollary 2.25. *Let A be a $\Lambda-set$ of a space (X,τ). Then A is $\delta-g-closed$ if and only if A is $\delta-closed$.*

Corollary 2.26. *Let A be a $\Lambda-set$ of a space (X,τ). Then A is $g-closed$ if and only if A is closed.*

Theorem 2.27. *Let (X,τ,I) be an ideal space and $A\subset X$. If A^{Λ} is $\delta-I_g-closed$, then A is also $\delta-I_g-closed$.*

Proof. Suppose that A^{Λ} is a $\delta-I_g-closed$ set. If $A\subset U$ and U is open, then $A^{\Lambda}\subset U$. Since A^{Λ} is $\delta-I_g-closed$, $cl^*_\delta(A^{\Lambda})\subset U$. But $cl^*_\delta(A)\subset cl^*_\delta(A^{\Lambda})$. Therefore, A is $\delta-I_g-closed$.

If we put $I=\{\phi\}$ in Theorem 2.27, we get Corollary 2.28. If we put $I=P(X)$ in Theorem 2.27, we get Corollary 2.29.

Corollary 2.28. *let $(X,\ \tau)$ be a topological space and $A\subset X$. If A^{Λ} is $\delta-g-closed$, then A is also $\delta-g-closed$.*

Corollary 2.29. *Let (X,τ) be a space and $A\subset X$. If A^{Λ} is $g-closed$ set, then A is also $g-closed$ set.*

Theorem 2.30. *For an ideal space (X,τ,I), the following are equivalent.*

(a) Every $\delta-I_g-closed$ set is $\delta-I-closed$.

(b) Every singleton of X is closed or $\delta-I-open$.

Proof. $(a)\Rightarrow(b)$. Let $x\in X$. If $\{x\}$ is not closed, then $A=X-\{x\}\notin\tau$ and then A is trivially $\delta-I_g-closed$. By (a), A is $\delta-I-closed$. Hence $\{x\} is$ $\delta-I-open$.

$(b)\Rightarrow(a)$. Let A be a $\delta-I_g-closed$ set and let $x\in cl^*_\delta(A)$. We have the following cases.

case (i). $\{x\}$ is closed. By Theorem 2.5, $cl^*_\delta(A) - A$ does not contain a nonempty closed set. This shows $x \in A$.

case (ii). $\{x\}$ is $\delta - I - open$. Then $\{x\} \cap A \neq \phi$. Hence, $x \in A$.

Thus in both the cases $x \in A$ and so $A = cl^*_\delta(A)$, that is, A is $\delta - I - closed$, which proves (a).

If we put $I = \{\phi\}$ in Theorem 2.30, we get Corollary 2.31. If we put $I = P(X)$ in Theorem 2.30, we get Corollary 2.32.

Corollary 2.31. *For a topological space (X, τ) the following are equivalent*

(a) Every $\delta - g - closed$ set is $g - closed$.

(b) Every singleton of X is closed or $\delta - open$.

Corollary 2.32. *For a topological space (X, τ), the following are equivalent.*

(a) Every $g - closed$ set is closed.

(b) Every singleton of X is closed or open.

Theorem 2.33. *Let (X, τ, I) be an ideal space and $A \subset X$. Then A is $\delta - I_g - closed$ if and only if $A = F - N$, where F is $\delta - I - closed$ and N contains no nonempty closed set.*

Proof. If A is $\delta - I_g - closed$, then by Theorem 2.5, $N = cl^*_\delta(A) - A$ contains no nonempty closed set. If $F = cl^*_\delta(A)$, then F is $\delta - I - closed$ such that $F - N = cl^*_\delta(A) - (cl^*_\delta(A) - A) = cl^*_\delta(A) \cap ((X - cl^*_\delta(A)) \cup A) = A$.

Conversely, suppose $A = F - N$, where F is $\delta - I - closed$ and N contains no nonempty closed set. let U be an open set such that $A \subset U$. Then, $F - N \subset U$, which implies that $F \cap (X - U) \subset N$. Now, $A \subset F$ and F is $\delta - I - closed$ implies that $cl^*_\delta(A) \cap (X - U) \subset cl^*_\delta(F) \cap (X - U) \subset F \cap (X - U) \subset N$. Since $\delta - I - closed$ sets are closed, $cl^*_\delta(A) \cap (X - U)$ is closed. By hypothesis, $cl^*_\delta(A) \cap (X - U) = \phi$ and $cl^*_\delta(A) \subset U$, which implies that A is $\delta - I_g - closed$.

If we put $I = \{\phi\}$, in Theorem 2.33, we get Corollary 2.34. If we put $I = P(X)$ in Theorem 2.33, we get Corollary 2.35.

