Odd Graceful Labeling of Some New Type of Graphs

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Abstract

In this paper some new type of odd graceful graphs are investigated. We prove that the graph obtained by joining a cycle C_8 with some star graphs $S_{1,r}$ keeping one vertex and three vertices gap between pair of vertices of the cycle admits odd graceful labelling. We also prove that the graph obtained by joining a cycle C_{12} with some star graphs $S_{1,r}$ keeping two, three and five vertices gap between pair of vertices of the cycle admits odd graceful labelling. We observed that $C_8 \Theta S^4_{1,r}$ with one vertex gap, $C_8 \Theta S^2_{1,r}$ three vertices gap between pair of vertices of the cycle C_8 are odd graceful , $C_{12} \Theta S^6_{1,r}$ with one vertex gap, $C_{12} \Theta S^4_{1,r}$ two vertices gap, $C_{12} \Theta S^3_{1,r}$ with three vertices gap and $C_{12} \Theta S^2_{1,r}$ with five vertices gap between pair of vertices are odd graceful graphs.

Keywords: *Graceful labeling, graceful graphs, odd graceful graphs, star graphs***.**

AMS Subject Classification (2010):05C78

1. Introduction

All graphs in this paper are finite, simple and undirected. Let G = (V, E) denotes a graph with n number of vertices and q number of edges. Here the terminology and notation we use are followed by Harary[1]. The symbols V and E will denote the vertex set and edge set of a graph G respectively. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph. In

1985, Lo[2] introduced the notion of edge-graceful graphs. Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang[3] introduced the k-edge-graceful graphs. Here we introduce *odd graceful graphs*. We provide some definitions and other information which are required for the present investigations.

Definition 1.1 If all the vertices of a graph are assigned some values subject to certain conditions then it is known as graph labelling

Definition 1.2 A simple graph $G(m,n)$ with m vertices and n edges is *graceful* if there is a labeling l of its vertices with distinct integers from the set $\{0,1,2,\ldots,n\}$ So that the induced edge labeling e defined by $e(uv)=|l(u)-l(v)|$ assigns each edge a different label. A graph which admits graceful labeling is called a *graceful graph*.

Definition 1.3 A graph $G=(V, E)$ with q edges is said to admit *odd graceful labeling* if $f: V \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ such that the induced function $f^*: E \rightarrow \{1,3,\ldots,2q-1\}$ defined as $f^*(ab) = |f(a) - f(b)|$ is bijective. A graph which admits odd graceful labeling is called an *odd graceful graph*.

Definition 1.4 A Star graph is a tree consisting of one vertex adjacent to all others. We denote here a star graph as $S_{1,r}$ where r number of pendant vertices are connected to one vertex.

The study of graph labeling and graceful graphs was introduced by Rosa[4].(Gnanajothi R.B,1991)[5] introduced the concept of odd graceful graphs and she has proved many results. Kathiresan K.M.,2008[6] has discussed odd gracefulness of ladders and graphs obtained fromthem by subdividing each step exactly once. Vaidya.S.K. e-tal 2010[7]proved the odd gracefulness of joining of even cycle with path and cycle sharing a common edge. Vaidya S.K. e-tal 2013[8] proved the odd gracefulness of splitting graph and the shadow graph of bister. Barrientos Christian 2009[9] discussed the odd gracefulness of Trees of Diameter 5.For detailed survey on graph labeling and related results we refer to Gallian J.A.,2015[10]

Main results:

Theorem-1: Graph obtained by joining the cycle C_8 with $\mathbf{C_8 \Theta S^4_{1,1}}$ with one vertex gap four star graphs $S_{1,r}$ keeping one vertex gap between pair of vertices of the cycle is odd graceful.

