Eulerian integral associated with product of two multivariable Aleph-functions, the multivariable $\bar{I}$-function and a class of polynomials

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of two multivariable Aleph-functions , the multivariable $\bar{I}-$function defined by Nambisan et al [1] and a class of multivariable polynomials with general arguments . We will study the cases concerning the multivariable I-function defined by Sharma et al [2] and Srivastava-Doust polynomial [4].

Keywords: Eulerian integral, multivariable I-function, generalized Lauricella function of several variables, multivariable Aleph-function, generalized hypergeometric function, class of polynomials, Srivastava-Daoust polynomial

Classification. 33C99, 33C60, 44A202010 Mathematics Subject

1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable Aleph-functions, the $\bar{I}$-function of several variables defined by Nambisan et al [1] and a class of polynomials with general arguments.
First time, we define the multivariable $\bar{I}$-function by :

$$\bar{I}(z_1''',\cdots,z_v''') = \bar{I}^{0,N:M_1,N_1;\cdots;M_v,N_v}_{P,Q:P_1,Q_1;\cdots;P_v,Q_v}\left(\begin{array}{c|c} z_1''' & (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(v)};A_j)_{N+1,P}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z_v''' & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(v)};B_j)_{M+1,Q}: \end{array}\right.$$

$$\left.\begin{array}{c} (c_j^{(1)},\gamma_j^{(1)};1)_{1,N_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{N_1+1,P_1};\cdots;(c_j^{(v)},\gamma_j^{(v)};1)_{1,N_u},(c_j^{(v)},\gamma_j^{(v)};C_j^{(v)})_{N_v+1,P_v} \\ (d_j^{(1)},\delta_j^{(1)};1)_{1,M_1},(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{M_1+1,Q_1};\cdots;(d_j^{(v)},\delta_j^{(v)};1)_{1,M_v},(d_j^{(v)},\delta_j^{(v)};D_j^{(v)})_{M_v+1,Q_v} \end{array}\right) \quad (1.1)$$

$$=\frac{1}{(2\pi\omega)^v}\int_{L_1}\cdots\int_{L_v}\phi_1(s_1,\cdots,s_v)\prod_{i=1}^{v}\xi_i'(s_i)z_i'''^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_v \quad (1.2)$$

where $\phi_1(s_1,\cdots,s_v),\xi_i'(s_i),i=1,\cdots,v$ are given by :

$$\phi_1(s_1,\cdots,s_v)=\frac{1}{\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{v}\alpha_j^{(i)}s_j\right)\prod_{j=M+1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{v}\beta_j^{(i)}s_j\right)} \quad (1.3)$$

$$\xi_i'(s_i) = \frac{\prod_{j=1}^{N_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{M_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=N_i+1}^{P_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=M_i+1}^{Q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \tag{1.4}$$

$i = 1, \cdots, v$

Serie representation

If $z_i''' \neq 0; i = 1, \cdots, v$

$\delta_{h_i}^{(i)}(d_j^{(i)}+k_i) \neq \delta_j^{(i)}(\delta_{h_i}^{(i)}+\eta_i) for j \neq h_i, j, h_i = 1, \cdots, m_i (i = 1, \cdots, v), k_i, \eta_i = 0, 1, 2, \cdots (i = 1, \cdots, v)$, then

$$\bar{I}(z_1''', \cdots, z_v''') = \sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \left[\phi_1\left(\frac{dh_1^{(1)}+k_1}{\delta h_1^{(1)}}, \cdots, \frac{dh_v^{(v)}+k_v}{\delta h_v^{(v)}}\right)\right] \prod_{j\neq h_i i=1}^{r} \frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i'''^{\frac{dh_i+k_i}{\delta h_i}} \tag{1.5}$$

This result can be proved on computing the residues at the poles :

$$s_i = \frac{dh_i^{(i)}+k_i}{\delta h_i^{(i)}}, (h_i = 1, \cdots, m_i, k_i = 0, 1, 2, \cdots) for i = 1, \cdots, v \tag{1.6}$$

We may establish the the asymptotic expansion in the following convenient form :

$$\bar{I}(z_1''', \cdots, z_v''') = 0(\,|z_1'''|^{\alpha_1}, \cdots, |z_v'''|^{\alpha_v}\,), max(\,|z_1'''|, \cdots, |z_v'''|\,) \rightarrow 0$$

$$\bar{I}(z_1''', \cdots, z_v''') = 0(\,|z_1'''|^{\beta_1}, \cdots, |z_v'''|^{\beta_u}\,), min(\,|z_1'''|, \cdots, |z_v'''|\,) \rightarrow \infty$$

where $k = 1, \cdots, v : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will note $\eta_{h_i,k_i} = \dfrac{dh_i^{(i)}+k_i}{\delta h_i}$, $(h_i = 1, \cdots, m_i, k_i = 0, 1, 2, \cdots) for i = 1, \cdots, v$ **(1.7)**