Corollary 2.34. *Let* (X, τ) *be a space and* $A \subset X$. *Then* A *is* $\delta - g - closed$ *subset of* X *if and only if* $A = F - N$, *where* F *is* $\delta - closed$ *and* N *contains no nonempty closed set.*

Corollary 2.35. *Let* (X, τ) *be a space and* $A \subset X$. *Then* A *is* $g - closed$ *if and only if* $A = F - N$, *where* F *is* $g - closed$ *and* N *contains no nonempty closed set.*

Theorem 2.36. *Let* (X, τ, I) *be an ideal space. If* A *is a* $\delta - I_g - closed$ *subset of* X *and* $A \subset B \subset cl^*_\delta(A)$, *then* B *is also* $\delta - I_g - closed$.

Proof. $cl^*_\delta(B) - B \subset cl^*_\delta(A) - A$ and since $cl^*_\delta(A) - A$ has no nonempty closed subset, neither does $cl^*_\delta(B) - B$. By Theorem 2.5, B is $\delta - I_g - closed$.

If we put $I = \{\phi\}$ in Theorem 2.36, we get Corollary 2.37. If we put $I = P(X)$ in Theorem 2.36, we get Corollary 2.38.

Corollary 2.37. *Let* (X, τ) *be a space. If* A *is a* $\delta - g - closed$ *subset of* X *and* $A \subset B \subset cl_\delta(A)$, *then* B *is also* $\delta - g - closed$.

Corollary 2.38. *Let* (X, τ) *be a space. If* A *is a* $g - closed$ *subset of* X *and* $A \subset B \subset cl(A)$, *then* B *is also* $g - closed$.

The following Theorem 2.39 gives characterization for $\delta - I_g - open$ sets.

Theorem 2.39. *A subset* A *of an ideal space* (X, τ, I) *is* $\delta - I_g - open$ *if and only if* $F \subset int^*_\delta(A)$ *whenever* F *is closed and* $F \subset A$.

Proof. Suppose A is a $\delta - I_g - open$ set and F is a closed set contained in A. Then $X - A \subset X - F$ and $X - F$ is open. Since $X - A$ is $\delta - I_g - closed$, $cl^*_\delta(X - A) \subset (X - F)$ and so $F \subset X - cl^*_\delta(X - A) = int^*_\delta(A)$.

Conversely, suppose $X - A \subset U$ and U is open. By hypothesis, $X - U \subset int^*_\delta(A)$, which implies that $cl^*_\delta(X - A) = X - int^*_\delta(A) \subset U$. Therefore, $X - A$ is $\delta - I_g - closed$ and hence A is $\delta - I_g - open$.

If we put $I = \{\phi\}$ in Theorem 2.39, we set Corollary 2.40. If we put $I = P(X)$ in Theorem 2.39, we get Corollary 2.41.

Corollary 2.40. *A subset A of a space* (X, τ) *is* $\delta - g - open$ *if and only if* $F \subset int_\delta(A)$ *whenever* F *is closed and* $F \subset A$.

Corollary 2.41. *A subset A of space* (X, τ) *is* $g - open$ *if and only if* $F \subset int(A)$ *whenever* F *is closed and* $F \subset A$.

The following Theorem 2.42 gives characterization of $\delta - I_g - closed$ sets in terms of $\delta - I_g - open$ sets.

Theorem 2.42. *Let* (X, τ, I) *be an ideal space and* $A \subset X$. *Then the following are equivalent*

(a) A *is* $\delta - I_g - closed$

(b) $A \cup (X - cl^*_\delta(A))$ *is* $\delta - I_g - closed$

(c) $cl^*_\delta(A) - A$ *is* $\delta - I_g - open$.

Proof. $(a) \Rightarrow (b)$. Suppose A is $\delta - I_g - closed$. If U is any open set containing $A \cup (X - cl^*_\delta(A)$ then $X - U \subset X - (A \cup (X - cl^*_\delta(A))) = cl^*_\delta(A) - A$. Since A is $\delta - I_g - closed$, by Theorem 2.5 (c), it follows that $X - U = \phi$ and so $X = U$. Since X is the only open set containing $A \cup (X - cl^*_\delta(A)), A \cup (X - cl^*_\delta(A))$ is $\delta - I_g - closed$.