Proof: Let $a_1, a_2, \ldots\ldots a_8$ be the vertices of the cycle C_8 and a_i^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_i , for j=1,2,...r. We define the vertex labelling as

f : V(G) → {0,1,2,....,2q-1,where q is the total number of edges} as follows

Case-1: If 'i' is odd, we define $f(x) = \{i-1, if\ x = a_i\}$

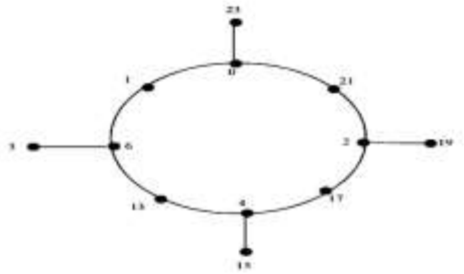
Let r-pendant vertices of the star graph $S_{1,r}$ are connected to each a_i keeping one vertex gap between pair of vertices of the cycle C_8 .

Case-2: If 'i' is even,We define $f(a_i) = 2q - i(r+1) + 1,\ if\ i < 8$ and

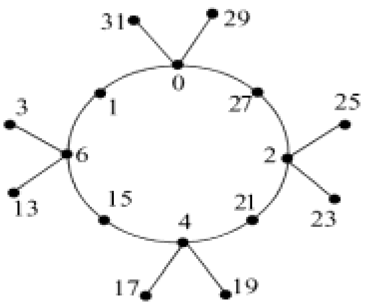
$f(a_8) = 1$

$f(a_i^{(j)}) = (2q-1) - (r+1)(i-1) - 2(j-1)$

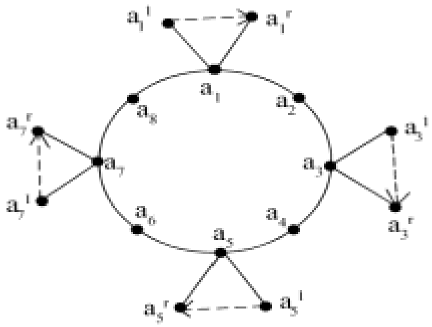
except $i = 7, j = r$ for $f(a_7^r) = 3$. The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,\ldots\ldots 2q-1\}$ Thus f is an edge graceful labeling of G = **C_8 Θ $S^4_{1,r}$**. Hence **C_8 Θ $S^4_{1,r}$** is an odd graceful graph . Example:



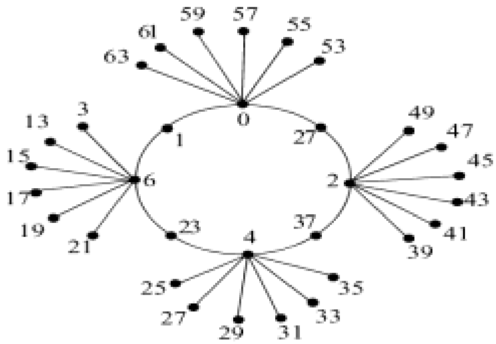
C_8 Θ $S^4_{1,1}$ with one vertex gap



C_8 Θ $S^4_{1,2}$ with one vertex gap is odd graceful



C_8 Θ $S^4_{1,r}$ with one vertex gap



C_{12} Θ $S^4_{1,6}$ with one vertex gap is odd graceful

<u>Theorem-2:</u>

Graph obtained by joining the cycle C_8 with three star graphs $S_{1,r}$ keeping three vertices gap between pair of vertices of the cycle is odd graceful.

Proof: Let $a_1, a_2, \ldots\ldots a_8$ be the vertices of the cycle C_8. a_i^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_i ,for j=1,2,...r. We define the vertex labeling as

f : V(G) → {0,1,2,....,2q-1,where q is the total number of edges} as follows

Case-1: If 'i' is odd

We define $f(x) = \{i-1,\ if\ x = a_i\}$

Let r-pendant vertices of the star graph $S_{1,r}$ are connected to each a_i keeping three vertices gap between pair of vertices of the cycle

