Simple Roots and Weyl Groups

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Abstract — *Its origins in the theory of Lie algebras are introduced, and then an axiomatic definition is provided. Simple roots, Bases, Weyl groups, and the transitive action of the latter on the former are explained and proven, respectively.*

Keywords — *Simple roots, Bases, Weyl groups.*

INTRODUCTION

This chapter brings the structure of the roots and Weyl group.Baker studied Matrix Groups: An Introduction to Lie Group Theory. Bourbaki studied Lie Groups and Lie Algebras. Erdmann and Wildon discussed an introduction to Lie algebras, Hall studied Lie Groups, Lie Algebras, and Representations, An Elementary Introduction. Humphrey studied Introduction to Lie Algebras and Representation Theory. Jacobson studied Lie Algebras. Rossmann studied Lie Groups: An Introduction through Linear Groups. Simon have studied Representations of Finite and Compact Groups.

Definition:

Let Φ denote a root system of rank ℓ in a Euclidean space E, with weyl group $\mathcal{W}$. A subset Δ of Φ is called a base if Δ is a linear basis of E and each root of β can be written as $\beta = \sum k_\alpha \alpha\ (\alpha \epsilon \Delta)$ with integral coefficients. k_α all non-negative or all non-positive. The roots in the base Δ are called simple roots.

Example:

A root having multiplicity n=1 is called simple root. $f(z) = (z-1)(z-2)$ has a simple root at $z_0 = 1$, but $g = (z-1)^2$ has a root of multiplicity 2 at $z_0 = 1$, which is therefore not a simple root.

Lemma:

Let α be a simple root. Then σ_α permutes the positive roots other than α.

Proof:

Given α is simple. Let $\beta \in \Phi \pm \{\alpha\}$.

$\therefore\ \beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ where $k_\gamma > 0$

$\beta \neq \alpha$. Both β & α are positive.

$\therefore \beta \neq -\alpha \therefore \beta \neq \pm\alpha$

$\therefore k_\gamma \neq 0$ for some $\gamma \neq \alpha$

For, if $k_\gamma = 0\ \forall\ \gamma \in \Delta, \gamma \neq \alpha$ then $\beta = k_\alpha \alpha$

$\therefore \beta \neq \pm\alpha$ because the only multiple of a root is $\pm\alpha$.

But $\beta \neq \pm\alpha$. This is contradiction.

$\therefore$ there exists γ such that $k_\gamma \neq 0$.

$\sigma_\alpha(\beta) = \beta - \langle \beta\ \alpha \rangle \alpha$

The coefficient of γ in $\sigma_\alpha(\beta)$ is the same as the coefficient of γ in β.

Since σ_α permutes the roots, $\sigma_\alpha(\beta)$ is also a root.

But the coefficient of γ in $\sigma_\alpha(\beta)$ is k_γ which is great than 0.

$\therefore$ all the coefficient in $\sigma_\alpha(\beta)$ must be positive

$\therefore\ \sigma_\alpha(\beta)$ is a positive root.

$\sigma_\alpha(-\alpha) = \alpha$ & $\beta \neq -\alpha$.

$\therefore\ \sigma_\alpha(\beta) \neq \alpha$.

$\therefore\ \sigma_\alpha$ permutes the positive roots other than α.

Definition:

A subset Φ of all Euclidean space E is called a root system in E if the following axioms are satisfied

i) Φ spans E, finite, $0 \notin \Phi$
ii) If $\alpha \in \Phi$, the only multiplies of α in Φ are $\pm\alpha$
iii) If $\alpha \in \Phi$, the reflexion σ_α leaves Φ invariance. Also if $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

Definition:

Let Φ be a root system in E. Let $\mathcal{W}$ denote the subgroup of $GL(E)$ (group of all invertible endomorphism of E) generated by the reflexion σ_α, $\alpha \in \Phi$. Then any $w \in \mathcal{W}$ is a finite product of reflexion of the form $\sigma_{\alpha_1}, \sigma_{\alpha_2}$, ----------$\sigma_{\alpha_n}$ where α_1, α_2,------- $\alpha_n \in \Phi$.

