

On general Eulerian integral of certain products of two multivariable

I-functions and a class of polynomials

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ABSTRACT

The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-function, defined by Nambisan et al [2], a general class of multivariable polynomials and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function defined by Srivastava et al [8] and the Srivastava-Daoust polynomial [5].

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function.

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1. Introduction

In this paper, we evaluate a general Eulerian integral concerning the product of two multivariable I-functions defined by Nambisan et al [2], a generalized hypergeometric function and a class of multivariable polynomials. We will give a serie expansion of a multivariable I-function. The multivariable I-function is an extension of the multivariable H-function defined by Srivastava et al [8]. We will given a contracted form.

$$\bar{I}(z_1''', \dots, z_v''') = \bar{I}_{P,Q:P_1,Q_1;\dots;P_v,Q_v}^{0,N:M_1,N_1;\dots;M_v,N_v} \left(\begin{matrix} z_1''' \\ \cdot \\ \cdot \\ z_v''' \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(v)}; A_j)_{N+1,P} : \\ \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(v)}; B_j)_{M+1,Q} : \end{matrix} \right. \left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)}; 1)_{1,N_1}, (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{N_1+1,P_1}; \dots; (c_j^{(v)}, \gamma_j^{(v)}; 1)_{1,N_v}, (c_j^{(v)}, \gamma_j^{(v)}; C_j^{(v)})_{N_v+1,P_v} \\ \\ (\bar{d}_j^{(1)}, \bar{\delta}_j^{(1)}; 1)_{1,M_1}, (\bar{d}_j^{(1)}, \bar{\delta}_j^{(1)}; D_j^{(1)})_{M_1+1,Q_1}; \dots; (\bar{d}_j^{(v)}, \bar{\delta}_j^{(v)}; 1)_{1,M_v}, (\bar{d}_j^{(v)}, \bar{\delta}_j^{(v)}; D_j^{(v)})_{M_v+1,Q_v} \end{matrix} \right) \quad (1.1)$$

$$= \frac{1}{(2\pi\omega)^v} \int_{L_1} \dots \int_{L_v} \phi_1(s_1, \dots, s_v) \prod_{i=1}^v \xi_i'(s_i) z_i''' s_i ds_1 \dots ds_v \quad (1.2)$$

where $\phi_1(s_1, \dots, s_v), \xi_i'(s_i), i = 1, \dots, v$ are given by :

$$\phi_1(s_1, \dots, s_v) = \frac{1}{\prod_{j=N+1}^P \Gamma^{A_j} (a_j - \sum_{i=1}^v \alpha_j^{(i)} s_i) \prod_{j=M+1}^Q \Gamma^{B_j} (1 - b_j + \sum_{i=1}^v \beta_j^{(i)} s_j)} \quad (1.3)$$

$$\xi_i'(s_i) = \frac{\prod_{j=1}^{N_i} \Gamma (1 - c_j^{(i)} + \gamma_j^{(i)} s_i) \prod_{j=1}^{M_i} \Gamma (\bar{d}_j^{(i)} - \bar{\delta}_j^{(i)} s_i)}{\prod_{j=N_i+1}^{P_i} \Gamma^{C_j^{(i)}} (c_j^{(i)} - \gamma_j^{(i)} s_i) \prod_{j=M_i+1}^{Q_i} \Gamma^{D_j^{(i)}} (1 - \bar{d}_j^{(i)} + \bar{\delta}_j^{(i)} s_i)} \quad (1.4)$$

$$i = 1, \dots, v$$

Serie representation

If $z_i''' \neq 0; i = 1, \dots, v$

$k_i, \eta_i = 0, 1, 2, \dots (i = 1, \dots, v)$, then $\delta^{(\bar{i})}_{h_i}(d^{(\bar{i})}_j + k_i) \neq \delta_j^{(\bar{i})}(\delta_{h_i}^{(\bar{i})} + \eta_i)$ for $j \neq h_i, j, h_i = 1, \dots, m_i (i = 1, \dots, v)$,

