Some Study of Na-Continuous Functions

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Abstract: *The purpose of this paper is to introduce a new class of continuous functions like NA continuous functions and study some of their properties.*

Keywords: *NA-continuity, δ-continuity, strongly continuous, β-continuous.*

I. INTRODUCTION

The concept of na-continuous functions was first introduced and studied by Gyu Ihn Chae, T. Noiri and Dowon Lee in their paper [4] in the year 1986. In the year 1982, B. M. Munshi and D. S. Bassan [5] introduced and studied the new class of functions called super-continuous and S. P. Arya and R. Gupta [1] introduced and studied the strongly continuous functions. Gyu Ihn Chae, T. Noiri and Dowon Lee [4] showed that na-continuous functions are weaker than the class of strongly continuous functions and stronger than super-continuous.

In this paper, we study the concept of na-continuous functions defined by Gyu Ihn Chae, T. Noiri and Dowon Lee [3]. Here we concentrate on basic properties similar to the properties of continuous functions.

II PRILIMINARIES

Throughout this paper ,We denote the family of all regular open sets δ-open sets and feebly-open sets in a topological space (X, $\Im$) by RO (X, $\Im$), DO (X, $\Im$) and FO (X, $\Im$). [Simply we denote by RO (X), DO (X) and FO (X)] and A is any subset of space X, then Cl(A) and Int(A) denote the closure of A and the interior of A in X respectively.

Definition 2.1 [9] : A subset A is said to be regular open if A=int(cl(A)) and regular closed if A = cl (int (A)).

Definition 2.2 [6] : A subset A is said to be α-open if A $\subseteq$ int(cl(int(A))).

Definition 2.3 [4]: A subset A is said to be feebly open if there exists an open set O such that O $\subseteq$ A $\subseteq$ scl(O), where scl(O) denoted the semi-closure of O.

Definition 2.4 : A subset G is said to be δ-open if for each x $\in$ G there exists a regular open set H such that x $\in$ H $\subseteq$ G. Or equivalently, if G is expressible as an arbitrary union of regularly open sets.

III *NA-CONTINUOUS FUNCTIONS*

Definition 3.1: A function g : X $\rightarrow$ Y is said to be NA-continuous if for each feebly open set V in Y g^{-1}(V) is δ-open set in X.

Theorem 3.2 : For a function g : X $\rightarrow$ Y, the following are equivalent.

(i) g is NA-continuous.

(ii) For each x $\in$ X and each V $\in$ FO (X) (that is, V is feebly open set in Y) containing g (x), there exists an δ-open set U in X containing x such that g (U) $\subset$ V.

(iii) For each x $\in$ X and each feebly open set V in Y containing g (x), there exists a regular open set U in X containing x such that g(U) $\subset$ V.

(iv) For each feebly closed set F of Y, g^{-1}(F) is δ-closed.

(v) g (δcl (A)) $\subset$ fcl (g (A)) for each subset A of X.

(vi) δcl (g^{-1}(B)) $\subset$ g^{-1} (fcl (B)) for each subset B of Y.

Proof: (i) $\Rightarrow$ (ii): Let for each x $\in$ X and let V be an feebly open set in Y containing g (x), that is, g (x) $\in$ V. Since g is NA-continuous, then g^{-1}(V) is feebly open set in X containing x. So x $\in$ g^{-1}(V). Take U = g^{-1}(V). Then we have, g (U) $\subset$ V. Hence (ii) holds.

(ii) $\Rightarrow$ (iii): Let x $\in$ X and let V be any feebly open set in Y containing g(x). Then there exists an δ-open set U_0 of X containing x such that g (U_0) $\in$ Y. Since a δ-open set is the union of regular open sets, there exists an U $\in$ RO(X) such that x $\in$ U $\subset$ U_0. Therefore we have g (U) $\subset$ V.

