Spectral Methods for Volterra Integral Equations

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Abstract- *The purpose of this paper is to introduce a novel numerical method called a legedre-collocation method to solve the Volterra integral equations of second kind. We also provide error analysis for the proposed method, which indicates that the numerical errors decay exponentially provided that the kernel function and the source function are sufficiently smooth. Numerical results confirm theoretical prediction of the exponential rate of convergence.*

Keywords- *Volterra integral equations of the second kind, Legendre-collocation method Convergence analysis.*

I. Introduction- Consider Volterra integral equations of the second kind

$$y(x)=f(x)+\int_0^x K(x,\,t)\,y(t)\,dt,\quad t\in[0,\,T] \qquad \text{-------- (1.1)}$$

where the source function f and the kernel function K are given and y is an unknown function. There are many numerical approximation methods available to solve Volterra integral equations of type (1.1) such as product integration method, Runge-Kutta method etc. But the problem is that the solution obtained by Numerical approximation method may not be as close to the theoretical solution as expected.

In 1995, H.C.Tain [6] applied spectral approximation method for Volterra integral equations. However no theoretical analysis is provided to justify the high accuracy of the solution obtained. In 1996, G.N.Elnagar and M.Kazemi [4] have applied Chebyshev spectral method to solve non-linear Volterra-Hammerstein integral equations. In 2006, H. Fujiwara [7] applied a spectral approximation method to solve Fredholm integral equations of the first kind under multiple precision arithmetic.

Our interest in this chapter is to apply spectral approximation methods to obtain highly accurate solutions of equations of type (1.1).

Fredholm integral equations behave more or less like boundary value problems. Spectral approximation methods can give highly accurate solutions for boundary value problems. Hence to solve Fredholm integral equations one can directly apply spectral approximation methods to obtain highly accurate solutions. Whereas Volterra integral equations of the second kind behave like initial value problems. Therefore it was not popular to apply spectral approximation methods to initial value problems. The main reason for this is that the functions involved in Volterra integral equations are local functions. Whereas the spectral methods use global basis functions.

The spectral methods for Volterra integral equations may be different from those for the standard initial value problems in the sense that the former requires storage of all values at grid points while the latter requires information only at a fixed number of previous grid points .

One of the main difficulties is how to implement the spectral methods to Volterra integral equations so that an accurate solution can be eventually obtained.

The storage requirement for equation (1.1) also makes use of global basis functions of spectral methods more acceptable.

In this paper we apply Legendre collocation method, which is a spectral method, to solve equation (1.1). We also make rigorous error analysis. The error indicates

that the solution obtained by Legendre collocation method decreases exponentially provided that the source function and the kernel function are sufficiently smooth.

II. Legendre-Collocation Method

Consider Volterra integral equations of the second kind

$$u(t)=g(t)+\int_0^t R(t,\,s)\,u(s)\,ds,\quad \mathrm{t}\in[0,\,\mathrm{T}]. \qquad \text{-- (2.1)}$$

where the source function g and the kernel function R are given and u is an unknown function . For ease of analysis we will transform the problem (2.1) to an equivalent problem defined in $[-1,1]$. More specifically we use the change of variable

$$t=\frac{T(1+x)}{2}\ ,\ \ x=\frac{2t}{T}-1,$$

to rewrite the Volterra equation (2.1) as follows

$$y(x)=f(x)+\int_0^{\frac{T(1+x)}{2}} R\left(\tfrac{T}{2}(1+x),\,s\right)y(s)\,ds,$$
$$x\in[-1,1] \qquad \text{----- (2.2)}$$

and

$y(x)=u\left(\frac{T}{2}(1+x)\right),\quad f(x)=g\left(\frac{T}{2}(1+x)\right).$

Furthermore, to transfer the interval $\left[0,\frac{T(1+x)}{2}\right]$ to the interval $[-1,\, x]$, we make a linear transformation $s=\frac{T(1+t)}{2},\qquad t\in[-1,\, x].$