$$\bar{I}(z_1''', \dots, z_v''') = \sum_{h_1=1}^{M_1} \dots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \dots \sum_{k_v=0}^{\infty} \left[\phi_1 \left(\frac{dh_1^{(1)} + k_1}{\delta h_1^{(1)}}, \dots, \frac{dh_v^{(v)} + k_v}{\delta h_v^{(v)}} \right) \right] \prod_{j \neq h_i, i=1}^r \frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i'''^{\frac{dh_i + k_i}{\delta h_i}} \quad (1.5)$$

This result can be proved on computing the residues at the poles :

$$s_i = \frac{dh_i^{(i)} + k_i}{\delta h_i^{(i)}}, (h_i = 1, \dots, m_i, k_i = 0, 1, 2, \dots) \text{ for } i = 1, \dots, v \quad (1.6)$$

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1''', \dots, z_v''') = O(|z_1'''|^{\alpha_1}, \dots, |z_v'''|^{\alpha_v}), \max(|z_1'''|, \dots, |z_v'''|) \rightarrow 0$$

$$\bar{I}(z_1''', \dots, z_v''') = O(|z_1'''|^{\beta_1}, \dots, |z_v'''|^{\beta_v}), \min(|z_1'''|, \dots, |z_v'''|) \rightarrow \infty$$

where $k = 1, \dots, v: \alpha_k = \min[Re(\bar{d}_j^{(k)}/\delta_j^{(k)})], j = 1, \dots, m_k$ and

$$\beta_k = \max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \dots, n_k$$

$$\text{We will note } \eta_{h_i, k_i} = \frac{dh_i^{(i)} + k_i}{\delta h_i^{(i)}}, (h_i = 1, \dots, m_i, k_i = 0, 1, 2, \dots) \text{ for } i = 1, \dots, v \quad (1.7)$$

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, \dots, z_r) = I_{p, q; p_1, q_1; \dots; p_r, q_r}^{0, n; m_1, n_1; \dots; m_r, n_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1, p} : \\ \\ \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1, q} : \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1, p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1, p_r} \\ \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1, q_1}; \dots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1, q_r} \end{matrix} \right) \quad (1.8)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi(s_1, \dots, s_r) \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} ds_1 \dots ds_r \quad (1.9)$$

where $\phi(s_1, \dots, s_r), \theta_i(s_i), i = 1, \dots, r$ are given by :

$$\phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma^{A_j} (1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} s_j)}{\prod_{j=n+1}^p \Gamma^{A_j} (a_j - \sum_{i=1}^r \alpha_j^{(i)} s_j) \prod_{j=1}^q \Gamma^{B_j} (1 - b_j + \sum_{i=1}^r \beta_j^{(i)} s_j)} \quad (1.10)$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}} (1 - c_j^{(i)} + \gamma_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}} (d_j^{(i)} - \delta_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}} (c_j^{(i)} - \gamma_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}} (1 - d_j^{(i)} + \delta_j^{(i)} s_i)} \quad (1.11)$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leq 0, i = 1, \dots, r \quad (1.12)$$

The integral (2.1) converges absolutely if

$$|arg(z_k)| < \frac{1}{2} \Delta_k \pi, k = 1, \dots, r \text{ where}$$

$$\Delta_k = - \sum_{j=n+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \quad (1.13)$$

Srivastava and Garg [6] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \dots, h_u} [z_1, \dots, z_u] = \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} (-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u) \frac{z_1^{R_1} \dots z_u^{R_u}}{R_1! \dots R_u!} \quad (1.14)$$

The coefficients are $B[E; R_1, \dots, R_u]$ arbitrary constants, real or complex.