The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of several variables defined by Srivastava et al [7]. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1, \cdots, z_r) = \aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$[(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}]$, $[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]$:
.... , $[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}]$:

$[(c_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots ; \; ; [(c_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$[(d'^{(1)}_j), \delta'^{(1)}_j)_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots ;; [(d'^{(r)}_j), \delta'^{(r)}_j)_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\Bigg)$

$$= \frac{1}{(2\pi\omega)^r} \int_{L'_1} \cdots \int_{L'_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.8}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k)]} \tag{1.9}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j'^{(k)} - \delta_j'^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]} \tag{1.10}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j'^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.11}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j'^{(k)} - \delta_j'^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j'^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)} \tag{1.12}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,)\to \infty$$

where $k=1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j=1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R\ ;\ V = m_1, n_1;\cdots;m_r, n_r \tag{1.13}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)},\cdots,p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.14}$$

$$A = \{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.15}$$

$$B = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.16}$$

$$C = \{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.17}$$

$$D = \{(d_j'^{(1)};\delta_j'^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,\{(d_j'^{(r)};\delta_j'^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.18}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & A : C \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \cdot & \\ z_r & B : D \end{array}\right) \tag{1.19}$$

Consider the Aleph-function of s variables

$$\aleph(z_1,\cdots,z_s) = \aleph_{p_i',q_i',\iota_i;r':p_{i^{(1)}}',q_{i^{(1)}}',\iota_{i^{(1)}};r^{(1)};\cdots;p_{i^{(s)}}',q_{i^{(s)}}';\iota_{i^{(s)}};r^{(s)}}^{0,n':m_1',n_1',\cdots,m_s',n_s'}\left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_s \end{array}\right.$$

$$\begin{array}{ll} [(u_j;\mu_j^{(1)},\cdots,\mu_j^{(r')})_{1,n'}] & ,[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r')})_{n'+1,p_i'}]: \\ \cdots\cdots\cdots\cdots & ,[\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(r')})_{m'+1,q_i'}]: \end{array}$$

$$\left.\begin{array}{l} [(a_j^{(1)});\alpha_j^{(1)})_{1,n_1'}], [\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n_1'+1,p_i'^{(1)}}];\cdots;[(a_j^{(s)});\alpha_j^{(s)})_{1,n_s'}], [\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n_s'+1,P_i^{(s)}}] \\ [(b_j^{(1)});\beta_j^{(1)})_{1,m_1'}], [\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m_1'+1,q_i'^{(1)}}];\cdots;[(b_j^{(s)});\beta_j^{(s)})_{1,m_s'}], [\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m_s'+1,Q_i^{(s)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L''_1} \cdots \int_{L''_s} \zeta(t_1, \cdots, t_s) \prod_{k=1}^{s} \phi_k(t_k) z_k^{t_k} \, \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.20}$$

with $\omega = \sqrt{-1}$

$$\zeta(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{n'} \Gamma(1 - u_j + \sum_{k=1}^{s} \mu_j^{(k)} t_k)}{\sum_{i=1}^{r'} [\iota_i \prod_{j=n'+1}^{P_i} \Gamma(u_{ji} - \sum_{k=1}^{s} \mu_{ji}^{(k)} t_k) \prod_{j=1}^{q'_i} \Gamma(1 - v_{ji} + \sum_{k=1}^{s} v_{ji}^{(k)} t_k)]} \tag{1.21}$$

and $$\phi_k(t_k) = \frac{\prod_{j=1}^{m'_k} \Gamma(b_j^{(k)} - \beta_j^{(k)} t_k) \prod_{j=1}^{n'_k} \Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}} [\iota_{i^{(k)}} \prod_{j=m'_k+1}^{Q_{i^{(k)}}} \Gamma(1 - b_{ji^{(k)}}^{(k)} + \beta_{ji^{(k)}}^{(k)} t_k) \prod_{j=n'_k+1}^{P_{i^{(k)}}} \Gamma(a_{ji^{(k)}}^{(k)} - \alpha_{ji^{(k)}}^{(k)} s_k)]} \tag{1.22}$$