Then, Equation (2.2) becomes

$$y(x)=f(x)+\int_{-1}^{x}K(x,\, t)y(t)dt,\quad x\in[-1,\, 1], \quad \text{-- (2.3)}$$

where

$$K(x,\, t)=\frac{T}{2}R\left(\frac{T}{2}(1+x),\ \frac{T}{2}(1+t)\right).$$

To solve equation (2.3) we apply Legendre collocation method. For this we choose Legendre Gauss or Gauss-Radau or Gauss-Lobatto points $\{x_i\}_{i=0}^{N}$ as collocation points. Assume that equation (2.3) holds at each x_i, then

$$y(x_i)=f(x_i)+\int_{-1}^{x_i}K(x_i,\, t)y(t)dt,\quad 0\le i\le N. \quad \text{-----(2.4)}$$

To obtaining high order accuracy for small values of x_i, there is little information available for $y(x)$. To overcome this difficulty, we will transfer the integral interval $[-1,\, x_i]$ to the fixed interval $[-1,\, 1]$ and then make use of some appropriate quadrature rule. More precisely, we first make a simple linear transformation:

$$t(x,\, \theta)=\frac{1+x}{2}\theta+\frac{x-1}{2},-1\le\theta\le 1. \quad \text{-----(2.5)}$$

Then (2.4) becomes

$$y(x_i)=f(x_i)+\frac{1+x_i}{2}\int_{-1}^{1}K(x_i,\, t(x_i,\, \theta))y(t(x_i,\, \theta))d\theta,$$

$$0\le i\le N. \quad \text{-- (2.6)}$$

Using $(N+1)$- point Gauss quadrature formula relative to the Legendre weights $\{\omega_k\}$ gives

$$y(x_i)=f(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{N}K(x_i,\, t(x_i,\, \theta_j))y(t(x_i,\, \theta_j))\omega_j,\quad 0\le i\le N, \quad \text{-- (2.7)}$$

where the set $\{\theta_j\}_{j=0}^{N}$ coincide with the collocation points $\{x_i\}_{i=0}^{N}$.

We now need to represent $y(t(x_i,\, \theta_j))$ using $y_i,\ 0\le i\le N,$ the values at all the grid points. To this end, we expand y using Lagrange interpolation polynomials,

$$\text{i.e, } y(\sigma)\approx\sum_{k=0}^{N}y_kF_k(\sigma), \quad \text{-- (2.8)}$$

where F_k is the k-th Lagrange basis function. Combining (2.8) and (2.7) yields

$$y_i=f(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{N}y_j\left(\sum_{p=0}^{N}K(x_i,\, t(x_i,\theta_p))F_j(t(x_i,\theta_p))\omega_p\right),$$

$$0\le i\le N. \quad \text{---- (2.9)}$$

Remark 2.1

It is seen from the numerical scheme (2.9) that to compute the approximation to $y(x_i)$, we require the entire solution information of $\{y(x_i)\}_{i=0}^{N}$ and the semi-local information of $\{K(x_i,\, t(x_i,\theta_j))\}_{j=0}^{i}$. Here $-1\le t(x_i,\ \theta_j)\le x_i$. This is different with the collocation methods or product integration methods which use the semi-local information of both the solution and K, namely $\{y(x_i)\}_{i=0}^{N}$ and $\{K(x_i,\, \beta_j)\}$, where $-1\le\beta_j\le x_i$ are some collocation points. It is because of this difference that we will be able to obtain, as to be demonstrated in the next section, a spectral rate of accuracy instead of an algebraic order of accuracy for the proposed scheme (2.9).

Implementation of the spectral collocation algorithm

Writing $Y_N=[y_0,\, y_1,\, ...,\, y_N]^T$ and $f_N=[f(x_0),\, f(x_1),\, ...,\, f(x_N)]^T$.

We can obtain an equation of the matrix form:

$$Y_N+AY_N=f_N. \quad \text{-- (2.10)}$$

where the entries in the matrix A is given by

$$A_{i,j} = \frac{1+x_i}{2} \sum_{p=0}^{N} K\left(x_i, t\left(x_i, \theta_p\right)\right) F_j\left(t\left(x_i, \theta_p\right)\right) \omega_p.$$

We now discuss an efficient computation of $F_j\left(t\left(x_i, \theta_p\right)\right)$. The idea is to express $F_j(t)$ in terms of the Legendre functions:

$$F_j(t) = \sum_{p=0}^{N} \alpha_{p,j} L_p(t), \qquad \text{----(2.11)}$$

where $\alpha_{p,j}$ is called the discrete polynomial coefficients of F_j.