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [7 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_P F_Q [(A_P); (B_Q); -(x_1 + \dots + x_r)]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \dots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \dots + s_r)} \Gamma(-s_1) \dots \Gamma(-s_r) x_1^{s_1} \dots x_r^{s_r} ds_1 \dots ds_r \quad (2.1)$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \dots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \dots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \dots, r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^l [1-\tau_j(t-a)^{h_j}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} dt = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \prod_{j=1}^k (af_j + g_j)^{\sigma_j}$$

$$F_{1:0, \dots, 0; 0, \dots, 0}^{1:1, \dots, 1; 1, \dots, 1} \left(\begin{matrix} (\alpha : h_1, \dots, h_l, 1, \dots, 1) : (\lambda_1 : 1), \dots, (\lambda_l : 1); (-\sigma_1 : 1), \dots, (-\sigma_k : 1) \\ \dots \\ (\alpha + \beta : h_1, \dots, h_l, 1, \dots, 1) : -, \dots, -; -, \dots, - \end{matrix} \right)$$

$$; \tau_1(b-a)^{h_1}, \dots, \tau_l(b-a)^{h_l}, -\frac{(b-a)f_1}{af_1 + g_1}, \dots, -\frac{(b-a)f_k}{af_k + g_k} \quad (2.2)$$

where $a, b \in \mathbb{R}(a < b), \alpha, \beta, f_i, g_i, \sigma_i, \tau_j, h_j \in \mathbb{C}, \lambda_j \in \mathbb{R}^+(i = 1, \dots, k; j = 1, \dots, l)$

$$\min(\operatorname{Re}(\alpha), \operatorname{Re}(\beta)) > 0, \max_{1 \leq j \leq l} \{|\tau_j(b-a)^{h_j}|\} < 1, \max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_j}{af_j + g_j} \right| \right\} < 1,$$

and $F_{1:0, \dots, 0; 0, \dots, 0}^{1:1, \dots, 1; 1, \dots, 1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[5, page 454] given by :

$$F_{1:0, \dots, 0; 0, \dots, 0}^{1:1, \dots, 1; 1, \dots, 1} \left(\begin{matrix} (\alpha : h_1, \dots, h_l, 1, \dots, 1) : (\lambda_1 : 1), \dots, (\lambda_l : 1); (-\sigma_1 : 1), \dots, (-\sigma_k : 1) \\ \dots \\ (\alpha + \beta : h_1, \dots, h_l, 1, \dots, 1) : -, \dots, -; -, \dots, - \end{matrix} \right)$$

$$; \tau_1(b-a)^{h_1}, \dots, \tau_l(b-a)^{h_l}, -\frac{(b-a)f_1}{af_1 + g_1}, \dots, -\frac{(b-a)f_k}{af_k + g_k} \Bigg) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \prod_{j=1}^l \Gamma(\lambda_j) \prod_{j=1}^k \Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}} \int_{L_1} \dots \int_{L_{l+k}} \frac{\Gamma(\alpha + \sum_{j=1}^l h_j w_j + \sum_{j=1}^k w_{l+j})}{\Gamma(\alpha + \beta + \sum_{j=1}^l h_j w_j + \sum_{j=1}^k w_{l+j})} \prod_{j=1}^l \Gamma(\lambda_j + w_j) \prod_{j=1}^k \Gamma(-\sigma_j + w_{l+j})$$

$$\prod_{j=1}^{l+k} \Gamma(-w_j) z_1^{w_1} \dots z_l^{w_l} z_{l+1}^{w_{l+1}} \dots, w_{l+k}^{w_{l+k}} dw_1 \dots dw_{l+k} \quad (2.3)$$

Here the contour L'_j s are defined by $L_j = L_{w\zeta_j\infty}(\operatorname{Re}(\zeta_j) = v'_j)$ starting at the point $v'_j - \omega\infty$ and terminating at the point $v'_j + \omega\infty$ with $v'_j \in \mathbb{R}(j = 1, \dots, l)$ and each of the remaining contour L_{l+1}, \dots, L_{l+k} run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^l [1-\tau_j(t-a)^{h_j}]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r (|z| < 1) \quad (2.4)$$

integrating term by term with the help of the integral given by Saigo and Saxena [3, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [5, page 454].