$\alpha \in \Phi \Rightarrow \sigma_\alpha$ leaves Φ invariance.

$\therefore\ w$ leaves Φ invariance.

$\therefore\ w$ is a permutation of Φ.

By (i) Φ is a finite set, spanning E.

Hence $\mathcal{W}$ is a subgroup of the symmetric group on Φ.

Hence $\mathcal{W}$ is finite.

This $\mathcal{W}$ is called a weyl group of Φ.

Note:

$\mathcal{W}$ is a normal subgroup of Aut Φ(automorphism of Φ).

Any element of $\mathcal{W}$ is a permutation of Φ.

$\therefore$ any element of $\mathcal{W}$ is an automorphism of Φ.

$\therefore$ $\mathcal{W}$ is a subgroup of Aut Φ.

Let $\psi \in$ Aut Φ. Then $\psi : \Phi \to \Phi$ is an isomorphism.

Let $\sigma \in \mathcal{W}$.

Consider the map $\theta_\psi : \mathcal{W} \to \mathcal{W}$ defined by $\theta_\psi(\sigma) = \psi \circ \sigma \circ \psi^{-1}$

Then θ_ψ is an isomorphism of $\mathcal{W}$ onto $\mathcal{W}$

$\therefore \psi \circ \sigma \circ \psi^{-1} \in \mathcal{W}$.

This true for all $\psi \in$ Aut Φ,

$\therefore$ $\mathcal{W}$ is a normal subgroup of Aut Φ.

Lemma:

For all $\sigma \in \mathcal{W}$, $\ell(\sigma) = n(\sigma)$ where $\ell(\sigma)$ is the length of σ and $n(\sigma)$ is the number os positive roots α for which $\sigma(\alpha) < 0$.

Proof:

We prove this result using induction on $\ell(\sigma)$.

Suppose $\ell(\sigma) = 0$. Then $\sigma = 1$.

$n(\sigma)$ = number of positive roots α for which $\sigma(\alpha) < 0$.

= number of positive roots α for which $1(\alpha) < 0$.

= number of positive roots α for which $\alpha < 0$.

$= 0 = \ell(\sigma)$

$\therefore$ $n(\sigma) = \ell(\sigma)$ when $\ell(\sigma) = 0$.

$\therefore$ the result is true when $\ell(\sigma) = 0$.

Next, we assume that the lemma is true for all $\tau \in \mathcal{W}$ for which $\ell(\tau) = \ell(\sigma)$.

Let $\sigma_{\alpha_1}, \sigma_{\alpha_2}$, ----------$\sigma_{\alpha_t}$ be the reduced expression for σ.

Let $\alpha = \alpha_t$.

Then we have $\sigma(\alpha) < 0$.

$n(\sigma\sigma_\alpha)$ = number of positive roots β for which $\sigma\sigma_\alpha(\beta) < 0$.

= number of positive roots β for which $\sigma(\beta) < 0$.

Since α is simple, we have σ_α permutes with the roots other than α.

$n(\sigma\sigma_\alpha) = n(\sigma) = 1$

$\ell(\sigma\sigma_\alpha) = \ell(\sigma_{\alpha_1}, \sigma_{\alpha_2}, --- \sigma_{\alpha_t}\ \sigma)$

$= \ell(\sigma_{\alpha_1}, \sigma_{\alpha_2}, --- \sigma_{\alpha_{t-1}}) = t - 1$

$\therefore \ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1$

Now $\ell(\sigma\sigma_\alpha) < \ell(\sigma)$

$\therefore \ell(\sigma\sigma_\alpha) = n(\sigma\sigma_\alpha)$ ie $\ell(\sigma) - 1 = n(\sigma) - 1$.

$\therefore \ell(\sigma) = n(\sigma)$

$\therefore$ By induction the result is true for all values of $\ell(\sigma)$.

Lemma:

Let Φ be a root system in E, with weyl group $\mathcal{W}$. If $\sigma \in GL(E)$, leaves Φ invariant, then

$\sigma = \sigma\,\sigma_\alpha \sigma^{-1}\ \forall\ \sigma \in \Phi$ & $\langle \beta, \alpha \rangle = \langle \sigma(\beta)\ \sigma(\alpha) \rangle$ for $\alpha, \beta \in \Phi$.