Case-2: If 'i' is even

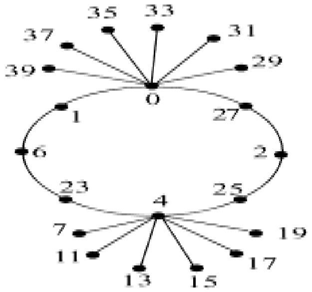
We define $f(a_i) = 2q - ir - 2i + 3, i \neq 8$ and $f(a_8) = 1$

$$f(a_i^{(j)}) = \begin{cases} (2q-1)-(r+1)(i-1)-2(j-1) \; if \; i=1 \\ (q-1)-2(j-1) \; if \; i=5, j \neq r \\ 7 \; if \; i=5, j=r \end{cases}$$

The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,.....,2q-1\}$

Thus f is an edge graceful labeling of G = C_8 **ʘ** $S^2_{1,r}$ .Hence C_8 **ʘ** $S^2_{1,r}$ is an odd graceful graph.

Example:



C_8 **ʘ** $S^2_{1,6}$ with three vertices gap is odd graceful

<u>Theorem-3:</u>

Graph obtained by joining the cycle_C_8 with a star graph $S_{1,r}$ is odd graceful.

Proof: Let $a_1, a_2, ... a_8$ be the vertices of the cycle C_8. a_1^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_1 ,for j=1,2,...r. We define the vertex labeling as

f : V(G) → {0,1,2,....,2q-1,where q is the total number of edges} as follows

Case-1: If 'i' is odd

We define $f(x) = \{i-1, \; if \; x = a_i\}$

Let r-pendant vertices of the star $S_{1,r}$ are connected to a_1

Case-2: If 'i' is even

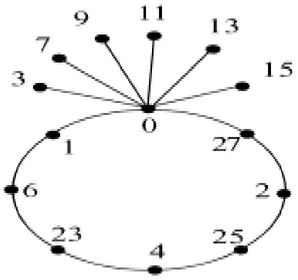
We define $f(a_i) = 2q - i + 1, i \neq 8$ and $f(a_8) = 1$

$$f(a_1^j) = 2j+1, \; j \neq 2 \; and \; 1 \leq j \leq r+1$$

The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,.....,2q-1\}$

Thus f is an edge graceful labeling of G = C_8 **ʘ** $S_{1,r}$ .Hence C_8 **ʘ** $S_{1,r}$ is an odd graceful graph.

Example:



C_8 **ʘ** $S_{1,6}$ is odd graceful .

<u>Theorem-4</u>

C_{12} ʘ $S^6_{1,r}$ (where r > 4) with one vertex gap between pair of vertices of the cycle C_{12} is an odd graceful graph.

Proof: Let $a_1, a_2, ... a_{12}$ be the vertices of the cycle C_{12}. a_i^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_i ,for j=1,2,...r. We define the vertex labelling as

f : V(G) → {0,1,2,....,2q-1,where q is the total number of edges} as follows

Case-1: If 'i' is odd,

We define $f(a_i) = i - 1$

Case-2: If i is even

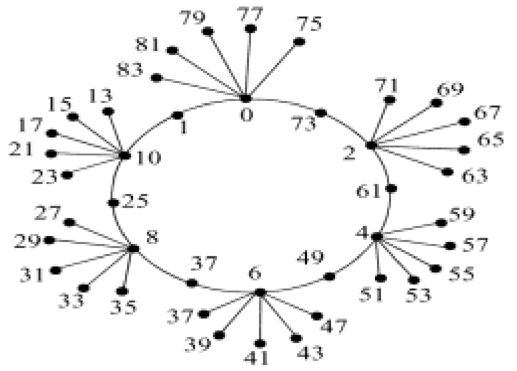
We define $f(a_i) = 2q - i(r+1) + 1$ and

$$f(a_{2n+1}^j) = 10 + 2j + 1, \; j \neq 4 \; and \; 1 \leq j \leq r+1, n = 4$$

The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,.....,2q-1\}$

Thus f is a graceful labeling of G = C_{12} **ʘ** $S^6_{1,r}$ .Hence C_{12} **ʘ** $S^6_{1,r}$ is an odd graceful graph.

Example:



C_{12} ʘ $S^6_{1,5}$ is odd graceful .