The inverse relation is (see, e.g., [4]):

$$\alpha_{p,j} = \frac{1}{\beta_p} \sum_{i=0}^{N} F_j(x_i) L_p(x_i) \omega_i = \frac{L_p(x_j)\omega_j}{\beta_p} \qquad \text{-- (2.12)}$$

where

$$\beta_p = \sum_{i=0}^{N} L_p^2(x_i)\omega_i = \left(p+\tfrac{1}{2}\right)^{-1}, \text{ for } p < N \qquad \text{-- (2.13)}$$

and $\beta_N = \left(N + \frac{1}{2}\right)^{-1}$ for the Gauss and Gauss-Radau formulas, and $\beta_N = \frac{2}{N}$ for the Gauss-Lobatto formula. It follows from (2.11) and (2.12) that

$$F_j(t) = \sum_{p=0}^{N} \frac{L_p(x_j) L_p(t)\omega_j}{\beta_p}$$

which, together with the known recurrence formulas for $L_p(t)$ can be used to evaluate $F_j\left(t\left(x_i, \theta_p\right)\right)$ in an efficient way.

III. Some Useful Lemmas

In this section, a convergence analysis for the numerical schemes for the Volterra equation (2.3) will be provided. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed spectral approximations.

Lemma 3.1 ([2], p.290. Integration Error from Gauss Quadrature). Assume that a $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto *quadrature formula relative to the Legendre weight is used to integrate the product* $y\phi$, *where* $y \in H^m(I)$ *with* $I := (-1, 1)$ *for some* $m \geq 1$ *and* $\phi \in P_N$. *Then there exists a constant C independent of N such that*

$$\left|\int_{-1}^{1} y(x)\phi(x)dx - (y, \phi)_N\right| \leq CN^{-m} |y|_{\tilde{H}m,\, N(I)} \|\phi\|_{L^2(I)}, \qquad \text{-- (3.1)}$$

where

$$|y|_{\tilde{H}m,\, N(I)} = \left(\sum_{j=\min(m,\, N+1)}^{m} \left\|y^{(j)}\right\|_{L^2(I)}^2\right)^{\frac{1}{2}}, \qquad \text{-- (3.2)}$$

$$(y, \phi)_N = \sum_{j=0}^{N} \omega_i y(x_i)\phi(x_i), \qquad \text{---(3.3)}$$

Lemma 3.2 ([2], p.289, *Estimates for the Interpolation Error*). Let $y \in H^m(I)$ and demote $I_N y$ its interpolation polynomial associated with the $(N+1)$-point Gauss, or Gauss-Radau, or Gauss Lobatto Points $\{x_i\}_{i=0}^{N}$, namely

$$I_N y = \sum_{i=0}^{N} y(x_i) F_i(x). \qquad \text{---(3.4)}$$

Then the following estimates hold

$$\|y - I_N y\|_{L^2(I)} \leq CN^{-m} |y|_{\tilde{H}m,\, N(I)}, \qquad \text{----(3.5)}$$

$$\|y - I_N y\|_{H^l(I)} \leq CN^{2l-\frac{1}{2}-m} |y|_{\tilde{H}m,\, N(I)}, \quad 1 \leq l \leq m. \qquad \text{-- (3.6)}$$

Lemma 3.3 ([3] Lebesgue Constant for the Legendre Series). Let $F_j(x)$ is the N-th Lagrange interpolation polynomials associated with the Gauss, or Gauss-Radau, or Gauss Lobatto points. Then

$$\max_{x\in(-1,\,1)} \sum_{j=0}^{N} \left|F_j(x)\right| = 1 + \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} N^{\frac{1}{2}} + B_0 + O\left(N^{\frac{-1}{2}}\right), \qquad \text{-- (3.7)}$$

where B_0 is a bounded constant.

Lemma 3.4 (Gronwall Inequality). If a non-negative integrable function $E(t)$ satisfies

$$E(s) \le C_1 \int_{-1}^{s} E(t)\,ds + G(s), \quad -1 < s \le 1, \qquad \text{-----(3.8)}$$

where $G(s)$ is an integrable function, then

$$\|E\|_{L_p(I)} \le C\|G\|_{L_p(I)}, \; p \ge 1. \qquad \text{-----(3.9)}$$

IV. Convergence *Analysis*

In this section, we will carry our convergence analysis in both L^2 and L^∞ spaces.