Proof:

σ leaves Φ invariance.

$\therefore \beta \in \Phi \Rightarrow \sigma(\beta) \in \Phi$

$\sigma \in GL(E) \Rightarrow \sigma$ is invertible.

ie σ is 1-1.

As β runs over Φ, $\sigma(\beta)$ runs over Φ.

Take $\alpha \in \Phi$. By (iii) σ_α leaves Φ invariant.

If $\beta \in \Phi$ then $\sigma_\alpha(\beta) \in \Phi$

For $\beta \in \Phi$, $(\sigma\,\sigma_\alpha \sigma^{-1})(\sigma(\beta)) = \sigma\,\sigma_\alpha\left(\sigma^{-1}(\sigma(\beta))\right)$

$= \sigma\,\sigma_\alpha(\beta) \in \Phi$

$\sigma(\sigma_\alpha(\beta)) = \sigma(\beta \langle \beta\ \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta\ \alpha \rangle \sigma(\alpha)$.

$\therefore (\sigma\sigma_\alpha\sigma^{-1})(\sigma(\beta)) = \sigma(\beta) - \langle \beta\ \alpha \rangle \sigma(\alpha)$.

$\therefore$ As β runs over Φ, $\sigma(\beta)$ runs over Φ and $\sigma(\alpha) \in \Phi\ \forall\ \alpha \in \Phi$

$\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant.

Consider the hyper plane $\sigma(P_\alpha)$.

Suppose $x \in P_\alpha$. Then $x \in \sigma(\beta)$ for some $\beta \in P_\alpha$.

$\beta \in P_\alpha \Rightarrow (\beta\ \alpha) = 0$.

$(\sigma\,\sigma_\alpha\sigma^{-1})(x) = (\sigma\,\sigma_\alpha\sigma^{-1})(\sigma(\beta))$

$= \sigma\,\sigma_\alpha\left(\sigma^{-1}(\sigma(\beta))\right)$

$= \sigma\,\sigma_\alpha(\beta)$

$= \sigma(\sigma_\alpha(\beta))$

$= \sigma(\beta - \langle \beta\ \alpha \rangle \alpha)$

$= \sigma(\beta) - \langle \beta\ \alpha \rangle \sigma(\alpha)$

$= \sigma(\beta) - \frac{2(\beta\ \alpha)}{(\alpha\ \alpha)} \sigma(\alpha)$

$= \sigma(\beta) = x$ ($\because$ $(\beta, \alpha) = 0$)

$\therefore (\sigma\,\sigma_\alpha\sigma^{-1})(x) = x$

$\therefore$ $\sigma\,\sigma_\alpha\sigma^{-1}$ fixes the hyper plane $\sigma(P_\alpha)$ point wise.

$(\sigma\,\sigma_\alpha\sigma^{-1})(\sigma(\alpha)) = \sigma\,\sigma_\alpha(\alpha) = \sigma(-\alpha)$

$\therefore$ $\sigma\,\sigma_\alpha\sigma^{-1}$ leaves Φ invariant, fixes $\sigma(P_\alpha)$ point wise and stands $\sigma(\alpha)$ to -$\sigma(\alpha)$

$\therefore$ $\sigma\,\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$

$(\sigma\,\sigma_\alpha\sigma^{-1})(\sigma(\beta)) = \sigma\,\sigma_\alpha(\beta)$

$= \sigma(\beta - \langle \beta\ \alpha \rangle \alpha)$

$= \sigma(\beta) - \langle \beta\ \alpha \rangle \sigma(\alpha)$ (1)

$\sigma\,\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma_{\sigma(\alpha)}(\sigma(\beta))$

$= \sigma(\beta) - \langle \sigma(\beta)\ \sigma(\alpha) \rangle \sigma(\alpha)$ (2)

By (1) & (2)

$\langle \sigma(\beta)\ \sigma(\alpha) \rangle = \langle \beta\ \alpha \rangle$

Bases and weyl chambers:

Definition:

Let Δ be a base of root systemΦ. Let β be any root. Let $\beta = \sum k_\alpha \alpha$. Then the height of β relative to Δ denoted by ht(β) is defined by $ht(\beta) = \sum_{\alpha \in \Delta} k_\alpha$,

β is called a positive root if $k_\alpha \geq 0$. β is called a negative root if $k_\alpha \leq 0$.

