A Study on Fuzzy Critical Path Method Based On Metric Distance Ranking Of Fuzzy Numbers with Conventional Method

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Abstract : *In this paper Algorithm is presented to perform critical path method in a fuzzy environment. The trapezoidal fuzzy number given by decision makers or characterized by historical data are utilized to assess the activity time in a project network. The fuzzy shortest path problem thus obtained is compared with conventional method.*

Keywords : *Critcal path method, Metric distance, Trapezoidal fuzzy number, Triangular Fuzzy number.*

1. Introduction

A project network is defined as a set of activities that must be performed according to precedence constraints stating which activities must start after the completion of specified other activities [2]. Such a project network can be represented as a directed graph. A path through a project network is one of the routes from the starting n
ode to the ending node. The length of a path is the sum of the duration of the activities on the path. The project duration equals the length of the longest path through the project network. The longest path is called the critical path in the network.

In many situations, projects can be complicated and challenging to manage. When the activity times in the project are deterministic and known, CPM has been demonstrated to
be a useful tool in managing projects in an efficient manner to meet the challenge [3]. However, there are many cases where the activity times may not be presented in a precise manner. To deal quantitatively with imprecise data, the program evaluation and review technique (PERT) [3] can be employed. However, there are critiques of PERT [4]. An alternative way to deal with imprecise data is to employ the concept of fuzziness [5], whereby the vague activity times can be represented by fuzzy sets. Several studies have investigated the case where activity times in a project are approximately known and are more suitably represented by fuzzy sets rather than crisp numbers [6,7,13]

2. Fuzzy Concepts

In this section, we briefly review the theory of fuzzy sets from [8-11]. In Fig.1, we see a graph of a crisp set and a fuzzy set. The fuzzy set A can look very different depending on the chosen membership function. Using this function, it is possible to assign a membership degree to each of the element in the universe of discourse X. Elements of the set could but are not required to be numbers as long as a degree of membership can be deduced from them. It is important to note the fact that membership grades are not probabilities. One important difference is that the summation of probabilities on a finite universal set must equal 1, while there is no such requirement for membership grades.

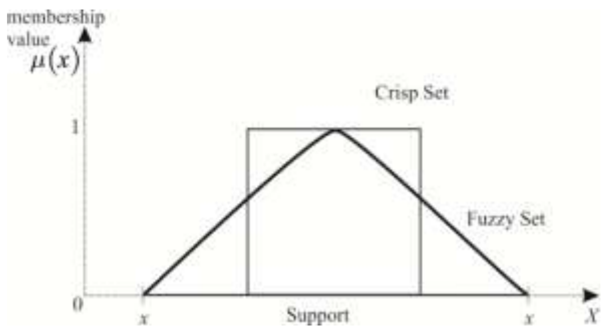

Fig. 1 Crisp set and Fuzzy set

Let X be the universe of discourse, $X = \{x_1, x_2, ..., x_n\}$. A fuzzy set A of X can be represented by

$$A = m_A(x_1)/X_1 + m_A(x_2)/x_2 + ... + m_A(x_n)/x_n \quad (1)$$

where μ_A is the membership function of the fuzzy set A and $\mu_A(x_1)$ indicates the grade of membership of x_i in the fuzzy set A, where $\mu_A(x_i) \in [0,1]$.

A fuzzy number is a fuzzy set which is both convex and normal. A fuzzy set A of the universe of discourse *X* is convex if and only if for all x_1, x_2 in *X*.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq Min(\mu_A(x_1), \mu_A(x_2)) \qquad (2)$$

where $\lambda \in [0,1]$. A fuzzy set of the universe of discourse X is called a normal fuzzy set if $\exists x_i \in X$, $\mu_A(x_i) = 1$. A trapezoidal fuzzy number A of the universe of discourse X can be characterized by a trapezoidal membership function parameterized by a quadruple (a,b,c,d) as shown in Fig 2, where *a*, *b*, *c* and *d* are real values.