4.1 Error Analysis in L^2 Space

Theorem 4.1 Let y be the exact solution of the Volterra integral equation (2.3) and assume that

$$Y^N(x) = \sum_{j=0}^{N} y_j F_j(x), \qquad \text{-- (4.1)}$$

where y_j is given by (2.9) and $F_j(x)$ is the j-th Lagrange basis function associated with the Gauss-points $\{x_i\}_{i=0}^{N}$. If $y \in H^m(I)$, then for $m \ge 1$,

$$\|y - Y\|_{L^\infty(I)} \le CN^{\frac{1}{2}-m} \max_{-1\le x\le 1} \left|K(x, \mathrm{t}(x, \cdot))\right|_{\tilde{H}m,\,N(I)} \|y\|_{L^2(I)} + CN^{-m}|y|_{\tilde{H}m,\,N(I)}, \qquad \text{-- (4.2)}$$

provided that N is sufficiently large, where $t(x_i, \theta)$ is defined by (2.5) and C is a constant independent of N.

Proof **:** Following the notations of (3.3), we consider

$$K\big((x, \mathrm{t}), \phi(t)\big)_{N,\,\mathrm{t}} = \sum_{j=0}^{N} K\big(x, \mathrm{t}(x, \theta_j)\big)\phi\big(t(x, \theta_j)\big)\omega_j \qquad \text{---(4.3)}$$

Then the numerical scheme (2.9) can be written as

$$y_i = f(x_i) + \frac{1+x_i}{2}\left(K(x_i, \mathrm{t}), \mathrm{Y}^N(t)\right)_{N,\,\mathrm{t}}, \qquad \text{----(4.4)}$$

which gives

$$y_i = f(x_i) + \frac{1+x_i}{2}\int_{-1}^{1} K\big(x_i, \mathrm{t}(x_i, \theta)\big)Y^N\big(t(x_i, \theta)\big)d\theta$$

$$= f(x_i) + J_1(x_i), \; 1 \le i \le N. \qquad \text{-- (4.5)}$$

where

$$J_1(x) = \frac{1+x}{2}\int_{-1}^{1} K\big(x, \mathrm{t}(x, \theta)\big)Y^N\big(t(x, \theta)\big)d\theta - \frac{1+x}{2}\left(K(x, \mathrm{t}), \mathrm{Y}^N(t)\right)_{N,\,\mathrm{t}}. \qquad \text{-- (4.6)}$$

Using Lemma 3.1 gives

$$|J_1(x)| \le CN^{-m}\left|K(x, \mathrm{t}(x, \cdot))\right|_{\tilde{H}m,\,N(I)} \left\|Y^N\right\|_{L^2(I)}. \qquad \text{-- (4.7)}$$

It follows from (4.5), (2.4) and (2.6) that

$$y_i + \int_{-1}^{x_i} K(x_i, \mathrm{t})Y^N(t)dt = f(x_i) + J_1(x_i), \; 1 \le i \le N. \qquad \text{-- (4.8)}$$

Multiplying $F_j(x)$ on both sides of (4.8) and summing up from 0 to N yield

$$Y^N(x) + I_N\left(\int_{-1}^{x} K(x, \mathrm{t})y(t)dt\right) + I_N\left(\int_{-1}^{x} K(x, \mathrm{t})e(t)dt\right) = I_N(f) + I_N(J_1), \qquad \text{-- (4.9)}$$

where Y^N is defined by (4.1), the interpolation operator I_N is defined by (3.4),

e denotes the error function,

i.e.

$$e(x) = Y^N(x) - y(x), \quad x \in [-1, 1]. \qquad \text{-- (4.10)}$$

It follows from (4.9) and (2.3) that

$$Y^N(x) + I_N(f - y) + I_N\left(\int_{-1}^{x} K(x, \mathrm{t})e(t)dt\right) = I_N(f) + I_N(J_1),$$

which gives

$$e(x) + (y - I_N y)(x) + I_N\left(\int_{-1}^{x} K(x, \mathrm{t})e(t)dt\right) = I_N(J_1). \qquad \text{-- (4.11)}$$

Consequently,

$$e(x) + \int_{-1}^{x} K(x, \mathrm{t})e(t)dt = I_N(J_1) + J_2(x) + J_3(x), \qquad \text{-- (4.12)}$$

where

$$J_2 = I_N y(x) - y(x),$$

$$J_3 = \int_{-1}^{x} K(x,\ t) e(t) dt - I_N \left(\int_{-1}^{x} K(x,\ t) e(t) dt \right). \qquad \text{-- (4.13)}$$