(iii) $\Rightarrow$ (iv): Let F be a feebly closed set of Y. Then Y – F is feebly open set in Y. For each x $\in$ g^{-1}(Y – F), there exists an regular open set U_x of X such that x $\in$ U_x $\subset$ g^{-1}(Y – F). Therefore we have g^{-1}(F) = $\cap$ { X – U_x : x $\in$ g^{-1}(Y – F) }. This means that g^{-1}(F) is δ-closed set in X.

(iv) $\Rightarrow$ (v): For each subset A of X, f cl(g(x)) is the smallest feebly closed set of Y containing g(A) [by the theorem in feebly closure, that is, fcl (H) is the smallest feebly closed set containing A]. Thus, A $\subset$ g^{-1}(fcl(g(A))) and hence δ-cl (A) $\subset$ g^{-1}(fcl (g(A))), by (d). Therefore we have g((δ-cl (A)) $\subset$ fcl (g(A)). Hence (v) holds.

(v) ⇒ (vi): For each subset B of Y, we have g (δ-cl (g^{-1}(B))) ⊂ fcl(g(g^{-1}(B))) ⊂ fcl (B) and hence δ-cl (g^{-1}(B)) ⊂ g^{-1} (fcl (B)). Hence (vi) holds.

(vi) ⇒ (i): Let V be any feebly open set in Y. Then Y – V is feebly closed set in Y. Then from (f), δ-cl (g^{-1}(Y – V)) ⊂ g^{-1} (fcl (Y – V)) = g^{-1} (Y – V). Thus g^{-1} (Y – V) is δ-closed set in X. Therefore g^{-1} (Y – V) = X – g^{-1}(V) is δ-closed set in X. So g^{-1}(V) is δ-open in X. Hence g is NA-continuous.

Definition 3.3: A function $g : X \rightarrow Y$ is said to be super-continuous if for each x ∈ X and each neighbourhood V of g(x), there exists a neighborhood U of x such that g (int (cl (U))) ⊂ V.

It was shown in [3] that the family of feebly open set in X is a topology on a space X, that is, (X, FO(X)) is a topological space. The family of regular open sets in X, that is, RO(X) is a basis for a topology which is called the semi-regularization of T for a space (X, ℑ) and is denoted by $\Im_s$.

Theorem 3.4 : For a function g : (X, ℑ) → (Y, ℑ′), the following are equivalent:

- (i) g is na-continuous.
- (ii) g_0 : (X, ℑ) → (Y, FO(Y)) is super-continuous, where g_0 (x) = g (x) for each x ∈ X.
- (iii) g_* : (X, $\Im_s$) → (Y, FO(Y)) is continuous, where g_* (x) = g (x) for each x ∈ X.

Proof : (i) ⇒ (ii) : Let V be an open set of (Y, FO(Y)). Then V is feebly open set in (Y, ℑ). From (a), g^{-1}(V) is δ-open set in (X, ℑ). It follows from the Theorem that g is super-continuous.

(ii) ⇒ (iii) : For each open set V of (Y, FO(Y)), g_0^{-1}(V) is δ-open set in (X, ℑ) and g_*^{-1}(V) is open in (X, $\Im_s$). Therefore g_* is continuous.

(iii) ⇒ (i) : Let V be any feebly open set in (Y, ℑ′). Then V is open in (Y, FO(Y)) and hence g_*^{-1}(V) is open in (X, $\Im_s$). Therefore g^{-1}(V) is δ-open set in (X, ℑ). Hence g is NA-continuous.

Definition 3.5 : A filter base β = { B_λ } on a space X is said to be δ-convergence to a point x in X, if for each V ∈ RO(X) (resp. V ∈ FO(X)) there exists a B_λ ∈ β such that B_λ ⊂ V.

A net { x_λ }$_{\lambda\in D}$ in X is said to be δ-convergence (resp. sf-convergence) to x ∈ X if the net is eventually in each regular open set containing x (resp. each feebly open set).