3.Eulerian integral

In this section , we evaluate a general Eulerian integral with the product of two multivariable I-functions, class of multivariable polynomials and generalized hypergeometric function

$$B_u = \frac{(-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u)}{R_1! \dots R_u!} \tag{3.1}$$

$$\text{and } B_{u,v} = (b-a)^{\sum_{i=1}^v (a'_i + b'_i) \eta_{G_i, g_i} + \sum_{i=1}^u (a_i + b_i) R_i} \left\{ \prod_{j=1}^h (af_j + g_j)^{-\sum_{i=1}^v \lambda_i''' \eta_{g_i, h_i} - \sum_{i=1}^u \lambda_i'' R_i} \right\} \tag{3.2}$$

$$\theta_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j^{(i)}}, \zeta_j^{(i)} > 0 (i = 1, \dots, r); \theta'_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j'^{(i)}}, \zeta_j'^{(i)} > 0 (i = 1, \dots, s) \tag{3.3}$$

$$\theta''_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0 (i = 1, \dots, u) \tag{3.4}$$

$$\theta'''_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j'''^{(i)}}, \zeta_j'''^{(i)} > 0 (i = 1, \dots, v) \tag{3.5}$$

$$X = m_1, n_1; \dots; m_r, n_r; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0 \tag{3.6}$$

$$Y = p_1, q_1; \dots; p_r, q_r; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1 \tag{3.7}$$

$$A = (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0; A_j)_{1,p} \tag{3.8}$$

$$B = (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0; B_j)_{1,q} \tag{3.9}$$

$$C = (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r}; (1, 0; 1) \dots; (1, 0; 1) \\ (1, 0; 1); \dots; (1, 0; 1); (1, 0; 1); \dots; (1, 0; 1) \tag{3.10}$$

$$D = (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r}; (0, 1; 1); \dots; (0, 1; 1); \\ (0, 1; 1); \dots; (0, 1; 1); (0, 1; 1); \dots; (0, 1; 1) \tag{3.11}$$

$$K_1 = (1 - \alpha - \sum_{i=1}^u R_i a_i - \sum_{i=1}^v \eta_{G_i, g_i} a'_i; \mu_1, \dots, \mu_r, \mu'_1, \dots, \mu'_s, h_1, \dots, h_l, 1, \dots, 1; 1) \tag{3.12}$$

$$K_2 = (1 - \beta - \sum_{i=1}^u R_i b_i - \sum_{i=1}^v \eta_{G_i, g_i} b'_i; \rho_1, \dots, \rho_r, \rho'_1, \dots, \rho'_s, 0, \dots, 0, 0, \dots, 0; 1) \tag{3.13}$$

$$K_P = [1 - A_j; 0, \dots, 0, 1, \dots, 1, 0, \dots, 0, 0, \dots, 0; 1]_{1,P} \tag{3.14}$$

$$K_j = [1 - \lambda_j - \sum_{i=1}^u R_i \zeta_j''^{(i)} - \sum_{i=1}^v \eta_{G_i, g_i} \zeta_j'''^{(i)}; \zeta_j^{(1)}, \dots, \zeta_j^{(r)}, \zeta_j'^{(1)}, \dots, \zeta_j'^{(s)},$$

$$0, \dots, 1, \dots, 0, 0 \dots, 0; 1]_{1,l} \tag{3.14}$$

$$K'_j = [1 + \sigma_j - \sum_{i=1}^u R_i \lambda_j^{(i)} - \sum_{i=1}^v \eta_{G_i, g_i} \lambda_j^{(i)}; \lambda_j^{(1)}, \dots, \lambda_j^{(r)}, \lambda_j^{(1)} \dots, \lambda_j^{(s)},$$