It follows from the Gronwall inequality (see Lemma 3.4) with $p = 2$ that

$$\|e\|_{L^2(I)} \le C \left(\|I_N(J_1)\|_{L^2(I)} + \|J_2\|_{L^2(I)} + \|J_3\|_{L^2(I)} \right). \qquad \text{-- (4.14)}$$

Using (4.7) and Lemma 3.3 gives

$$\|I_N(J_1)\|_{L^2(I)} \le C N^{-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \|Y^N\|_{L^2(I)} \max_{x \in I} \sum_{j=0}^{N} |F_j(x)|$$

$$\le C N^{\frac{1}{2}-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \|Y^N\|_{L^2(I)}$$

$$\le C N^{\frac{1}{2}-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \left(\|e\|_{L^2(I)} + \|y\|_{L^2(I)} \right). \qquad \text{-- - (4.15)}$$

Using the L^2 - error bounds for the interpolation polynomials (i.e., Lemma 3.2) gives

$$\|J_2\|_{L^2(I)} \le C N^{-m} |y|_{\tilde{H}m,\ N(I)}, \qquad \text{-----(4.16)}$$

and, by letting $m = 1$ in (3.5), yields

$$\|J_3\|_{L^2(I)} \le C N^{-1} \left\| K(x,\ x) e(x) + \int_{-1}^{x} K_x(x,\ t) e(t) dt \right\|_{L^2(I)}$$

$$\le C N^{-1} \|e\|_{L^2(I)}. \qquad \text{----(4.17)}$$

The above estimates, together with (4.14), yield

$$\|e\|_{L^2(I)} \le C N^{\frac{1}{2}-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \left(\|e\|_{L^2(I)} + \|y\|_{L^2(I)} \right)$$

$$+ C N^{-m} |y|_{\tilde{H}m,\ N(I)} + C N^{-1} \|e\|_{L^2(I)}, \qquad \text{----(4.18)}$$

which leads to (4.2) provided that N is sufficiently large. This completes the proof of this theorem.

4.2 Error Analysis in L^∞ Space

Below we will extend the L^2 space error estimate in the last subsection to the L^∞ space. The key technique is to use an extrapolation between L^2 space and H^1 space.

Theorem 4.2 Let y be the exact solution of the Volterra integral equation (2.3) and Y^N be defined by (4.1). If $y \in H^m(I)$, then for $m \ge 1$,

$$\|y - Y^N\|_{L^\infty(I)}$$

$$\le C N^{\frac{1}{2}-m} \max_{x \in I} |K(x, t(x, .))|_{\tilde{H}m, N(I)} \|y\|_{L^2(I)}$$

$$+ C N^{\frac{1}{2}-m} |y|_{\tilde{H}m,\ N(I)}, \qquad \text{---(4.19)}$$

provided that N is sufficiently large, where $t(x_i,\ \theta)$ is defined by (2.5) and C is a constant independent of N.

Proof **:** Following the same procedure as in the proof of Theorem 4.1 we have

$$e(x) + \int_{-1}^{x} K(x,\ t) e(t) dt = I_N(J_1) + J_2(x) + J_3(x), \qquad \text{-----(4.20)}$$

where $I_N(J_1)$, J_2 and J_3 are defined by (4.6) and (4.13), respectively. It follows from the Gronwall inequality (see Lemma 3.4) that

$$\|e\|_{L^\infty(I)} \le C \left(\|I_N(J_1)\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)} + \|J_3\|_{L^\infty(I)} \right). \qquad \text{----(4.21)}$$

Using (4.7) and Lemma 3.3 gives

$$\|I_N(J_1)\|_{L^\infty(I)}$$

$$\le C N^{-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \|Y^N\|_{L^2(I)} \max_{x \in I} \sum_{j=0}^{N} |F_j(x)|$$

$$\le C N^{\frac{1}{2}-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \|Y^N\|_{L^2(I)}$$

$$\le C N^{\frac{1}{2}-m} \max_{x \in I} |K(x,\ t(x,\ \theta))|_{\tilde{H}m,\ N(I)} \left(\|e\|_{L^\infty(I)} + \|y\|_{L^2(I)} \right). \qquad \text{---(4.22)}$$