Theorem 3.6 : For a function g : (X, ℑ) → (Y, ℑ′), the following are equivalent:

- (i) g is NA-continuous.
- (ii) For each x ∈ X and each set { x_λ }$_{\lambda\in D}$ δ-convergence to x, g (β) converges to g (x) in (Y, FO(Y)).
- (iii) For each x ∈ X and each set { x_λ }$_{\lambda\in D}$ δ-convergence to x, the set {g (x_λ) }$_{\lambda\in D}$ converges to g (x) in (Y, FO(Y)).
- (iv) For each x ∈ X and each filterbase β δ-converges to x, g (β) sf-converges to g (x) in (Y, ℑ′).
- (v) For each x ∈ X and each net { x_λ }$_{\lambda\in D}$ δ-convergence to x, the set {g (x_λ) }$_{\lambda\in D}$ sf-converges to g (x) in (Y, ℑ′).

Proof : (i)⇔ (ii)⇔ (iii) proof follows from the Theorem 3.2 and (i) ⇔ (iv) follows from Theorem 3.2.

IV *Properties of NA-Continuous Functions*

Theorem 4.1 : If g : X → Y is NA-continuous and A is open, then the restriction $g_{/A}$: A → Y is na-continuous.

Proof : Let V be any feebly open set in Y. Since g is na-continuous, g^{-1}(V) is δ-open set in X. Since the δ-open set g^{-1}(V) is the union of regular open sets V_i of X. Since A is open in X, V_i ∩A is regular open in the subspace A [7]. Therefore $(g_{/A})^{-1}$(V) is the union of g^{-1}(V_i) ∩A and hence $(g_{/A})^{-1}$(V) is δ-open set in A. Hence $g_{/A}$ is NA-continuous.

Theorem 4.2 : Composition of two NA-continuous functions is NA-continuous.

Proof : Let g : X → Y and f : Y → Z are na-continuous. Let V be any feebly open set in Z. Since f is NA-continuous, f^{-1}(V) δ-open set in Y. Since every δ-open set is feebly open, so f^{-1}(V) is feebly open in Y. Again since g is NA-continuous, $g^{-1}(f^{-1}(V)$ is δ-open set in X. So (g o f) $^{-1}$(V) is feebly open set in X. Hence g o f) is NA-continuous.

Lemma 4.3 : Let { X_λ : λ ∈ D } be a family of spaces and $U_{\lambda i}$ be a subset of $X_{\lambda i}$ for each i = 1, 2, ……..,n. Then $U = \prod_{i=1}^{n} U_{\lambda i} \times \prod_{\lambda \neq \lambda i} X_\lambda$ is δ-open (resp. feebly open) in $\prod_{\lambda\in D} X_\lambda$ if and only if $U_{\lambda i}$ ∈ DO($X_{\lambda i}$) (resp. $U_{\lambda i}$ ∈ FO($X_{\lambda i}$)), that is, $U_{\lambda i}$ is δ-open in $X_{\lambda i}$ (resp. $U_{\lambda i}$ is feebly open in $X_{\lambda i}$) for each i = 1, 2, ……, n.

Proof : we know that ($\prod X_\lambda$)$_i$ = $\prod (X_\lambda)_i$. Thus U is δ-open set in ($\prod X_\lambda$)$_i$ if and only if U is open in ($\prod X_\lambda$)$_i$, where (X_λ)$_i$ is the semi-regularization of X_λ. Therefore U is δ-open in $\prod X_\lambda$ if and only if $U_{\lambda i}$ is open in (X_λ)$_i$, that is, $U_{\lambda i}$ is δ-open in (X_λ)$_i$, for each i = 1, 2, ….., n.

Next, assume that U be an feebly open set in $\prod X_\lambda$. Then we have, U ⊂ int (cl (int (U))) ⊂ { $\prod_{i=1}^{n} \text{int}(cl(\text{int}(U_{\lambda i})))$ }× $\prod_{\lambda \neq \lambda i} X_\lambda$. Therefore, we obtain $U_{\lambda i}$ ⊂ int (cl (int ($U_{\lambda i}$) for each i = 1, 2, …., n. Thus, $U_{\lambda i}$ is feebly open set in $X_{\lambda i}$ for each i = 1, 2, …., n.