$(b) \Rightarrow (a)$. Suppose $A \cup (X - cl^*_\delta(A))$ is $\delta - I_g - closed$. If F is any closed set contained in $cl^*_\delta(A) - A$, then $A \cup (X - cl^*_\delta(A)) \subset X - F$ and $X - F$ is open. Therefore, $cl^*_\delta(A \cup (X - cl^*_\delta(A)) \subset X - F$, which implies that $cl^*_\delta(A) \cup$

$cl^*_\delta(X - cl^*_\delta(A)) \subset X - F$ and so $X \subset X - F$, it follows that $F = \phi$. Hence A is $\delta - I_g - closed$.

The equivalence of (b) and (c) follows from the fact that $X - (cl^*_\delta(A) - A) = A \cup (X - cl^*_\delta(A))$.

If we put $I = \{\phi\}$ in Theorem 2.42, we get Corollary 2.43. If we put $I = P(X)$ in Theorem 2.42, we get Corollary 2.44.

Corollary 2.43. *Let (X, τ) be a space and $A \subset X$. Then the following are equivalent.*

(a) A is $\delta - g - closed$

(b) $A \cup (X - cl_\delta(A))$ is $\delta - g - closed$

(c) $cl_\delta(A) - A$ is $\delta - g - open$.

Corollary 2.44. *Let (X, τ) be a space and $A \subset X$. Then the following are equivalent.*

(a) A is $g - closed$

(b) $A \cup (X - cl(A))$ is $g - closed$

(c) $cl(A) - A$ is $g - open$.

3 Characterization of $T_{1/2}$ and $T_I - spaces$

Theorem 3.1. *In an ideal space (X, τ, I), the following are equivalent.*

(a) Every $\delta - g - closed$ set is closed

(b) (X, τ) is a $T_{1/2} - space$

(c) Every $\delta - I_g - closed$ is closed.

Proof. $(a) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then $B = X - \{x\}$ is not open. Therefore, X is the only open set containing B and hence B is $\delta - g - closed$. So by (a), B is closed and hence $\{x\}$ is open. Thus every singleton in X is either open or closed. Therefore by Corollary 2.32, (X, τ) is a $T_{1/2} - space$.

$(b) \Rightarrow (a)$. Let $A \subset X$ be $\delta - g - closed$. Then A is $g - closed$. Therefore by hypothesis, A is closed. This proves (a).

$(b) \Rightarrow (c)$. Let A be a $\delta - I_g - closed$ set. Since every $\delta - I_g - closed$ set is $g - closed$, A is $g - closed$. By hypothesis, A is closed.

$(c) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then $B = X - \{x\}$ is not open. So B is $\delta - I_g - closed$. By hypothesis, B is closed and so $\{x\}$ is open. By Corollary 2.32, (X, τ) is $T_{1/2} - space$.

Theorem 3.2. *In an ideal space* (X, τ, I) *the following are equivalent.*

(a) Every $\delta - g - closed$ *set is* $\star - closed$.

(b) (X, τ, I) *is a* $T_I - space$.

(c) Every $\delta - I_g - closed$ *set is* $\star - closed$.

Proof. $(a) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then X is the only open set containing $X - \{x\}$ and hence $X - \{x\}$ is $\delta - g - closed$. By hypothesis, $X - \{x\}$ is $\star - closed$. Therefore, {x} is $\star - open$. Thus every singleton in X is either $\star - open$ or $closed$. By Theorem 3.3 [2], (X, τ, I) is a $T_I - space$.

$(b) \Rightarrow (a)$. The proof follows from the fact that every $\delta - g - closed$ set is $I_g - closed$.

$(b) \Rightarrow (c)$. The proof follows from the fact that every $\delta - I_g - closed$ set is $I_g - closed$.

$(c) \Rightarrow (b)$. Let $x \in X$. if $\{x\}$ is not closed, then X is the only open set containing $X - \{x\}$ and hence $X - \{x\}$ is $\delta - I_g - closed$. By hypothesis, $X - \{x\}$

is $\star - closed$. Thus $\{x\}$ is $\star - open$. Therefore, every singleton in X is either $\star - open$ or $closed$. Therefore, by Theorem 3.3 [2], (X, τ, I) is a $T_I - space$.

The proof of Corollary 3.3 follows from Theorem 3.2 and Theorem 3.10 [12]. If we put $I = \{\phi\}$ in Corollary 3.3, we get Corollary 3.4.

Corollary 3.3. *In an ideal space (X, τ, I), the following are equivalent.*

(a) Every $\delta - g - closed$ set is $\star - closed$.

(b) Every $\delta - I_g - closed$ set is $\star - closed$

(c) Every $I_g - closed$ set is an $I - locally * -closed$ set.

Corollary 3.4. *In a topological space (X, τ), the following are equivalent.*

(a) Every $\delta - g - closed$ set is closed.

(b) Every $g - closed$ set is locally closed set.

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