Using the inequality in the Sobolev Space ([4], p.496)

$$\|\omega\|_{L^\infty(a,\ b)} \le \left(\frac{1}{b-a} + 2 \right)^{\frac{1}{2}} \|\omega\|_{L^2(a,\ b)}^{\frac{1}{2}} \|\omega\|_{H^1(a,\ b)}^{\frac{1}{2}},$$

$$\forall\ u \in H^1(a,\ b). \qquad \text{-- (4.23)}$$

and Lemma 3.2, we have

$$\|J_2\|_{L^\infty(I)} = \|y - I_N y\|_{L^\infty(I)}$$

$$\le C \|y - I_N y\|_{L^2(I)}^{\frac{1}{2}} \|y - I_N y\|_{H'(I)}^{\frac{1}{2}}$$

$$\le CN^{\frac{3}{4}-m} |y|_{\tilde{H}m,\, N(I)}. \qquad \text{-- (4.24)}$$

It follows again from Lemma 3.2 and (4.17) that

$$\|J_3\|_{L^2(I)} \le CN^{-1} \|e\|_{L^2(I)}$$

$$\le CN^{-1} \|e\|_{L^\infty(I)}, \qquad \text{-- (4.25)}$$

and

$$\|J_3\|_{H^1(I)} \le CN^{\frac{1}{2}} \left\| K(x, x)e(x) + \int_{-1}^{x} K_x(x, t)e(t)dt \right\|_{L^2(I)}$$

$$\le CN^{\frac{1}{2}} \|e\|_{L^2(I)} \le CN^{\frac{1}{2}} \|e\|_{L^\infty(I)}. \qquad \text{----(4.26)}$$

Using the Sobolev inequality (4.23) gives

$$\|J_3\|_{L^\infty(I)} \le CN^{\frac{-1}{4}} \|e\|_{L^\infty(I)}. \qquad \text{---(4.27)}$$

The desired estimate (4.19) follows from the above estimates and (4.21).

V. Numerical Experiments

Without lose of generality, we will only use the Legendre-Gauss-Lobatto points (i.e., the zeros of $\left(1-x^2\right)L'_N(x)$) as the collocation points. Our numerical evidences show that the other two kinds of Legendre-Gauss points produce results with similar accuracy. For the Legendre-Gauss-Lobatto points, the corresponding weights are

$$\omega_j = \frac{2}{\left(1-x_j^2\right)\left[L'_{N_H}(x_j)\right]^2}, \quad 0 \le j \le N.$$

Example 5.1

Our first example is concerned with an one-dimensional Volterra integral equation of the second kind. More precisely, consider the Volterra integral equation (2.3) with

$$K(x, t) = e^{xt},$$

$$f(x) = e^{4x} + \frac{1}{x+4}\left(e^{x(x+4)} - e^{-(x+4)}\right).$$

The corresponding exact solution is given by $y(x) = e^{4x}$.

Table 5.1: Example 5.1: The maximum point-wise error

N Error	6 3.66e-01	8 1.88e-02	10 6.57e-04	12 1.65e-05	14 3.11e-07
N Error	16 4.57e-09	18 5.37e-11	20 5.19e-13	22 5.68e-14	24 4.26e-14

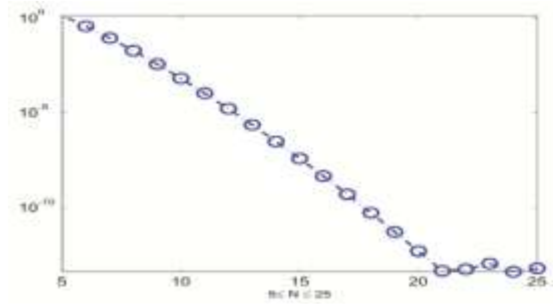

Fig. 5.1 Example 5.1: maximum error for the $1-D$ linear Volterra integral equation

We use the numerical scheme(2.9) Numerical errors with several values of N are displayed in Table (5.1) and Fig (5.1). These results indicate that the desired spectral accuracy is obtained.

In practice, many Volterra integral equations are usually nonlinear. However, the nonlinearity adds rather little to the difficulty of obtaining a numerical solution. The methods described above remain applicable. Although in this work our convergence theory does not cover the non linear case, it should be quite straightforward to establish a convergence result similar to Theorem 4.1 provided that the kernel k in (5.1) is Lipschitz continuous with its third argument. A similar technique for the collocation methods to the nonlinear Volterra integral equations was used by Brunner and Tang [9]. Below we will provide a numerical example using the spectral technique proposed in this work.