Suppose , as usual , that the parameters

$u_j, j = 1, \cdots, p'; v_j, j = 1, \cdots, q';$

$a_j^{(k)}, j = 1, \cdots, n'_k; a_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p'_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j = m'_k + 1, \cdots, q'_{i^{(k)}}; b_j^{(k)}, j = 1, \cdots, m'_k;$

with $k = 1 \cdots, s, i = 1, \cdots, r', i^{(k)} = 1, \cdots, r^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i'^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} + \iota_i \sum_{j=n'+1}^{p'_i} \mu_{ji}^{(k)} + \sum_{j=1}^{n'_k} \alpha_j^{(k)} + \iota_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} - \iota_i \sum_{j=1}^{q'_i} v_{ji}^{(k)} - \sum_{j=1}^{m'_k} \beta_j^{(k)}$$

$$-\iota_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.23}$$

The reals numbers τ_i are positives for $i = 1, \cdots, s$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to m'_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to n'_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} B_i^{(k)} \pi$, where

$$B_i^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} - \iota_i \sum_{j=n'+1}^{p'_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{q'_i} v_{ji}^{(k)} + \sum_{j=1}^{n'_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m'_k}\beta_j^{(k)} - \iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i(k)}}\beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k=1,\cdots,s, i=1,\cdots,r \ , i^{(k)}=1,\cdots,r^{(k)} \tag{1.24}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_s) = 0(\,|z_1|^{\alpha'_1},\cdots,|z_s|^{\alpha'_s}\,), max(\,|z_1|,\cdots,|z_s|\,) \rightarrow 0$$

$$\aleph(z_1,\cdots,z_s) \ = 0(\,|z_1|^{\beta'_1},\cdots,|z_s|^{\beta'_s}\,), min(\,|z_1|,\cdots,|z_s|\,) \rightarrow \infty$$

where $k=1,\cdots,z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j=1,\cdots,m'_k$ and

$$\beta'_k = max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})], j=1,\cdots,n'_k$$

We will use these following notations in this paper

$$U' = p'_i, q'_i, \iota_i; r' ; \ V' = m'_1, n'_1; \cdots; m'_s, n'_s \tag{1.25}$$

$$W' = p'_{i(1)}, q'_{i(1)}, \iota_{i^{(1)}}; r^{(1)}, \cdots, p'_{i(r)}, q'_{i(r)}, \iota_{i^{(s)}}; r^{(s)} \tag{1.26}$$

$$A' = \{(u_j;\mu_j^{(1)},\cdots,\mu_j^{(s)})_{1,n'}\}, \{\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)})_{n'+1,p'_i}\} \tag{1.27}$$

$$B' = \{\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(s)})_{m'+1,q'_i}\} \tag{1.28}$$

$$C' = (a_j^{(1)};\alpha_j^{(1)})_{1,n'_1} \ , \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p'_{i(1)}}, \cdots, (a_j^{(s)};\alpha_j^{(s)})_{1,n'_s} \ , \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n'_s+1,p'_{i(s)}} \tag{1.29}$$

$$D' = (b_j^{(1)};\beta_j^{(1)})_{1,m'_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q'_{i(1)}}, \cdots, (b_j^{(s)};\beta_j^{(s)})_{1,m'_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m'_s+1,q'_{i(s)}} \tag{1.30}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_s) = \aleph_{U':W'}^{0,n':V'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A' : C'} \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_s & \mathrm{B' : D'} \end{array}\right) \tag{1.31}$$

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.32}$$

The coefficients are $B[E;R_1,\ldots,R_v]$ arbitrary constants, real or complex.

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [6 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[5,page 454] and [6] given by :

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+s_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+s_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-s_j)z_1^{s_1}\cdots z_l^{s_l}z_{l+1}^{s_{l+1}}\cdots,z_{l+k}^{s_{l+k}}\,\mathrm{d}s_1\cdots\mathrm{d}s_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j = L_{w\zeta_j\infty}(Re(\zeta_j) = v_j'')$ starting at the point $v_j'' - \omega\infty$ and terminating at the point $v_j'' + \omega\infty$ with $v_j'' \in \mathbb{R}(j = 1, \cdots, l)$ and each of the remaining contour $L_{l+1}, \cdots, L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [3, page 454].