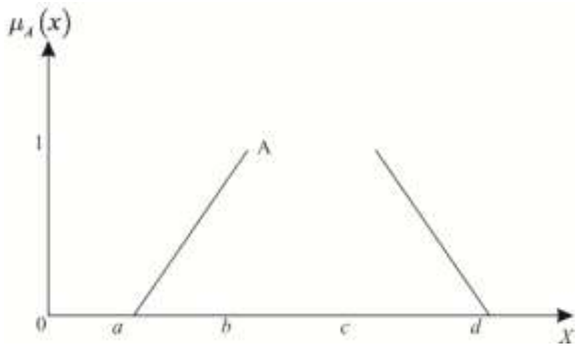

Fig.2. Membership function curve of trapezoidal Fuzzy number A

From Fig.2, we can see that if $a = b$ and $c = d$, then *A* is called a crisp interval; if $a = b = c = d$, then A is a crisp value. In Fig.2, if $b = c$, then A becomes a triangular fuzzy number as shown in Fig.3, and it can be parameterized by a triplet (*a*, *b*, *d*).

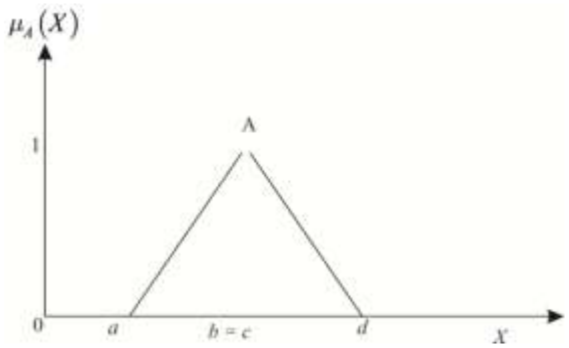

Fig.3. Membership function curve of triangular fuzzy number A

Let A_1 and A_2 be two trapezoidal fuzzy numbers parameterized by the quadruple (a_1,b_1,c_1,d_1) and (a_2,b_2,c_2,d_2) respectively. The simplified fuzzy number arithmetic operations between the trapezoidal fuzzy numbers A_1 and A_2 are as follows :

Fuzzy numbers addition $\oplus$:

$$(a_1,b_1,c_1,d_1) \oplus (a_2,b_2,c_2,d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2) \qquad (3)$$

Fuzzy numbers subtraction $\ominus$:

$$(a_1,b_1,c_1,d_1) \ominus (a_2,b_2,c_2,d_2) = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2) \qquad (4)$$

For example : Let A_1 and A_2 be two trapezoidal fuzzy numbers, where $A_1 = (16,20,22,24)$ and $A_2 = (3,4,5,6)$.

Then

$$A_1 \oplus A_2 = (16,20,22,24) \oplus (3,4,5,6) = (19,24,27,30)$$

$$A_1 \ominus A_2 = (16,20,22,24) \ominus (3,4,5,6) = (10,15,18,21)$$

3. Metric Distance Ranking

Chen and Cheng [12] proposed a metric distance method to rank fuzzy numbers. Let *A* and *B* be two fuzzy numbers defined as follows :

$$f_A(x) = \begin{cases} f_A^L(x), & x < m_A \\ f_A^R(x), & x \geq m_A \end{cases} \qquad (5)$$

$$f_B(x) = \begin{cases} f_B^L(x), & x < m_B \\ f_B^R(x), & x \geq m_B \end{cases} \qquad (6)$$

where m_A and m_B are the mean of A and B. The metric distance between A and B can be calculated as follows :

$$D(A,B) = \left[\int_0^1 \left(g_A^L(y) - g_B^L(y)\right)^2 dy + \int_0^1 \left(g_A^R(y) - g_B^R(y)\right)^2 dy \right]^{\frac{1}{2}} \qquad (7)$$

where g_A^L, g_A^R, g_B^L and g_B^R are the inverse functions of f_A^L, f_A^R, f_B^L and f_B^R respectively.