Lemma:

Let α, β∈E we can say that β∝α if only if either α-β is the sum of positive roots or α=β. This relation ∝ defined above is a partial order on E.

Proof:

(i) For all α∈Φ, α= α. ∴ α∝α.
∴∝ is∝ reflexive.

(ii) Let α∝β and β∝α
α∝β ⇒ either (β-α) is the sum of positive roots or β=α.
β∝α ⇒ either (α -β) is the sum of positive roots or α=β.
∴ 0=(β-α)+ (α -β) is the sum of positive roots.
∴ (β-α)=0 & (α -β)=0 is the sum of positive roots.
ie β=α & α=-β.
∴ ∝ is antisymmetric.

(iii) Let Let α∝β and β∝γ.
α∝β ⇒ either (β-α) is the sum of positive roots or β=α.
β∝γ ⇒ either (γ -β) is the sum of positive roots or γ=β.

case (i): Suppose β-α & γ-β are sum of positive roots
Then (β-α)+ (γ-β) is also a sum of positive roots.
ie γ- α is a sum of positive root. ∴α ∝γ.

case(ii): Suppose β-α is a sum of positive root and γ=β, then γ - α is a sum of positive root.
∴α ∝γ.

case(iii):Suppose β=α & γ-β are sum of positive root, then γ- α is a sum of positive root. ∴α ∝γ.

case(iv): Suppose β=α & γ=β, then γ= α. ∴α ∝γ.
∴ ∝ is transitive. ∴ ∝ is a partial order of E.

Note:

Any positive root is a linear combination of simple roots with nonnegative coefficient. Hence sum positive roots are also sum of simple roots with nonnegative coefficient. Therefore the above definition of partial order can be replaced by the following definition.

Definition:

For any two elements α, β∈E we can say that β∝α if only if either α-β is the sum of simple root or α=β.

Lemma II:

Let Δ be a base ofΦ.

Let α, β∈Δ be such that α≠β.
Then (i) (α -β)≤0 and (ii) (α -β) is not a root.

Proof:

Suppose (α -β)>0. Then Δ is a base of Φ and α, β∈Δ.

∴ α &β are linearly independent.
∴ α≠-β.
∴ α &β are non-proportional roots with (α -β)>0.
∴ (α -β) is a root.
∴ α +(-1)β is a root. Which is contradiction to each root of β can be written as $\beta = \sum k_\alpha \alpha \ (\alpha \epsilon \Delta)$ with integral coefficients.
∴ our assumption that (α -β)>0 is wrong.
ie (α -β)≤0 and (α -β) is not a root.

Definition:

For each vector γ∈E, we define $\Phi^+(\gamma) = \{\alpha \in \Phi / (\alpha, \gamma) > 0\}$. The set of all roots lying in the positive side of the hyper plane orthogonal to γ.

Theorem:

Let γ∈E be regular then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ and every base is obtained in this way.

Proof:

Let γ be regular and $\Delta(\gamma)$ be the set of all indecomposable roots in $\Phi^+(\gamma)$.

Step I:

We claim that each root in $\Phi^+(\gamma)$ is a non negative $\lneq$-linear combination of elements of $\Delta(\gamma)$

Suppose not

Then there exists an $\alpha \in \Phi^+(\gamma)$, which cannot be written as a non negative $\lneq$-linear combination of elements of $\Delta(\gamma)$

Choose an α such that $(\gamma \alpha)$ is as small as possible.

Suppose$\alpha \epsilon \Delta(\gamma)$. Then α=1.α is a non negative $\lneq$ linear combination of elements of $\Delta(\gamma)$

This is a contradiction. ∴ $\alpha \notin \Delta(\gamma)$

∴ α is decomposable.