<u>Theorem-5</u>:C_{12} ʘ $S^4_{1,r}$ with two vertices gap between pair of vertices of the cycle C_{12} is an odd graceful graph.

Proof: Let $a_1, a_2, \ldots . a_{12}$ be the vertices of the cycle C_{12}. a_i^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_i ,for j=1,2,...r. We define the vertex labeling as

f : V(G) $\rightarrow$ {0,1,2,....,2q-1,where q is the total number of edges} as follows

Case-1: If 'i' is odd

We define $f(a_i) = i - 1$

If 'i' is even

We define $f(a_i) = 2q - 2i + 1,\ i \neq 12$ and $f(a_{12}) = 1$

$$f(a^j_{2n+1}) = 2q - 2n(r+1) - 2j + 1, except\ n = 4$$

For r=1 i.e. if only one pendant vertex is attached to each vertex of the cyclic graph C_{12} keeping two vertices gap in between pair of vertices.

We define

$$f(a_i^{(j)}) = \begin{cases} 2q-1\,, & for\ i=1 \\ 2q-3i\,, & for\ i=4,7 \\ q+2\,, & for\ i=10 \end{cases}$$

For r=2 i.e. if only two pendant vertices are attached to each vertex of the cyclic graph C_{12} keeping two vertices gap in between pair of vertices.

We define

$$f(a_i^{(j)}) = \begin{cases} 2q-2j+1\,, & for\ i=1 \\ q+2j-4\,, & for\ i=4 \\ q-4j+1\,, & for\ i=7 \\ q+2j\ \ , & for\ i=10 \end{cases}$$

For r>2 i.e. if more than two pendant vertices are attached to each vertex of the cyclic graph C_{12} keeping two vertices gap in between pair of vertices.

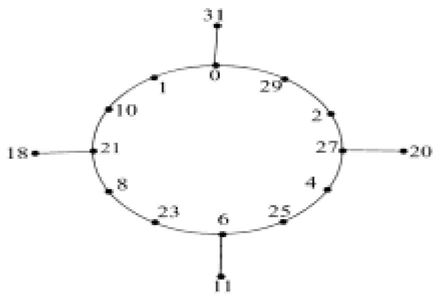
We define

$$f(a_i^{(j)}) = \begin{cases} 2q-2j+1\,, & for\ i=1 \\ 4i+2j\ \ , & for\ i=4 \\ 2r+2j+9\,, & for\ i=7 \\ i+4r+2j\ \ , & for\ i=10 \end{cases}$$

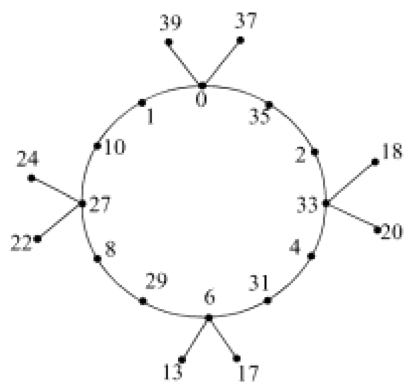
The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,\ldots..,2q-1\}$

Thus f is an edge graceful labeling of G = **C_{12} ʘ $S^4_{1,r}$** .Hence **C_{12} ʘ $S^4_{1,r}$** is an odd graceful graph.

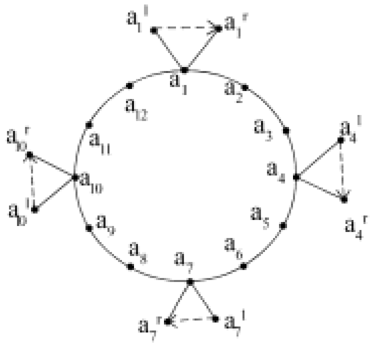
Examples:



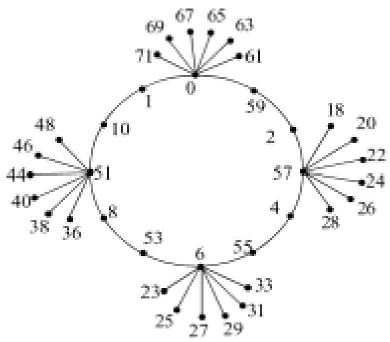
C_{12} ʘ $S^4_{1,1}$ with two vertices gap is odd graceful