Conversely, assume that $U_{\lambda i}$ is feebly open set in $X_{\lambda i}$ for each i = 1, 2, …., n. Then, U ⊂ { $\prod_{i=1}^{n} \text{int}(cl(\text{int}(U_{\lambda i})))$ } × $\prod_{\lambda \neq \lambda i} X_\lambda$ ⊂ int (cl (int (U))). Hence the proof.

Theorem 4.4 : Let $g_\lambda : X_\lambda \to Y_\lambda$ be a function for each λ ∈ D and g : $\prod X_\lambda \to \prod Y_\lambda$ be a function defined by g ({ X_λ }) = { g_λ (X_λ) } for each { X_λ } ∈ $\prod X_\lambda$. If g is NA-continuous, then g_λ is NA-continuous for each λ ∈ D.

Proof : Let β ∈ D and V_β be any feebly open set in Y_β. Then by Lemma 3.3.3, V = $V_\beta \times \prod_{\lambda \neq \beta} Y_\lambda$ is feebly open in $\prod Y_\lambda$. Since g is NA-continuous, $g^{-1}(V) = g_\beta^{-1}(V_\beta) \times \prod_{\lambda \neq \beta} X_\lambda$ is δ-open in $\prod X_\lambda$. Then from Lemma 4.3, $g_\beta^{-1}(V_\beta)$ is δ-open in X. Therefore g_β is NA-continuous, that is, g_λ is na-continuous.

Theorem 4.5 : Let g : X → Y be a function and G : X → X × Y be a graph function of g defined by G (x) = (x, g (x)) for each x ∈ X. If G is NA-continuous then g is NA -continuous.

Proof : Let x ∈ X and V be feebly open set in Y containing g (x). Then by Lemma 4.3, X × V is feebly open set in X × Y containing g (x). Since G is NA -continuous by Theorem 3.2, there exists an δ-open set U in X containing x such that G (U) ⊂ X × Y. Hence g (U) ⊂ Y.

Remark 4.6 : The converse of the theorem 4.5 may not be true in general. It was known that an feebly open set V in X × Y may not, generally, be a union of sets of the form A × B in the product space X × Y, where A and B are feebly open sets in X and Y respectively.

Definition 4.7 : A function g : X → Y is said to be strongly continuous (briefly STC) if g (cl (A)) ⊆ g (A) for each subset A of X.

Definition 4.8 : A function g : X → Y is said to be completely continuous (briefly CC) [1] (resp. β-continuous (briefly βC) [3]) if the inverse image of each open (resp. regular open) set V of Y is regular open in X. D. Carnahan [2] called βC-functions as R-maps.

Definition 4.9 : A function g : X → Y is said to be strongly θ-continuous (written STθ) , if for each x ∈ X and each open neighborhood V of g (x), there exists an open neighborhood U of x such that g (cl (U)) ⊂ V.

Definition 4.10 : A function g : X → Y is said to be δ-continuous (briefly δC) [2], if for each x ∈ X and each open neighborhood V of g (x), there exists an open neighborhood U of x such that g (int (cl (U))) ⊂ int (cl (V)).

Definition 4.11 : A function g : X → Y is said to be almost continuous (briefly AC) [8], if for each x ∈ X and each open neighbourhood V of g (x), there exists an open neighbourhood U of x such that g (U) ⊂ int (cl (V)).

Theorem 4.12 : Every strongly continuity is NA-continuity

Proof : Let g : X → Y be a strongly continuity. Let V be a feebly open subset of Y. Since g is strongly continuous, then from the Remark g^{-1} (V) is both open and closed in X. Since every open set is δ-open, g^{-1} (V) is δ-open in X. Hence g is NA-continuous.

However the converse of the above theorem need not be true as seen from the following example.

Example 4.13 : Let X = {a, b, c} be a space with ℑ = {X, ϕ, {a}, {b}, {a, b}}. Let g : X → X be a function defined by g(a) = a g(b) = a and g (c) = c. Now FO(X) = X, ϕ, {a}, {b}, {a, b}. DO(X) = P(X).Then g is NA -continuous but not STθ-continuous and hence not strongly continuous.