3. Eulerian integral

In this section , we note :

$$\theta_i = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}, \zeta_j^{(i)} > 0(i=1,\cdots,r)\,;\, \theta_i' = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}}, \zeta_j'^{(i)} > 0(i=1,\cdots,s)$$

$$\theta_i'' = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0(i=1,\cdots,u)$$

$$\theta_i''' = \prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'''^{(i)}}, \zeta_j'''^{(i)} > 0(i=1,\cdots,v) \tag{3.1}$$

$$K_1 = (1-\alpha-\sum_{i=1}^{u}R_ia_i - \sum_{i=1}^{v}\eta_{G_i,g_i}a_i'; \mu_1,\cdots,\mu_r,\mu_1',\cdots,\mu_s',h_1,\cdots,h_l,1,\cdots,1) \tag{3.2}$$

$$K_2 = (1-\beta-\sum_{i=1}^{u}R_ib_i - \sum_{i=1}^{v}\eta_{G_i,g_i}b_i'; \rho_1,\cdots,\rho_r,\rho_1',\cdots,\rho_s',0,\cdots,0,0\cdots,0) \tag{3.3}$$

$$K_j = [1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j''^{(i)} - \sum_{i=1}^{v}\eta_{G_i,g_i}\zeta_j'''^{(i)}; \zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},$$

$$0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l} \tag{3.4}$$

$$K_j' = [1+\sigma_j-\sum_{i=1}^{u}R_i\lambda_j''^{(i)} - \sum_{i=1}^{v}\eta_{G_i,g_i}\lambda_j'''^{(i)}; \lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},$$

$$0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \tag{3.5}$$

$$L_1 = (1-\alpha-\beta-\sum_{i=1}^{u}R_i(a_i+b_i) - \sum_{i=1}^{v}(a_i'+b_i')\eta_{G_i,g_i}; \mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu_1'+\rho_1',\cdots,\mu_r'+\rho_r',$$

$$h_1,\cdots,h_l,1,\cdots,1) \tag{3.6}$$

$$L_j = [1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j''^{(i)} - \sum_{i=1}^{s}\zeta_j'''^{(i)}\eta_{G_i,g_i}; \zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l} \tag{3.7}$$

$$L'_j = [1+\sigma_j - \sum_{i=1}^{u} R_i \lambda_j''^{(i)} - \sum_{i=1}^{v} \lambda_j'''^{(i)} \eta_{G_i,g_i}; \lambda_j^{(1)}, \cdots, \lambda_j^{(r)}, \lambda_j'^{(1)} \cdots, \lambda_j'^{(s)}, 0, \cdots, 0, 0, \cdots, 0]_{1,k} \quad (3.8)$$

$$P_1 = (b-a)^{\alpha+\beta-1} \left\{ \prod_{j=1}^{h} (af_j + g_j)^{\sigma_j} \right\} \quad (3.9)$$

$$B_{u,v} = (b-a)^{\sum_{i=1}^{v}(a'_i+b'_i)\eta_{G_i,g_i} + \sum_{i=1}^{u}(a_i+b_i)R_i} \left\{ \prod_{j=1}^{h} (af_j+g_j)^{-\sum_{i=1}^{v}\lambda_i'''\eta_{g_i,h_i} - \sum_{i=1}^{u}\lambda_i'' R_i} \right\} \quad (3.10)$$

$$B_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots, R_u)}{R_1! \cdots R_u!} \quad (3.11)$$

$$V_1 = V; V'; 1,0; \cdots; 1,0; 1,0; \cdots; 1,0 \ ; W_1 = W; W'; 0,1; \cdots; 0,1; 0,1; \cdots; 0,1 \quad (3.12)$$

$$C_1 = C; C'; (1,0), \cdots, (1,0); (1,0), \cdots, (1,0) \ ; D_1 = D; D'; (0,1), \cdots, (0,1); (0,1), \cdots, (0,1) \quad (3.13)$$

We have the general Eulerian integral

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j} \prod_{j=1}^{k} (f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u} \left(\begin{array}{c} \mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array} \right)$$

$$\bar{I} \left(\begin{array}{c} \mathrm{z}_1'''\theta_1'''(t-a)^{a'_1}(b-t)^{b'_1} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_v'''\theta_v'''(t-a)^{a'_v}(b-t)^{b'_v} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array} \right)$$

$$\aleph \left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r} \prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

$$\aleph\left(\begin{array}{c} z'_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ z'_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t$$

$$= P_1\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}\frac{(-)^{k_i}}{\delta h_i^{(i)}k_i!}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}[\phi_1(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r})]_{j\neq h_i}$$

$$\aleph^{0,n+n'+l+k+2:V_1}_{U;U';l+k+2,l+k+1:W_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K'_j:C_1 \\ \cdots & . \\ \cdots & . \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & . \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \tau_1(b-a)^{h_1} & . \\ \cdots & . \\ \cdots & . \\ \tau_l(b-a)^{h_l} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L}_1,L_j,L'_j:D_1 \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (3.14)$$

We obtain the Aleph-function of $r+s+k+l$ variables. The quantities A,A',B,B',C,C',C_1,D_1,V_1 and W_1 are defined above.