$$0, \dots, 0, 0 \dots, 1, \dots, 0; 1]_{1,k} \tag{3.15}$$

$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^u R_i (a_i + b_i) - \sum_{i=1}^v (a'_i + b'_i) \eta_{G_i, g_i}; \mu_1 + \rho_1, \dots, \mu_r + \rho_r, \mu'_1 + \rho'_1, \dots, \mu'_r + \rho'_r,$$

$$h_1, \dots, h_l, 1, \dots, 1; 1) \tag{3.16}$$

$$L_Q = [1 - B_j; 0, \dots, 0, 1, \dots, 1, 0, \dots, 0, 0 \dots, 0; 1]_{1,Q} \tag{3.17}$$

$$L_j = [1 - \lambda_j - \sum_{i=1}^u R_i \zeta_j^{(i)} - \sum_{i=1}^s \zeta_j^{(i)} \eta_{G_i, g_i}; \zeta_j^{(1)}, \dots, \zeta_j^{(r)}, \zeta_j^{(1)} \dots, \zeta_j^{(s)}, 0, \dots, 0, 0 \dots, 0; 1]_{1,l} \tag{3.18}$$

$$L'_j = [1 + \sigma_j - \sum_{i=1}^u R_i \lambda_j^{(i)} - \sum_{i=1}^v \lambda_j^{(i)} \eta_{G_i, g_i}; \lambda_j^{(1)}, \dots, \lambda_j^{(r)}, \lambda_j^{(1)} \dots, \lambda_j^{(s)}, 0, \dots, 0, 0 \dots, 0; 1]_{1,k} \tag{3.19}$$

We have the general Eulerian integral

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^l [1 - \tau_j (t-a)^{h_j}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1, \dots, h_u} \left(\begin{matrix} z_1'' \theta_1'' (t-a)^{a_1} (b-t)^{b_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_u'' \theta_u'' (t-a)^{a_u} (b-t)^{b_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}} \end{matrix} \right)$$

$$\bar{I} \left(\begin{matrix} z_1''' \theta_1''' (t-a)^{a'_1} (b-t)^{b'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)'}} \\ \vdots \\ z_v''' \theta_v''' (t-a)^{a'_v} (b-t)^{b'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)'}} \end{matrix} \right)$$

$$I \left(\begin{matrix} z_1 \theta_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r \theta_r (t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{matrix} \right)$$

$a_j(j = 1, \dots, p), b_j(j = 1, \dots, q), c_j^{(i)}(j = 1, \dots, p_i, i = 1, \dots, r), d_j^{(i)}(j = 1, \dots, q_i, i = 1, \dots, r) \in \mathbb{C}$

The exponents $A_j(j = 1, \dots, p), B_j(j = 1, \dots, q), C_j^{(i)}(j = 1, \dots, p_i; i = 1, \dots, r), D_j^{(i)}(j = 1, \dots, q_i; i = 1, \dots, r)$ of various gamma function involved in (1.3) and (1.4) may take non integer values.

$$(C) \max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1$$

$$(D) \operatorname{Re} \left[\alpha + \sum_{i=1}^v a'_i \min_{1 \leq j \leq M_i} \frac{\bar{d}_j^{(i)}}{\bar{\delta}_j^{(i)}} + \sum_{i=1}^r \mu_i \min_{1 \leq j \leq m^{(i)}} \frac{d_j^{(i)}}{\delta_j^{(i)}} \right] > 0 \text{ and}$$

$$\operatorname{Re} \left[\beta + \sum_{i=1}^v b'_i \min_{1 \leq j \leq M_i} \frac{\bar{d}_j^{(i)}}{\bar{\delta}_j^{(i)}} + \sum_{i=1}^r \rho_i \min_{1 \leq j \leq m^{(i)}} \frac{d_j^{(i)}}{\delta_j^{(i)}} \right] > 0$$

