Estimation of Reliability Function of Lomax Distribution via Bayesian Approach

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Abstract: *In this paper Bayes estimation of the reliability function of the Lomax distribution have been obtained by taking non-informative and beta prior distributions. The loss function used is squared error, linex, precautionary and entropy.*

Keywords and Phrases: *Lomax distribution, non-informative and beta prior distributions, loss function*.

Introduction The Lomax distribution, conditionally also called the Pareto Type II distribution, is a heavy-tail probability distribution used in business, economics, actuarial science, queueing theory and Internet traffic modeling. Lomax distribution was introduced by Lomax (1974), Abdullah and Abdullah (2010) estimates the parameters of Lomax distribution based on generalized probability weighted moment.

Let us consider the probability density function (pdf) of the Lomax distribution is given by

$f(x;k;\theta) = k\theta(1+kx)^{-(\theta+1)}$; $\theta > 0,\ k > 0$ and $0 < x < k$ (1.1)

where θ is the shape and k is scale parameter.

Cumulative distribution function: The cumulative distribution function of a Lomax distribution is

$$F(x;k;\theta) = \int_0^x f(x)dx$$
$$= 1-(1+kx)^{-(\theta)} \quad \theta > 0,\ k > 0 \text{ and } 0 < x < k \qquad (1.2)$$

Reliability function: The reliability function of a Lomax distribution is

$$R(t) = 1 - F(t)$$
$$= (1+kt)^{-(\theta)} \quad \theta > 0,\ k > 0 \text{ and } 0 < t < k \qquad (1.3)$$

Hazard function: The hazard function of a Lomax distribution is

$$H(t) = \frac{f(t)}{R(t)}$$
$$= \frac{k\theta}{(1+kt)} \quad \theta > 0,\ k > 0 \text{ and } 0 < t < k \qquad (1.4)$$

Bayes estimation under asymmetric loss functions

The Bayesian inference procedures have been developed generally under squared error loss function

$$L(\hat{R}(t), R(t)) = (\hat{R}(t) - R(t))^2 \qquad (1.5)$$

Where $\hat{R}(t)$ is an estimate of R(t). Clearly SELF is a symmetrical loss function and assign losses to over estimation and underestimation. The Bayes estimator under the above loss function, say $\hat{R}(t)_s$, is the posterior mean, i.e., $\hat{R}(t)_s = E_\pi[R(t)]$ (1.6)

Where E_π , is the posterior expectation. The is often used also because SELF it does not lead to extensive numerical computations but several authors Varian(1975), Aitchison & Dunsmore (1975), Zellner(1986) and Basu and Ebrahimi (1991)), have recognized that the inappropriateness of using symmetric loss function in several estimation problems and proposed different loss functions e.g. linex and many variant forms of it.

(i) Linex loss function : An underestimate of the failure rate results in more serious consequences than an overestimation of the failure rate. This lead to the statistician to think about asymmetrical loss functions which have been proposed in statistical literature. Varian (1975) introduced the following convex loss function known as LINEX (Linear-exponential) loss function

$$L(\Delta) = be^{a\Delta} - c\Delta - b;\quad a, c \neq 0, b > 0, \qquad (1.7)$$

Where $\Delta = \hat{R}(t) - R(t)$. It is clear that L (0) =0 and minimum occurs when a b = c, therefore, L (Δ) can be written as

$$L(\Delta) = b(e^{a\Delta} - a\Delta - 1); \quad a \neq 0, b > 0, \quad (1.8)$$

where a and b are the parameters of the loss function may be defined as shape and scale respectively. The Bayes estimator under precautionary loss function is denoted by $\hat{R}(t)_A$, and is obtained by solving the following equation

$$\exp(-a\hat{R}(t)_A) = [E_\pi\{\exp(-aR(t)] \quad (1.9)$$

(ii) ***Precautionary loss function*** : Norstrom(1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. These loss functions approach infinitely near the origin to prevent underestimation and thus giving conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is

$$L(\hat{R}(t), R(t)) = \frac{(\hat{R}(t) - R(t))^2}{\hat{R}(t)}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{R}(t)_p$, and is obtained by solving the following equation

$$\hat{R}(t)_p = \left[E_\pi\{R(t)^2\}^{\frac{1}{2}}\right] \quad (1.11)$$

(i) ***Entropy loss function*** **:** In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{R}(t)}{R(t)}$. In this case, Calabria and Pulcini (1994) points out that a useful asymmetric loss function is the entropy loss given by

$$L(\delta) \propto \left[\delta^p - p\log_e(\delta) - 1\right]$$

where

$$\delta = \frac{\hat{R}(t)}{R(t)}$$

and whose minimum occurs at $\hat{R}(t)$ = R(t). Also, the loss function L (δ) has been used in Dey et al. (1987) and Dey and Liu (1992), in the original form having p=1. Thus L (δ) can be written as

$$L(\delta) = b\left[\delta - \log_e(\delta) - 1\right] \quad ; b > 0$$

The Bayes estimator under entropy loss function is denoted by $\hat{R}(t)_e$, and is obtained by solving the following equation

$$\hat{R}(t)_e = \left[E_\pi\left(\frac{1}{R(t)}\right)\right]^{-1} \quad (1.13)$$

Let us consider the two prior distributions of R (t) to obtain the Bayes estimators which are given by

(i) ***Non-informative Prior*** **:** For the situation where the experimenter has no information about R (t), we may use the non-informative prior distribution

$$h_1(R(t)) = \frac{1}{R(t)\log R(t)} \qquad 0 < R(t) \leq 1. \quad (1.14)$$

(ii) ***Beta Prior*** **:** The most widely used prior distribution for R(t) is a beta distribution with parameters α, β > 0, with density function given by

$$h_2(R(t)) = \frac{1}{B(\alpha,\beta)}[R(t)]^{\alpha-1}[1-R(t)]^{\beta-1} \qquad 0 < R(t) \leq 1. \quad (1.15)$$

2. *ESTIMATION OF RELIABILITY FUNCTION OF LOMAX DISTRIBUTION*

Let us consider the probability density function (pdf) of the Lomax distribution is given by (1.10)

$$f(x;k;\theta) = k\theta(1+kx)^{-(\theta+1)} \quad ; \theta > 0, \; k > 0 \text{ and } 0 < x < k$$

where θ is the shape and k is scale parameter.