For the nonlinear Volterra integral equations of the second kind in the form

$$y(x) = f(x) + \int_{-1}^{x} K(x, t, y(t))dt, \quad x \in [-1, 1], \qquad \text{----(5.1)}$$

we can design a spectral collocation method similar to the linear case. More precisely, we assume that (2.10) holds at the Legendre collocation points and transform the interval $[-1, x]$ and $[-1, 1]$. This gives

$$y(x_i) = f(x_i) + \frac{1+x_i}{2}\int_{-1}^{1} K\left(x_i, t(x_i, \theta) y\left(t(x_i, \theta)\right)\right)d\theta,$$

$$0 \le i \le N. \qquad \text{-- (5.2)}$$

Similar to (2.9) we obtain

$$y_i = f(x_i) + \frac{1+x_i}{2}\sum_{j=0}^{N} k\left(x_i,\ t(x_i,\ \theta_j),\ \sum_{p=0}^{N} y_p F_p\left(t(x_i,\ \theta_j)\right)\right)\omega_j,$$

$$0 \le i \le N. \qquad \text{-- (5.3)}$$

This is a nonlinear problem. On the other hand, the numerical scheme (5.3) leads to a nonlinear system for $\{y_i\}_{i=1}^{N}$, and a proper solver for the nonlinear system (e.g., Newton method) should be used. In our computations, we just use a simple Jacobi type iteration method to solve the nonlinear system, which takes about 5 to 6 iterations. The numerical results can be seen from table 5.2 and Fig 5.2. Again the exponential rate of convergence is observed for the nonlinear problem.

Example 5.2

Our second example is about a nonlinear problem in one-dimension. Consider the Volterra integral equation (5.1) with

$$f(x) = \frac{-1}{2(1+36\pi^2)}\left(e^{-x} + 36\pi^2 e^{-x} - e^{-x}\cos 6\pi x\right.$$

$$\left. + 6\pi e^{-x}\sin 6\pi x - 36e\pi^2\right)e^{x} + e^{x}\sin 3\pi x,$$

$$K(x,\ t,\ y(t)) = e^{x-3t}y^2(t).$$

The exact solution is $y(x) = e^{x}\sin 3\pi x.$

Table 5.2: Example 5.2: The maximum point-wise error.

N Error	6 2.33e-02	8 7.22e-04	10 1.82e-05	12 3.15e-07	14 4.06e-09
N Error	16 3.98e-11	18 3.05e-13	20 3.86e-15	22 3.33e-15	24 3.98e-15

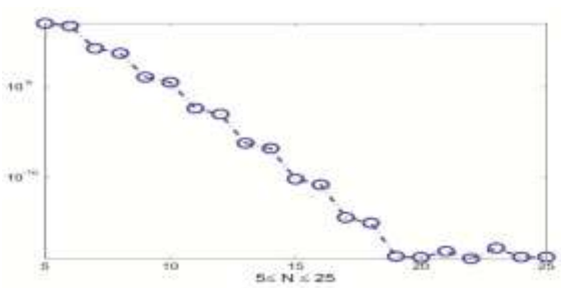

Fig - 5.2 Example 5.2: maximum error for the $1-D$ nonlinear Volterra integral equation.

5.3 Two - Dimensional Extension

Consider a nonlinear Volterra integral equation of the second kind in 2D :

$$y(x,\ z) = f(x,z) + \int_{-1}^{x}\int_{-1}^{z} K\left(x,\ y,\ t,\ s,\ y(t,\ s)\right)dtds,$$

$$(x,\ z) \in [-1,\ 1]^2. \qquad \text{-- (5.4)}$$

Letting the above equation hold at the Legendre point pairs (x_i, z_j), and then using the linear transformation and tricks used in $1D$ case yields

$$y_{i,j} = f(x_i, z_j) + \frac{1+x_i}{2}\frac{1+x_j}{2}$$

$$\sum_{p=0}^{N}\sum_{l=0}^{N} k\left(x_i,\ z_j,\ t(x_i,\ \theta_p),\ t(x_i,\ \theta_l),\ y\left(t(x_i,\ \theta_p),\ t(x_i,\ \theta_l)\right)\right)\omega_p\omega_l$$

----(5.5)

The values of $y\left(t(x_i,\ \theta_p),\ t(x_i,\ \theta_l)\right)$ can also be approximated by $y_{i,j}$ with the use of the relationship between the Lagrange interpolation polynomials associate with the Legendre collocation points, as demonstrated in the one-dimensional case. It is expected that the analysis techniques proposed in this work can be used to extend Theorem 4.1 to obtain a spectral convergence rate for (5.5).