C_{12} ʘ $S^4_{1,2}$ with two vertices gap is odd graceful



$C_{12}\,\Theta\, S^4_{1,r}$ with two vertices gap is odd graceful



$C_{12}\,\Theta\, S^4_{1,6}$ with two vertices gap is odd graceful

Theorem-6

$C_{12}\,\Theta\, S^3_{1,r}$ with three vertices gap between pair of vertices of the cycle C_{12} is an odd graceful graph.

Proof: Let $a_1, a_2, ... a_8$ be the vertices of the cycle C_8. a_i^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_i ,for j=1,2,...r. We define the vertex labeling as

$f^*: V(G) \rightarrow \{0,1,2,.....2q-1\}$, where q is the total number of edges, as follows

If 'i' is odd, we define $f(a_i) = i-1$

If 'i' is even, we define

$$f(a_i) = \begin{cases} 2q-2r-i+1, & i \le 4 \\ 2q-4r-i+1, & 4 \le i \le 8 \end{cases},$$

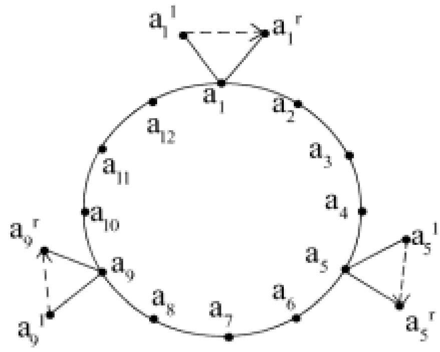
$$f(a_{10}) = 15 \quad and \quad f(a_{12}) = 1$$

and

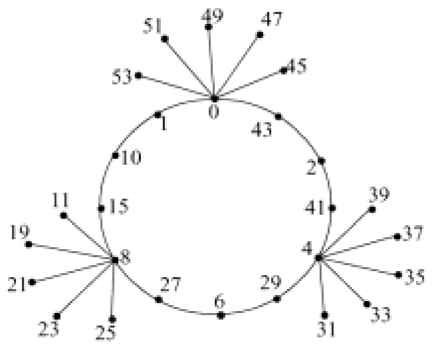
$$f(a_i^{(j)}) = \begin{cases} 2q-2j+1, & for\ i=1\ and\ 1 \le j \le r \\ 2q-2r-2j-3, & for\ i=5\ and\ 1 \le j \le r \\ 2q-4r-2j-7, & for\ i=9\ and\ 1 \le j < r \\ 11, & for\ i=9\ and\ j=r \end{cases}$$

The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,.....2q-1\}$

Thus f is an edge graceful labeling of G = C_8U P_r .Hence C_8U P_r is an odd graceful graph.



$C_{12}\,\Theta\, S^4_{1,r}$ with three vertices gap



$C_{12}\,\Theta\, S^4_{1,5}$ with three vertices gap is odd graceful

Theorem-7

$C_{12}\,\Theta\, S^4_{1,r}$ with five vertices gap between pair of vertices of the cycle C_{12} is an odd graceful graph.

Proof: Let $a_1, a_2, ... a_{12}$ be the vertices of the cycle C_{12} and a_i^j are pendant vertices adjacent to a_i ,for j=1,2,...r. If q is the total number of edges of the graph G, we define the vertex labeling as

$f: V(G) \rightarrow \{0,1,2,......2q-1\}$ as follows

We define $f(a_i)=\begin{cases} i-1, & if\ i\ is\ odd \\ 2q-i+1, if\ i\ is\ even\ and\ i<12 \end{cases}$

Case-1: **For r=1 (only one pendant vertex is attached to each vertex of the cyclic graph C_{12} keeping five vertices gap in between pair of vertices).**

We define $f(a_{12})=3$, $f(a_1^1)=1$, $f(a_7^1)=11$

Case-2 : For r=2 (two pendant vertices are attached to each vertex of the cyclic graph C_{12} keeping five vertices gap in between pair of vertices).