The reliability function of a Lomax distribution is

$$R(t) = (1+kt)^{-(\theta)} \qquad \theta > 0, k > 0 \text{ and } 0 < t < k$$

or -log R (t) =θ {log (1+kt)}

or $\theta = \frac{-\log R(t)}{\{\log(1+kt)\}}$

Substituting the value of θ in equation (1.1), we get

$$f(x;k;R(t)) = k\,\frac{-\log R(t)}{\log(1+kt)}(1+kt)^{\left\{\frac{\log R(t)}{\log(1+kt)}-1\right\}} \qquad 0 < R(t) < 1 \quad (2.1)$$

Let us suppose that n items are put to life test and terminate the experiment when r (< n) items have failed. If $x_1, \dots, x_r$ denote the first r observations having a common density function as given in equation (2.1), then the joint probability density function is given by

$$f(x;k;R(t)) = = \frac{n!}{(n-r)!}\left(k\,\frac{-\log R(t)}{\log(1+kt)}\right)^r e^{\left\{\frac{-\log R(t)}{\log(1+kt)}-1\right\}T_r} \qquad 0 < R(t) < 1 \quad (2.2)$$

where $T_r = -[\sum_{i=1}^{r}\log(1+kx_i) + (n-r)\log(1+kx_r)]$

Thus, the maximum likelihood estimator of R(t) is given by

$$\hat{R}(t) = (1+kt)^{\frac{r}{T_r}} \quad (1.12)$$

(2.3)

Using the equation (2.2), the posterior pdf of R(t) is obtained as

$$f(R(t)|\underline{x}) = \frac{w^r}{\Gamma(r)}[R(t)]^{w-1}[-\log R(t)]^{r-1}$$

(2.4)

where $w = -\frac{T_r}{\log(1+kt)}$

The Bayes estimator of $R(t)$ under squared error loss function is given by

$$\hat{R}(t)_s = \left(1+\frac{1}{w}\right)^{-r}$$

(2.5)

Using equation (1.9), the Bayes estimator of R(t) under linex loss function is obtained as

$$\hat{R}(t)_A = \left(-\frac{1}{a}\right)\log\left[\sum_{j=0}^{\infty}\frac{(-a)}{j!}\left(1+\frac{j}{w}\right)^{-r}\right]$$

(2.6)

Using (1.10), the Bayes estimator of R(t) relative to precautionary loss function comes out to be

$$\hat{R}(t)_P = \left(1+\frac{2}{w}\right)^{\frac{-r}{2}}$$

(2.7)

The Bayes estimator of R(t) relative to entropy loss function, using (1.13) is obtained as

$$\hat{R}(t)_e = \left(1-\frac{2}{w}\right)^{r}$$

(2.8)

3. Bayes Estimator under $h_2(R(t))$

Using equation (2.2), the posterior pdf of R(t) under $h_2(R(t))$ is obtained as

$$f(R(t)|\underline{x}) = \frac{[[R(t)]^{w+\alpha-1}\,[1-R(t)]^{\beta-1}][-\log R(t)]^r}{\Gamma(r+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k}\right)^{r+1}\right]};\quad 0 < R(t) \le 1.$$

(3.1)

The Bayes estimator of R(t) under squared error loss function is given by

$$\hat{R}(t)_S = \left[\frac{\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k+1}\right)^{r+1}\right]}{\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k}\right)^{r+1}\right]}\right]$$

(3.2)

Using (1.9), the Bayes estimator of R(t) under linex error loss function is given

$$\hat{R}(t)_A = \left(-\frac{1}{a}\right)\left[\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{\beta-1}\frac{(-a)}{j!}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k+2}\right)^{r+1}}{\left[\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k}\right)^{r+1}\right]\right]}\right] \quad (3.3)$$

Using (1.10), the Bayes estimator of R(t) under precautionary error loss function is given by

$$\hat{R}(t)_P = \left[\frac{\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k+2}\right)^{r+1}\right]}{\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k}\right)^{r+1}\right]}\right]$$

(3.4)

The Bayes estimator of R(t) under squared error loss function, using (1.13), is obtained as

$$\hat{R}(t)_e = \left[\frac{\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k}\right)^{r+1}\right]}{\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(\frac{1}{w+\alpha+k-1}\right)^{r+1}\right]}\right]$$

(3.5)

4. *Conclusion*

In this paper, we have obtained a number of Bayes estimators of reliability function R(t) of Lomax distribution . In equations (2.5)-(2.8) we have obtained the Bayes estimators by using non-informative prior and in equation (3.2)-(3.5) under beta prior. In the above equation, it is clear that the Bayes estimators depends upon the parameter of the prior distribution. In this case the risk functions and corresponding Bayes risks do not exist.

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