Let $\alpha = \beta_1 + \beta_2$ where $\beta_1, \beta_2 \in \Phi^+(\gamma)$

$(\gamma \ \alpha) = (\gamma \quad \beta_1 + \beta_2) = (\gamma \quad \beta_1) + (\gamma \quad \beta_2)$

$\beta_1, \beta_2 \in \Phi^+(\gamma)$ implies $(\gamma \quad \beta_1) > 0, \ (\gamma \quad \beta_2) > 0$

Also $(\gamma \quad \beta_1) < (\gamma \ \alpha)$ and $(\gamma \quad \beta_2) < (\gamma \ \alpha)$

∴ by the choice of $\alpha, \beta_1 \ \& \beta_2$ must be non-negative $\lneq$-linear combination of elements of $\Delta(\gamma)$

∴ $\beta_1 + \beta_2$ is a non-negative $\lneq$-linear combination of elements of $\Delta(\gamma)$ ie α is a non-negative $\lneq$-linear combination of elements of $\Delta(\gamma)$.

This is a contradiction.

∴ each root of $\Phi^+(\gamma)$ is a non-negative $\lneq$-linear combination of elements of $\Delta(\gamma)$.

Hence our claim.

Step II:

If $\alpha, \beta \in \Delta(\gamma)$ then $\alpha - \beta \leq 0$ unless$\alpha = \beta$.

Suppose $\alpha, \beta \in \Delta(\gamma), \alpha \neq \beta$.

Suppose $(\alpha, \beta) > 0$. Then Let α, β be non-proportional roots. If $(\alpha, \beta) > 0$ (ie if the angle between α, β is acute) then $\alpha - \beta$ is a root. If $(\alpha, \beta) > 0$ then $\alpha + \beta$ is a root.) $\alpha - \beta$ is a root.

Suppose $\beta = -\alpha$.
Then $(\alpha, \beta) = (\alpha, -\alpha) = -(\alpha, \alpha) \leq 0$.
Which is contradiction to $(\alpha, \beta) > 0$.
$\therefore \beta \neq -\alpha$.
$\therefore$ either $\beta - \alpha$ or $\alpha - \beta$ is in $\Phi^+(\gamma)$.
case (i) Suppose $\alpha - \beta \in \Phi^+(\gamma)$.
Then $\alpha = \beta + (\alpha - \beta)$.
$\therefore$ α is decomposable.
$\therefore \alpha \notin \Phi^+(\gamma)$. This is contradiction.
case(ii) Suppose $\beta - \alpha \in \Phi^+(\gamma)$.
Then $\beta = \beta + (\alpha - \beta)$.
$\therefore$ β is decomposable. $\therefore \beta \notin \Phi^+(\gamma)$.
This is contradiction.
$\therefore \alpha - \beta \leq 0$
Thus if $\alpha, \beta \in \Delta(\gamma)$ then $\alpha - \beta \leq 0$ unless $\alpha = \beta$.

Step III:

$\Delta(\gamma)$ is linearly independent.

Suppose $\sum r_\alpha \alpha = 0$ where $\alpha \in \Delta(\gamma)$ & $r_\alpha \in \mathbb{R}$

Separate the indices α for which $r_\alpha > 0$ and those for which $r_\alpha < 0$.

Then we can write this as $\sum r_\alpha \alpha = \sum t_\beta \beta$ where $r_\alpha > 0$ & $t_\beta = 0$.

The set of $\alpha's$ and $\beta's$ are distinct.

$\therefore \alpha \neq \beta \ \forall \alpha, \beta$

Let $\varepsilon = \sum r_\alpha \alpha$

$(\varepsilon, \varepsilon) = (\sum r_\alpha \alpha, \ \sum r_\alpha \alpha)$

$= \left(\sum r_\alpha \alpha, \ \sum t_\beta \beta\right) \quad = \sum r_\alpha t_\beta (\alpha, \beta) \leq 0$ (by stepII)

$\therefore 0 \leq (\varepsilon, \varepsilon) \leq 0$

$\therefore (\varepsilon, \varepsilon) = 0 \Rightarrow \varepsilon = 0$.