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\mu'_u,\rho_i,\rho'_u,\lambda_j^{(i)},\lambda_j'^{(u)},h_v\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j,\lambda_v\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;k;$

$u=1,\cdots,s;v=1,\cdots,l),a_i,b_i,\lambda_j''^{(i)},\zeta_j''^{(i)}\in\mathbb{R}^+,(i=1,\cdots,u;j=1,\cdots,k)$

$a'_i,b'_i,\lambda_j'''^{(i)},\zeta_j'''^{(i)}\in\mathbb{R}^+,(i=1,\cdots,v;j=1,\cdots,k)$

(B) See the section 1

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1$

(D) $$Re\big[\alpha+\sum_{j=1}^{v}a'_j\min_{1\leqslant k\leqslant M_i}\frac{d_k^{(j)}}{\delta_k^{(j)}}+\sum_{j=1}^{r}\mu_j\min_{1\leqslant k\leqslant m_i}\frac{d_k'^{(j)}}{\delta_k'^{(j)}}+\sum_{j=1}^{s}\mu'_i\min_{1\leqslant k\leqslant m'_i}\frac{b_k^{(j)}}{\beta_k^{(j)}}\big]>0$$

$$Re\big[\beta+\sum_{j=1}^{v}b'_j\min_{1\leqslant k\leqslant M_i}\frac{d_k^{(j)}}{\delta_k^{(j)}}+\sum_{j=1}^{r}\rho_j\min_{1\leqslant k\leqslant m_i}\frac{d_k'^{(j)}}{\delta_k'^{(j)}}+\sum_{j=1}^{s}\rho'_j\min_{1\leqslant k\leqslant m'_i}\frac{b_k^{(j)}}{\beta_k^{(j)}}\big]>0$$

(E) $$Re\left(\alpha+\sum_{i=1}^{v}\eta_{G_i,g_i}a'_i+\sum_{i=1}^{u}R_ia_i+\sum_{i=1}^{r}\mu_is_i+\sum_{i=1}^{s}t_i\mu'_i\right)>0$$

$$Re\left(\beta+\sum_{i=1}^{v}\eta_{G_i,g_i}b'_i+\sum_{i=1}^{u}R_ib_i+\sum_{i=1}^{r}v_is_i+\sum_{i=1}^{s}t_i\rho'_i\right)>0$$

$$Re\left(\lambda_j+\sum_{i=1}^{v}\eta_{G_i,g_i}\lambda_j'''^{(i)}+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\zeta_j^{(i)}+\sum_{i=1}^{s}t_i\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);$$

$$Re\left(-\sigma_j+\sum_{i=1}^{v}\eta_{G_i,g_i}\lambda'''^{(i)}+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\lambda_j^{(i)}+\sum_{i=1}^{s}t_i\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(F) $$U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j'^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0$$

$$U_i'^{(k)}=\sum_{j=1}^{n'}\mu_j^{(k)}+\iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)}+\sum_{j=1}^{n'_k}\alpha_j^{(k)}+\iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}-\iota_i\sum_{j=1}^{q'_i}\upsilon_{ji}^{(k)}-\sum_{j=1}^{m'_k}\beta_j^{(k)}$$

$$-\iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}\leqslant 0$$

(G) $$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j'^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}-\mu_k-\rho_k>0,$$ with $k=1\cdots,r,$

$i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)}$

$$B_i^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} - \iota_i \sum_{j=n'+1}^{p'_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{q'_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{n'_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m'_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} - \sum_{l=1}^{k} \lambda_j'^{(i)} - \sum_{l=1}^{l} \zeta_j'^{(i)} \quad -\mu'_k - \rho'_k > 0, \quad \text{with } k = 1, \cdots, s,$$

$i = 1, \cdots, r \ , i^{(k)} = 1, \cdots, r^{(k)}$

(H) $$\left| arg \left(z_i \prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{1}{2} A_i^{(k)} \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, r)$$

$$\left| arg \left(z'_i \prod_{j=1}^{l} \left[1 - \tau'_j (t-a)^{h'_i}\right]^{-\zeta_j'^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}} \right) \right| < \frac{1}{2} B_i^{(k)} \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, s)$$

(I) The multiple series occuring on the right-hand side of (3.14) is absolutely and uniformly convergent.