Example 5.3

The third example is concerned with a $2D$ - nonlinear Volterra integral equation with second kind. Consider the equation (5.4) with

$$f(x,z) = \frac{-1}{16}e^{x+z}\left(\sin(4x+2z) + \sin(2z-4)\right.$$

$$\left. + \sin(4x-2) + \sin 6\right) + \sin(2x+z).$$

$$K\left(x,\ z,\ t,\ s,\ y(t,\ s)\right) = e^{x+z}\cos(2t+s)\,y(t,\ s),$$

This problem has a unique solution $y(x,\ z) = \sin(2x+z).$

Table 5.3: Example 5.3:The maximum point-wise error for the $2D$ - nonlinear problem.

N Error	5 6.21e-04	6 2.02e-04	7 8.16e-06	8 1.78e-06	9 7.77e-08
N Error	12 1.73e-10	14 2.89e-13	16 1.30e-14	18 2.94e-15	20 1.67e-15

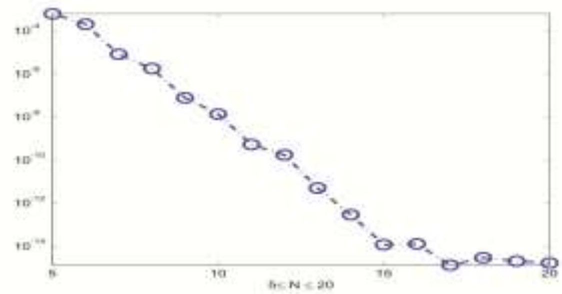

Fig-5.3. Example 5.3: Maximum error for the 2D - non - linear Volterra integral equation.

Table 5.3 and Fig 5.3 present the maximum point wise errors with difference values of N.

Again, it is observed clearly that the errors decay exponentially.

VI. Conclusions

This paper proposes a numerical method for the Volterra type integral equations based on spectral methods.

The most important contribution of this work is that we are able to demonstrate rigorously that the errors of the spectral approximations decay exponentially. More precisely it is proved that if the kernel function and solutions of the underlying Volterra integral equations are smooth, then errors obtained by the proposed spectral method decay exponentially which is a desired feature for a spectral method.

This work seems to be the first successful numerical method for the Volterra integral equations having exponential rate of convergence, which can be demonstrated theoretically and numerically. The tools used in establishing the error estimates include the standard estimates for the quadrature rule and the L^2 - error bounds for the interpolation function.

References

[1] B.Guo and L.Wang, Jacobi, *Interpolation Approximations and their Applications to Singular Differential Equations*, Adv. Comput. Math., 14(2001), 227-276.

[2] C.Canuto, M.Y. Hussaini, A.Quarteroni and T.A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag 2006.

[3] C.K.Qu, and R.Wong, Szego's *Conjecture on Lebesgue Constants for Legendre Series*, Pacific J. Math, 135 (1988), 157 -188.

[4] G.N. Elnagar and M.Kazemi, Chebyshev, *Spectral Solution of Nonlinear Volterra-Hammerstein Integral Equations*, J.Comput. Appl. Math., 76(1996), 147 -158.

[5] H.Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations Methods*, Cambridge University Press 2004.

[6] H-C. Tian, *Spectral-Method for Volterra Integral Equation*, MSc Thesis, Simon Fraser University 1995.

[7] H. Fujiwara, *High-Accurate Numerical Method for Integral Equations of the First Kind under Multiple-Precision Arithmetic*, Preprint, RIMS, Kyoto University, 2006.

[8] H.Brunner , J.P.Kauthen, *The Numerical Solution of Two-dimensional Volterra Integral Equation*, IMA J.Numer. Anal., 9(1989), 45-59.

[9] H.Brunner and T.Tang. *Polynomial Spline Collocation Methods for the Nonlinear Basset Equation*, Comput. Math . Appl., 18 (1989), 449-457.

[10] L.M. Delves, J.L. Mohammed, *Computational Methods for Integral Equations*, Cambridge University Press 1985.