$$(E) \operatorname{Re} \left(\alpha + \sum_{i=1}^v \eta_{G_i, g_i} a'_i + \sum_{i=1}^u R_i a_i + \sum_{i=1}^r \mu_i s_i + \sum_{i=1}^l h_i w_i \right) > 0; \operatorname{Re} \left(\beta + \sum_{i=1}^v \eta_{G_i, g_i} b'_i + \sum_{i=1}^u R_i b_i + \sum_{i=1}^r \rho_i s_i \right) > 0$$

$$\operatorname{Re} \left(\lambda_j + \sum_{i=1}^v \eta_{G_i, g_i} \lambda_j^{(i)} + \sum_{i=1}^u R_i \lambda_j^{(i)} + \sum_{i=1}^r s_i \zeta_j^{(i)} \right) > 0 (j = 1, \dots, l)$$

$$\operatorname{Re} \left(-\sigma_j + \sum_{i=1}^v \eta_{G_i, g_i} \lambda_j^{(i)} + \sum_{i=1}^u R_i \lambda_j^{(i)} + \sum_{i=1}^r s_i \lambda_j^{(i)} \right) > 0 (j = 1, \dots, k)$$

$$(F) U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leq 0, i = 1, \dots, r$$

$$(G) \Delta_k = - \sum_{j=n+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)}$$

$$-\mu_i - \rho_i - \sum_{l=1}^k \lambda_l^{(i)} > 0 \quad (i = 1, \dots, r)$$

$$(H) \left| \arg \left(z_i \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j^{(i)}} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{1}{2} \Delta_i \pi \quad (a \leq t \leq b; i = 1, \dots, r)$$

(I) $P \leq Q + 1$. The equality holds, when , in addition,

$$\text{either } P > Q \text{ and } \left| \left(z'_i \sum_{i=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j^{(i)}} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1 \quad (a \leq t \leq b)$$

$$\text{or } P \leq Q \text{ and } \max_{1 \leq i \leq k} \left[\left| \left(z'_i \sum_{i=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j^{(i)}} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| \right] < 1 \quad (a \leq t \leq b)$$

(J) The multiple series occurring on the right-hand side of (3.20) is absolutely and uniformly convergent.

Proof

To prove (3.20), first, we express in serie the multivariable I-function defined by Nambisan et al [2] with the help of (1.5), a class of multivariable polynomials defined by Srivastava et al [6] $S_L^{h_1, \dots, h_u} [.]$ in serie with the help of (1.14), the I-functions of r-variables defined by Nambisan et al [1] and in terms of Mellin-Barnes type contour integral with the help of (1.9), the generalized hypergeometric function ${}_pF_Q(\cdot)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of $[1 - \tau_j(t - a)^{h_i}]$ with $(i = 1, \dots, r; j = 1, \dots, l)$ and collect the power of $(f_j t + g_j)$ with $j = 1, \dots, k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r + s + k + l)$ dimensional Mellin-Barnes integral to multivariable I-function, we obtain the equation (3.20).

4. Particular cases

a) If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, the multivariable I-function defined by Nambisan et al [2] reduces to multivariable H-function defined by Srivastava et al [8].

The following generalized Eulerian integral concerning the multivariable H-function under the same notations and conditions that (3.20) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$:

$$\int_a^b (t - a)^{\alpha - 1} (b - t)^{\beta - 1} \prod_{j=1}^l [1 - \tau_j(t - a)^{h_i}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1, \dots, h_u} \left(\begin{matrix} z_1'' \theta_1'' (t - a)^{a_1} (b - t)^{b_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_u'' \theta_u'' (t - a)^{a_u} (b - t)^{b_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}} \end{matrix} \right)$$