Proof

To prove (3.14), first, we express in serie the multivariable I-function defined by Nambisan et al [2] with the help of (1.5), a class of multivariable polynomials defined by Srivastava $S_L^{h_1,\cdots,h_u}[.]$ in serie with the help of (1.32), the Aleph-functions of r-variables and s-variables in terms of Mellin-Barnes type contour integral with the help of (1.8) and (1.20) respectively. Now collect the power of $\left[1 - \tau_j (t-a)^{h_i}\right]$ with $(i = 1, \cdots, r; j = 1, \cdots, l)$ and collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable Aleph-function ,we obtain the equation (3.14).

Remarks

If a) $\rho_1 = \cdots, \rho_r = \rho'_1 = \cdots, \rho'_s = 0$; b) $\mu_1 = \cdots, \mu_r = \mu'_1 = \cdots, \mu'_s = 0$, we obtain the similar formulas that (3.14) with the corresponding simplifications.

4. Particular cases

a) If $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}}, \iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \rightarrow 1$, the multivariable Aleph-functions of r and s-variables reduces to multivariable I-functions of r and s-variables defined by Sharma and al [3] respectively and we have

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\lambda_j} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u} \begin{pmatrix} z''_1 \theta''_1 (t-a)^{a_1} (b-t)^{b_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_u \theta''_u (t-a)^{a_u} (b-t)^{b_u} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j''^{(u)}} \end{pmatrix}$$

$$\bar{I}\left(\begin{array}{c} z_1'''\theta_1'''(t-a)^{a_1'}(b-t)^{b_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_v'''\theta_v'''(t-a)^{a_v'}(b-t)^{b_v'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right) I\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right) \mathrm{d}t$$

$$= P_1\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}\frac{(-)^{k_i}}{\delta h_i^{(i)}k_i!}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}[\phi_1(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r})]_{j\neq h_i}$$

$$I^{0,n+n'+l+k+2:V_1}_{U;U';l+k+2,l+k+1:W_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K_j':C_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \mathrm{B}\ ;\mathrm{B}';\ \mathrm{L}_1,L_j,L_j':D_1 \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.14) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}},\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$

b) If $B(L;R_1,\cdots,R_u)=\dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi'_j+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [4]. We have

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c}z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}}\end{array}\right.$$

$$\left.\begin{array}{c}[(\text{-L});\text{R}_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}]\\ [(\text{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}]\end{array}\right)$$

$$\bar{I}\left(\begin{array}{c}z'''_1\theta'''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z'''_v\theta'''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}}\end{array}\right)$$

$$\aleph\left(\begin{array}{c}z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}}\end{array}\right)$$

$$\aleph\left(\begin{array}{c}z'_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z'_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}}\end{array}\right)\text{d}t$$

$$= P_1 \sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} \prod_{i=1}^{v} \frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i'''^{\eta_{h_i,k_i}} \prod_{k=1}^{u} z''^{R_k} B'_u B_{u,v} [\phi_1(\eta_{h_1,k_1}, \cdots, \eta_{h_r,k_r})]_{j \neq h_i}$$

$$\aleph_{U;U';l+k+2,l+k+1:W_1}^{0,n+n'+l+k+2:V_1} \left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A} ; \mathrm{A} ; \mathrm{K}_1, K_2, K_j, K'_j : C_1 \\ \cdot \cdot \cdot & \cdot \\ \cdot \cdot \cdot & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdot \cdot \cdot & \cdot \\ \cdot \cdot \cdot & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdot \cdot \cdot & \cdot \\ \cdot \cdot \cdot & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdot \cdot \cdot & \mathrm{B} ; \mathrm{B}' ; \mathrm{L}_1, L_j, L'_j : D_1 \\ \cdot \cdot \cdot & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array} \right) \quad (4.3)$$

under the same notations and conditions that (3.14)

where $B'_u = \dfrac{(-L)_{h_1R_1+\cdots+h_uR_u} B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions and a class of multivariable polynomials defined by Srivastava et al [5].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable Aleph-function, a expansion of multivariable I-function defined by Nambisan et al [1] and a class of multivariable polynomials defined by Srivastava et al [5] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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