$0 = (\gamma, \varepsilon) = (\gamma, \sum r_\alpha \alpha)$

$= \sum r_\alpha (\gamma, \alpha)$

$\sum r_\alpha (\gamma, \alpha) = 0$ (3)

$\alpha \in \Delta(\gamma)$. $\therefore (\gamma, \alpha) > 0 \ \forall \alpha$

Also $r_\alpha > 0$ for each α.

$\therefore \sum r_\alpha (\gamma, \alpha) > 0$ (4)

By (3) and (4) we get $0 > 0$. This is contradiction.

Similarly $t_\beta = 0 \ \forall \beta$

$\therefore \Delta(\gamma)$ is linearly independent.

Step IV:

$\Delta(\gamma)$ is a base of Φ

Now γ is regular.

$\therefore \Phi = \Phi^+(\gamma) \cup -\left(\Phi^+(\gamma)\right)$

$\beta \in \Phi$. Then $\beta \neq 0$ & $\beta \in \Phi^+(\gamma)$ or $\beta^+ \in \Phi^+(\gamma)$ but not both,

Suppose $\beta \in \Phi^+(\gamma)$

Them by step I, $\beta = \sum k_\alpha \alpha$ where $\alpha \in \Delta(\gamma)$, $k_\alpha \in \mathbb{S}$, $k_\alpha>0$.

Suppose $\beta \in -\Phi^+(\gamma)$

Then $-\beta \in \Phi^+(\gamma)$ and so by step I, $-\beta = \sum t_\alpha \alpha$ where $\alpha \in \Delta(\gamma)$, $t_\alpha \in \mathbb{S}$, $t_\alpha > 0$.

$\therefore \beta = \sum t_\alpha \alpha$ where $t_\alpha \in \mathbb{S}$, $t_\alpha > 0$.

Thus in either case, we have $\beta = \sum k_\alpha \alpha$ where $\alpha \in \Delta(\gamma)$ $k_\alpha \in \mathbb{S}$ such that k_α are all non- negative or non-positive.

Each $\beta \in \Phi^+(\gamma)$ is a linear combination of elements of $\Delta(\gamma)$.

$\therefore \Delta(\gamma)$ spans $\Phi^+(\gamma)$

Φ spans E.

$\therefore$ every element of E is a linear combination of elements of Φ.

Let $x \in E$. Then $\sum_{\alpha \in \Phi} r_\alpha \alpha$

$$x = \sum_{\alpha \in \Phi^+(\nu)} r_\alpha \alpha + \sum_{\beta \in \Phi^-(\nu)} t_\beta \beta$$

$$= \sum_{\alpha \in \Phi^+(\nu)} r_\alpha \alpha + \sum_{\beta \in -\Phi^+(\nu)} t_\beta \beta$$

$$= \sum_{\alpha \in \Phi^+(\nu)} r_\alpha \alpha + \sum_{-\beta \in \Phi^+(\nu)} (-t_\beta)(-\beta)$$

= a linear combination of elements of $\Phi^+(\gamma)$

$\therefore \Phi^+(\gamma)$ spans E

$\Delta(\gamma)$ spans $\Phi^+(\gamma)$

$\therefore \Delta(\gamma)$ spans E

$\therefore \Delta(\gamma)$ is a linear combination of elements of E

$\therefore \Delta(\gamma)$ is a base of Φ

So, Φ has a basis $\Delta(\gamma)$

Step V:

Each base Δ of Φ is of the form $\Delta(\gamma)$

Let $\Delta = \{\gamma_1, \gamma_2, --- \gamma_n\}$ be a base of Φ

Then δ_i be the projection of γ_i on the sub space E_i spanned by all basis vectors except γ_i

$t_i = \langle \gamma_1, \gamma_2, --- \gamma_{i-1}, \gamma_i, --- \gamma_n \rangle$

Each E_i is a hyper plane of E containing δ_i.