In order to rank fuzzy numbers, Chen and Cheng [12] let the fuzzy number $B = 0$ then the metric distance between A and 0 is calculated as follows :

$$D(A,0) = \left[\int_0^1 \left(g_A^L(y)\right)^2 dy + \int_0^1 \left(g_A^R(y)\right)^2 dy \right]^{\frac{1}{2}} \qquad (8)$$

The larger the value of $D(A,0)$, the better the ranking of *A*.

According to [12], a trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$ can be approximated as a symmetry fuzzy number $S[m, \sigma]$, m denotes the mean of *A*, σ denotes the standard deviation of *A*, and the membership function of *A* is defined as follows :

$$f_A(x) = \begin{cases} \dfrac{x - (m - \sigma)}{\sigma}, & if\ m - \sigma \le x \le m \\ \dfrac{(m + \sigma) - x}{\sigma}, & if\ m \le x \le m + \sigma \end{cases} \qquad (9)$$

where m and σ are calculated as follows :

$$\sigma = \frac{2(a_4 - a_1) + a_3 - a_2}{4} \qquad (10)$$

$$m = \frac{a_1 + a_2 + a_3 + a_4}{4} \qquad (11)$$

If $a_2 = a_3$, then A becomes a triangular fuzzy number, where $A = (a_1, a_2, a_4)$ and m and σ can be calculated as follows :

$$\sigma = \frac{a_4 - a_1}{2} \qquad (12)$$

$$m = \frac{a_1 + 2a_2 + a_4}{4} \qquad (13)$$

The inverse functions g_A^L and g_A^R of f_A^L and f_A^R respectively, are shown as follows :

$$g_A^L(y) = (m - \sigma) + \sigma \times y \qquad (14)$$

$$g_A^R(y) = (m + \sigma) - \sigma \times y \qquad (15)$$

Definition 3.2A fuzzy number $A = (a,b,c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x - a)}{(b - a)}, & a \le x \le b \\ \dfrac{(x - c)}{(b - c)}, & b \le x \le c \end{cases}$$

where $a, b, c \in R$

4. Fuzzy CPM Based on Metric Distance Ranking of Fuzzy Numbers

The operation time for each activity in the fuzzy project network is characterized as a positive trapezoidal fuzzy number. In accordance with CPM, the forward pass yields the fuzzy earliest-start and earliest-finish times :

$$\tilde{E}_i^s = \max_{j \in P(i)} \left\{ \tilde{E}_j^s \oplus \tilde{t}_j \right\} \qquad (16)$$

$$\tilde{E}_i^f = \tilde{E}_i^s \oplus \tilde{t}_i \qquad (17)$$

where $\tilde{E}_i^s$ is the fuzzy earliest –start time with $\tilde{E}_A^s = (0,0,0)$ at the initial node $i = A$, $\tilde{E}_i^f$ is the fuzzy earliest finish time with $\tilde{E}_z^f$ equal to the fuzzy project network completion time $\tilde{T}$ at the ending node $i = Z$, $P(i)$ is the set of predecessors for activity *i*, and $\tilde{t}_i$ is the operation time for activity *i*.

The backward pass is performed to calculate the fuzzy latest-start and latest-finish times :

$$\tilde{L}_i^f = \min_{j \in P(i)} \left\{ \tilde{L}_j^f \ominus \tilde{t}_j \right\} \qquad (18)$$

$$\tilde{L}_i^s = \tilde{L}_i^f \ominus t_i \qquad (19)$$

where $\tilde{L}_i^f$ is the fuzzy latest-finish time with $\tilde{L}_z^f = \tilde{T}$ at the end node $i = Z$, $\tilde{L}_i^s$ is the fuzzy latest start time and $S(i)$ is the set of successors for activity *i*.

Once $\tilde{E}_i^s, \tilde{E}_i^f, \tilde{L}_i^s$ and $\tilde{L}_i^f$ have been determined for the i^{th} activity, the fuzzy float time is either

$$T_i^F = \tilde{L}_i^s \ominus \tilde{E}_i^s \qquad (20)$$

(or)

$$T_i^F = \tilde{L}_i^f \ominus \tilde{E}_i^f \qquad (21)$$

We can easily compute the fuzzy float times of all activities in a project network. In Crisp CPM, activity *i* is said to be a critical activity if its float time is zero. This concept implies that the criticality rises as the fuzzy float time decreases.