$$\bar{I} \left(\begin{matrix} z_1''' \theta_1''' (t - a)^{a_1'} (b - t)^{b_1'} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)'}} \\ \vdots \\ z_v''' \theta_v''' (t - a)^{a_v'} (b - t)^{b_v'} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)'}} \end{matrix} \right)$$

$$H \left(\begin{matrix} z_1 \theta_1 (t - a)^{\mu_1} (b - t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r \theta_r (t - a)^{\mu_r} (b - t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{matrix} \right)$$

$${}_pF_Q \left[(A_P); (B_Q); - \sum_{i=1}^s z_i' \theta_i' (t - a)^{\mu_i'} (b - t)^{\rho_i'} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(i)'}} \right] dt = (b - a)^{\alpha + \beta - 1} \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{j=1}^k (a f_j + g_j)^{\sigma_j}$$

$$\sum_{h_1=1}^{M_1} \dots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \dots \sum_{k_v=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} \prod_{i=1}^v \frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i''' \eta_{h_i, k_i} \prod_{k=1}^u z_k'' R_k B_u B_{u,v} [\phi_1 (\eta_{h_1, k_1}, \dots, \eta_{h_r, k_r})]_{j \neq h_i}$$

$$\bar{I} \begin{pmatrix} z_1''' \theta_1''' (t-a)^{a'_1} (b-t)^{b'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_v''' \theta_v''' (t-a)^{a'_v} (b-t)^{b'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)}} \end{pmatrix}$$

$$I \begin{pmatrix} z_1 \theta_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r \theta_r (t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{pmatrix}$$

$${}_p F_Q \left[(A_P); (B_Q); - \sum_{i=1}^s z_i' \theta_i' (t-a)^{\mu_i'} (b-t)^{\rho_i'} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(i)}} \right] dt = (b-a)^{\alpha+\beta-1} \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{j=1}^k (af_j + g_j)^{\sigma_j}$$

$$\sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} \prod_{i=1}^v \frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i''' \eta_{h_i, k_i} \prod_{k=1}^u z''^{R_k} B_u' B_{u,v} [\phi_1 (\eta_{h_1, k_1}, \dots, \eta_{h_r, k_r})]_{j \neq h_i}$$

$$I_{p+P+l+k+2, q+Q+l+k+1; X}^{0, n+P+l+k+2; Y} \left(\begin{array}{c} \frac{z_1 (b-a)^{\mu_1 + \rho_1}}{\prod_{j=1}^k (af_j + g_j)^{\lambda_j^{(1)}}} \\ \dots \\ \frac{z_r (b-a)^{\mu_r + \rho_r}}{\prod_{j=1}^k (af_j + g_j)^{\lambda_j^{(r)}}} \\ \frac{z_1' (b-a)^{\mu_1' + \rho_1'}}{\prod_{j=1}^k (af_j + g_j)^{\lambda_j'^{(1)}}} \\ \dots \\ \frac{z_s' (b-a)^{\mu_s' + \rho_s'}}{\prod_{j=1}^k (af_j + g_j)^{\lambda_j'^{(s)}}} \\ \tau_1 (b-a)^{h_1} \\ \dots \\ \tau_l (b-a)^{h_l} \\ \frac{(b-a) f_1}{af_1 + g_1} \\ \dots \\ \frac{(b-a) f_k}{af_k + g_k} \end{array} \middle| \begin{array}{l} A \quad K_1, K_2, K_3, K_j, K_j' : C \\ \dots \\ B, L_1, L_2, L_j, L_j' : D \end{array} \right) \quad (4.3)$$

under the same notations and conditions that (3.20)

$$\text{where } B'_u = \frac{(-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u)}{R_1! \dots R_u!}, B[E; R_1, \dots, R_v] \text{ is defined by (4.2)}$$

Remark:

By the following similar procedure, the results of this document can be extended to product of any finite number of multivariable I-functions and a class of multivariable polynomials defined by Srivastava et al [6].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-function, defined by Nambisan et al [2], a class of multivariable polynomials and generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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