Example 4.14 : Let R be the usual space of reals and i : R → R be the identity function. Since R is regular, i is strongly θ-continuous. However, since there exists a feebly open set which is not open in R [3], i is not NA -continuous. Therefore a super-continuous function need not be NA-continuous.

Remark 4.15 : NA-continuity and β-continuity are independent of each other as seen from the following examples.

Example 4.16 : Let X = {a, b, c} be a space with $\Im$ = {X, ϕ, {a}}. Then (X, $\Im$) be a topological space. Then the identity map i : X $\rightarrow$ X is β-continuous but not na-continuous.

$\Im$-open sets : X, ϕ, {a}
Closed sets : X, ϕ, {b, c}
Feebly open sets : X, ϕ, {a}, {a, b}, {a, c}
Regular open sets : X, ϕ
δ-open sets : X, ϕ

Since for each regular open set V of Y, $i^{-1}(V)$ is regular open in X. Hence i is β- continuous. But i is not na-continuous., as, A = {a} is feebly open set in X , i^{-1}\{a\} = {a} which is not δ-open in X.

Example 4.17 : Let R be the usual space of reals and Y = {a, b, c, d} be a space with $\Im$ = {Y, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}. Define a function *g* : R $\rightarrow$ Y by *g* (x) = a if x < p; *g* (x) = b is p < x < q; *g* (x) = c if q < x < r; *g* (x) = d if x = p, q and r $\leq$ x, where p, q and r are distinct reals. Then *g* is NA-continuous but not β-continuous and not completely continuous.

Remark 4.18 : Completely continuity and NA-continuity are independent as seen from the following examples.

Example 4.19 : In the Example 4.18, *g* is NA -continuous but not completely continuous.

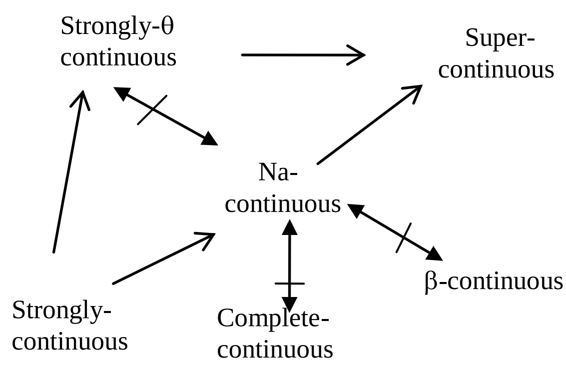
Example 4.20 : Let X = {a, b, c, d} and $\Im$ = {X, ϕ, {c}, {a, b}, {a, b, c}}. Let Y = {x, y, z} and μ = {Y, ϕ, {x}}. Then (X, $\Im$ } and (Y, μ) are topological spaces. Define a map *g* : X $\rightarrow$ Y by *g* (a) = *g* (b) = x, *g* (c) = y and *g* (d) = z. Then *g* is completely continuous but not NA -continuous.

$\Im$-open sets : X, ϕ, {c}, {a, b}, {a, b, c}
$\Im$-closed set : X, ϕ, {d}, {c, d}, {a, b, d}
$\Im$-regular open sets : X, ϕ, {c}, {a, b}
$\Im$ δ-open sets : X, ϕ, {c}, {a, b}
μ-open sets : Y, ϕ, {x}
μ-closed sets : Y, ϕ, {y, z}
μ feebly open sets : Y, ϕ, {x}, {x, y}, {x, z}, Y

Then every open set in Y, its inverse image is regular open in X.

Let A = {x} is open in Y, $g^{-1}(\{x\})$ = {a, b} is regular open in X. Hence *g* is completely continuous. But *g* is not NA -continuous. Since A = {x, z} is feebly open in Y, then $g^{-1}(\{x, z\})$ = {a, b, d} is not δ-open in X. Therefore *g* is not NA-continuous.

Remark 4.21 : From the above observations we have the following implications.



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