Proposed Method

Consider the fuzzy project network, where the duration time of each activity in a fuzzy project network is represented by a trapezoidal fuzzy number.

Step 1 : Calculate $\tilde{E}_i^s$'s and $\tilde{E}_i^f$'s using Eq.(16) and Eq.(17)

Step 2 : Calculate $\tilde{L}_i^f$'s and $\tilde{L}_i^s$'s using Eq.(18) and Eq.(19)

Step 3 : Calculate T_i^F 's for each activity (i, j) using Eq.(20) or Eq.(21)

Step 4 : Find all the possible paths and calculate the total slack fuzzy time of each path.

Step 5 : Rank the total slack fuzzy time of each path using metric distance ranking.

Step 6 : The path having minimum rank in step 5 is the critical path.

5. An Example

Fig.4 shows the network representation of a fuzzy project network. Table I represents the total float of each activity in the fuzzy project network.

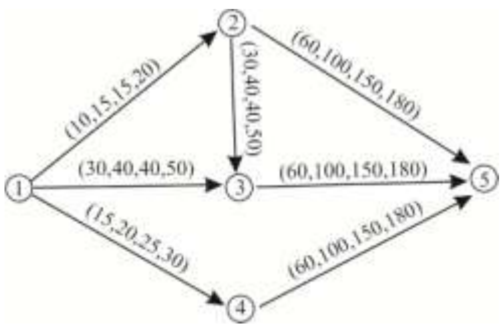

Fig.4. A fuzzy project network

Table I : Total float of each activity in the fuzzy project network

Activity	Fuzzy Activity time	Total float (a,b,c,d)	Total Float (b, c, b-a, c-d)
1-2	(10,15,15,20)	(-160,-60,60,160)	(-60,60,100,-100)
1-3	(30,40,40,50)	(-130,-35,75,170)	(-35,75,120,200)
2-3	(30,40,40,50)	(-160,-60,60,160)	(-60,60,100,-100)
1-4	(15,20,25,30)	(-110,-20,95,185)	(-20,95,90,-90)
2-5	(60,100,150,180)	(-100,-10,100,190)	(-10,100,90,-90)
3-5	(60,100,150,180)	(-160,-60,60,160)	(-60,60,100,-100)
4-5	(60,100,150,180)	(-110,-20,95,185)	(-20,95,90,-90)

The possible paths of fuzzy project network (Fig.4) are 1-2-3-5, 1-2-5, 1-3-5 and 1-4-5.

Metric distance rank of total fuzzy slack time for each path in fuzzy project network (Fig.4) are computed and presented in Table II.

Case (i) Path : 1-2-3-5

Total Fuzzy slack time for the path 1-2-3-5 is $(a,b,c,d) = (-480,-180,180,480)$. Then $(b,c,b-a,c-d) = (-180,180,300,-300)$

Let $A = (a_1,a_2,a_3,a_4) = (-180,180,300,-300)$

$$s = \frac{2(a_4 - a_1) + a_3 - a_2}{4}$$
$$= \frac{2(-300+180)+300-180}{4} = -30$$
$$m = \frac{a_1 + a_2 + a_3 + a_4}{4}$$
$$= \frac{-180+180+300+-300}{4} = 0$$
$$g_A^L(y) = (m - s) + s \times y$$
$$= 30(-y+1)$$
$$g_A^R(y) = (m + s) - s \times y$$
$$= 30(-1+y)$$
$$D(A,0) = \left[\int_0^1 \left(g_A^L(y)\right)^2 dy + \int_0^1 \left(g_A^R(y)\right)^2 dy\right]^{\frac{1}{2}}$$
=24.49.