Let $\gamma = \sum r_i \delta_i, r_i > 0$

$(r, r_i) = \left(\sum r_i \delta_i, \gamma_i\right)$

$= \sum r_i \left(\delta_i, \gamma_i\right) > 0$

Hence $\exists \ \gamma \in E$ such that $\left(\gamma, \gamma_i\right) > 0 \forall i$

Let $\beta \in \Phi$. Then $\beta = \sum_{i=1}^{n} k_i \nu_i$, $\gamma_i \in \Delta$ & $k_i \neq 0$

$$(\gamma,\beta) = \left(\gamma, \sum_{i=1}^{n} k_i v_i\right)$$

$$= \sum_{i=1}^{n} k_i \, (\gamma,\gamma_i) > 0 \text{ if the } k_i\text{'s are positive}$$

$$< 0 \text{ if the } k_i\text{'s are negative}$$

$\therefore (\gamma,\beta) \neq 0. \therefore \gamma \notin P_\beta \ \forall \beta \in \Phi$

$\therefore \gamma \in E \setminus \bigcup_{\beta\in\Phi} P_\beta$

$\therefore \gamma$ is regular.

Let $\beta \in \Phi^+$. Then β is a positive root.

Let $\beta = \sum_{i=1}^{n} k_i v_i$ where $k_i \in \lesseqgtr$, $k_i > 0$

$\forall i = 1,2,---n$

$(\gamma,\beta) = (\sum k_i\gamma_i) = \sum k_i(\gamma,\gamma_i) > 0$

$\therefore \beta \in \Phi^+(\gamma)$

$\beta \in \Phi^+ \Rightarrow \beta \in \Phi^+(\gamma)$

$\therefore \Phi^+ \subset \Phi^+(\gamma)$

Let $\beta \in \Phi^+(\gamma)$. Then $(\gamma,\beta) > 0$.

Let $\beta = \sum k_i\gamma_i$ where $k_i's$ are all non- negative or non-positive.

$\therefore (\gamma, \sum k_i\gamma_i) > 0$

Suppose all the $k_i's$ are non- positive

Then $\beta \in \Phi^+ \subset -\Phi^+ \subset -\Phi^+(\gamma)$

$\therefore \beta \in -\Phi^+(\gamma)$. This is contradiction for $\beta \in \Phi^+(\gamma)$

$\therefore$ all the $k_i's$ are non- negative.

$\therefore \beta \in \Phi^+$

$\beta \in \Phi^+(\gamma) \Rightarrow \beta \in \Phi^+$

$\therefore \Phi^+(\gamma) \subset \Phi^+$

Hence $\Phi^+ = \Phi^+(\gamma)$

Let $\alpha \in \Delta$. Then $\alpha = \gamma_i$ for some i.

$=0.\gamma_1 + 0.\gamma_2 + ---+ 1.\gamma_i + ---0.\gamma_n$

$\therefore \alpha \in \Phi^+$ for the coefficient are all non-negative.

$\therefore \alpha \in \Delta \Rightarrow \alpha \in \Phi^+ = \Phi^+(\gamma)$

$\therefore \Delta \subseteq \Phi^+(\gamma)$

Let us suppose that $\alpha \in \Delta$ & α is decomposable.

Then $\alpha = \alpha_1 + \alpha_2, \alpha_1, \alpha_2 \in \Phi^+(\gamma)$

Let $\alpha_1 = \sum_{i=1}^{n} t_i \gamma_i, \quad \alpha_2 = \sum_{i=1}^{n} s_i \gamma_i$

Then $\alpha = \sum_{i=1}^{n} (t_i + s_i)\gamma_i, \ (t_i + s_i) > 0$

$\therefore$ any element α of Δ is a linear combination of other elements of Δ which is a contradiction for Δ is a linearly independent.

$\therefore$ α is indecomposable

$\alpha \in \Delta(\gamma)$

$\alpha \in \Delta \Rightarrow \alpha \in \Delta(\gamma) \ \therefore \Delta \subseteq \Delta(\gamma)$

But Card Δ = n = Card $\Delta(\gamma)$

$\therefore \Delta = \Delta(\gamma)$

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