We define $f(a_{12})=1$, $f(a_1^j)=2j+1,\ for\ 1\le j\le 2$,

$$f(a_7^j)=\begin{cases} 13\ for\ j=1 \\ 17\ for\ j=2 \end{cases}$$

Case-3: **For r > 2 (if more than two pendant vertices are attached to each vertex of the cyclic graph C_{12} keeping five vertices gap in between pair of vertices)**

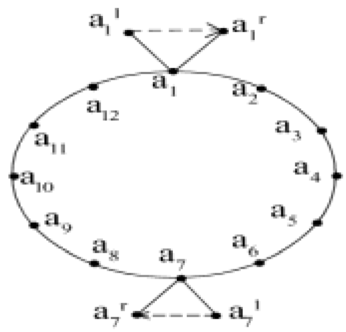
We define $f(a_{12})=1$

$$f(a_1^j)=\begin{cases} 2j+1, & for\ j<4 \\ 2j+3, & for\ j\ge 4 \end{cases}$$

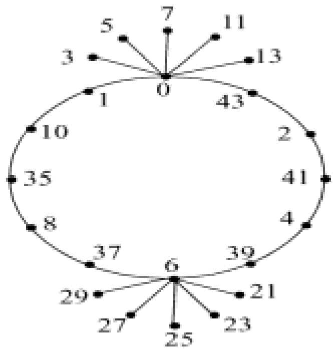
$$f(a_7^j)=q+2j-3,\ for\ 1\le j\le r$$

The vertex function defined above induces a bijective edge function $f^*:E(G)\to\{1,3,5,\dots\dots,2q-1\}$

Thus f is an edge graceful labeling of G = $\mathbf{C_{12}\ \Theta\ S^4_{1,r}}$ .Hence $\mathbf{C_{12}\ \Theta\ S^4_{1,r}}$ is an odd edge graceful graph.



$\mathbf{C_{12}\ \Theta\ S^4_{1,r}}$ with five vertices gap



$\mathbf{C_{12}\ \Theta\ S^4_{1,5}}$ with five vertices gap is odd graceful

Theorem-8: Graph obtained by joining the cycle C_{12} with a star graph $S_{1,r}$ is odd graceful.

Proof: Let $a_1, a_2, \dots\dots a_8$ be the vertices of the cycle C_{12} and a_1^j are pendant vertices of the star graph $S_{1,r}$ adjacent to a_1 ,for j=1,2,...r. We define the vertex labeling as

f : V(G) $\to$ {0,1,2,....,2q-1,where q is the total number of edges} as follows

Case-1: If 'i' is odd

We define $f(a_i)=i-1$

Let r-pendant vertices of the star $S_{1,r}$ are connected to each a_1

Case-2: If 'i' is even

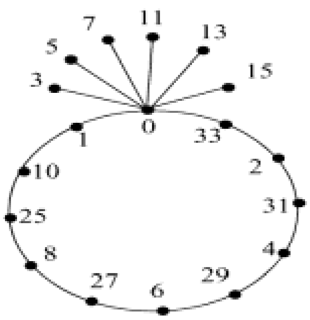
We define $f(a_i)=2q-i+1, i\ne 12$ and $f(a_{12})=1$

$$f(a_1^j)=2j+1,\ \ j\ne 4\ and\ 1\le j\le r+1$$

The vertex function defined induces a bijective edge function $f^*:E(G)\to\{1,3,5,\dots\dots,2q-1\}$

.Thus f is an edge graceful labeling of G = $C_{12} \Theta S_{1,r}$ .Hence $C_{12} \Theta S_{1,r}$ is an odd graceful graph.

Examples:



$C_{12} \Theta S_{1,6}$ is odd graceful .

Application: Graph labeling has wide application in coding theory, communication networks, optimal circuits layouts and graph decomposition problems.

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