Case (ii) Path : 1-2-5

Total fuzzy slack time for the path 1-2-5 is $(a,b,c,d) = (-260,-70,160,350)$. Then $(b,c,b-a,c-d) = (-70,160,190,-190)$

Let $A = (a_1,a_2,a_3,a_4) = (-70,160,190,-190)$

$$s = \frac{2(a_4 - a_1) + a_3 - a_2}{4}$$
$$= -52.5$$
$$m = \frac{a_1 + a_2 + a_3 + a_4}{4}$$
$$= 22.5.$$
$$g_A^L(y) = (m - s) + s \times y$$
$$= 75 - 52.5y$$
$$g_A^R(y) = (m + s) - s \times y$$
$$= -30 + 52.5y.$$

$$D(A,0)=\left[\int_0^1 \left(g_A^L(y)\right)^2 dy+\int_0^1 \left(g_A^R(y)\right)^2 dy\right]^{\frac{1}{2}}$$
=53.38.

Case (iii) Path : 1-3-5

Total fuzzy slack time for the path 1-3-5 is $(a,b,c,d)=(-290,-95,135,330)$. Then

$(b,c,b-a,c-d)=(-95,135,195,-195)$

Let $A=(a_1,a_2,a_3,a_4)=(-95,135,195,-195)$

$$s=\frac{2(a_4-a_1)+a_3-a_2}{4}=-35$$

$$m=\frac{a_1+a_2+a_3+a_4}{4}=10$$

$$g_A^L(y)=(m-s)+s\times y$$
$$=45-35y$$
$$g_A^R(y)=(m+s)-s\times y$$
$$=-25+35y.$$

$$D(A,0)=\left[\int_0^1 \left(g_A^L(y)\right)^2 dy+\int_0^1 \left(g_A^R(y)\right)^2 dy\right]^{\frac{1}{2}}$$
= 31.58.

Case (iv) Path : 1-4-5

Total fuzzy slack time for the path 1-4-5 is $(a,b,c,d)=(-220,-40,190,370)$. Then

$(b,c,b-a,c-d)=(-40,190,180,-180)$

Let $A=(a_1,a_2,a_3,a_4)=(-40,190,180,-180)$

$$s=\frac{2(a_4-a_1)+a_3-a_2}{4}=-72.5$$

$$m=\frac{a_1+a_2+a_3+a_4}{4}=37.5$$

$$g_A^L(y)=(m-s)+s\times y$$
$$=110-72.5y$$
$$g_A^R(y)=(m+s)-s\times y$$
$$=-35+72.5y.$$

$$D(A,0)=\left[\int_0^1 \left(g_A^L(y)\right)^2 dy+\int_0^1 \left(g_A^R(y)\right)^2 dy\right]^{\frac{1}{2}}$$
= 79.48.

Table II : Metric distance rank of total fuzzy slack time for each path in fuzzy project network.

Path	Total Fuzzy Slack time (a,b,c,d)	Total Fuzzy slack time (b,c,b-a,c-d)	Metric distance rank
1-2-3-5	(-480,-180, 180,480)	(-180,180, 300,-300)	24.49
1-2-5	(-260,-70, 160,350)	(-70,160, 190,-190)	53.38
1-3-5	(-290,-95, 135,330)	(-95,135, 195,-195)	31.58
1-4-5	(-220,-40, 190,370)	(-40,190, 180,-180)	79.48

Here, the path having minimum rank is 1-2-3-5. Therefore, the required critical path for the fuzzy project is 1-2-3-5.

Possibility of meeting a fuzzy project in a specified time is calculated for set of 12 different projects having different number of activities using fuzzy critical path method based on signed distance ranking of fuzzy numbers and also using proposed method. One of these project networks (D) is shown in Fig.4. The comparisons are shown in Table III. The comparison reveal that the method proposed is more accurate.

Table III : Possibility of meeting a fuzzy project in a specified time.

Project	Activities	Possiblities of meeting	
		Signed distance method [13]	Proposed Method
A	5	0.87	0.9
B	7	0.91	0.93
C	12	0.89	0.92
D	10	0.65	0.72
E	6	0.71	0.75
F	15	0.92	0.95
G	10	0.71	0.73
H	8	0.62	0.69
I	6	0.63	0.74
J	13	0.73	0.79
K	5	0.78	0.84
L	10	0.84	0.89

5.1 Method of Converting Trapezoidal Fuzzy Number Into Triangular Fuzzy Numbers

Step1: if (a,b,c,d) is the given trapezoidal fuzzy numbers

Step2: taking average of $(b+c)/2=b'$.

(i.e average of the second and third number) the resulting number substituted in that position. That gives the triangular fuzzy number (a,b',c).

6. Conventional Method

Algorithm 2

Step 1: Construct the network diagram according to Fulkerson rule.

Step 2: select the component wise fuzzy number (a,b,c) in that order treated as a time between nodes.

We obtained three stages of critical path problem.

Step 3: find the number of possible ways (path from initial vertex to end vertex).

Step 4: from the possible ways select the minimum value that is required critical path.

6.1 An Example

Fig.4 shows the Triangular network representation of a fuzzy project network. Table I represents the total float of each activity in the fuzzy project network.

Applying Algorithm2 calculation in three stages;

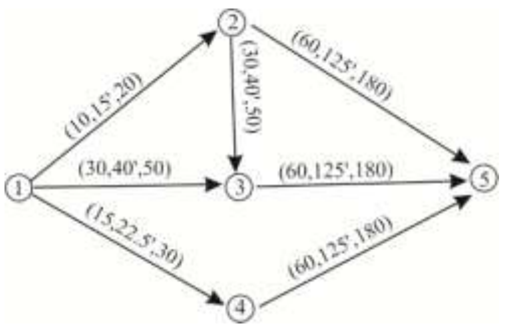

Fig.5 : Triangular Fuzzy Network – Applying Algorithm

Stage1: Component wise the first number applying the conventional method algorithm we obtained he following critical path (1® 2® 3® 5).

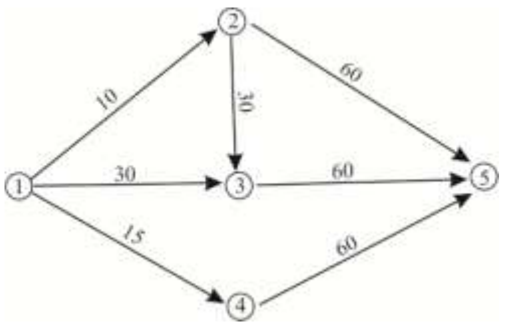

Fig.6 : Applying Algorithm – Stage 1

Stage 2: Component wise the second fuzzy number applying the conventional method algorithm we obtained he following shortest path (1® 2® 3® 5)

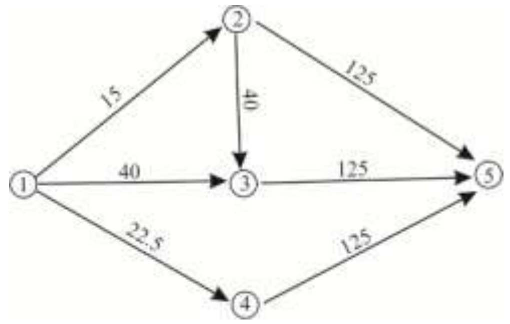

Fig.7 : Applying Algorithm – Stage 2

Stage 3: component wise the third fuzzy number applying the conventional method algorithm we obtained he following shortest path (1® 2® 3® 5)

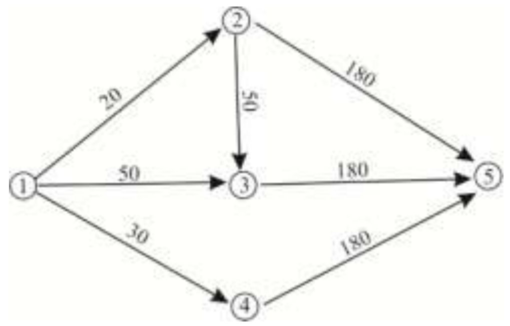

Fig.8 : Applying Algorithm – Stage 3

6. Conclusion

In the fuzzy critical path method the path is (1® 2® 3® 5). in conventional method we find the critical path in three stages we obtain The same unique solution hence The execution of algorithm2 is best one for the implementation by the algoriThm1.

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