Selberg integral involving a extension of the Hurwitz-Lerch Zeta function , a class of polynomial and the multivariable I-functions

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ABSTRACT
In the present paper we evaluate the modified Selberg integral involving the product of two multivariable I-functions defined by Nambisan et al [3], a extension of the Hurwitz-Lerch Zeta function and a general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the I-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.We will study two particular cases.

Keywords: General class of polynomials, modified Selberg integral, a extension of the Hurwitz-Lerch Zeta function , multivariable I-function, multivariable H-function.

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1.Introduction

First time, we define the multivariable $\bar{I}$-function by : (see Nambisan et al [3])

$$\bar{I}(z_1,\cdots,z_r) = \bar{I}^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{n+1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{m+1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};1)_{1,n_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{n_1+1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};1)_{1,n_r},(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{n_r+1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};1)_{1,m_1},(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{m_1+1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{1,m_r},(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{m_r+1,q_r} \end{array}\right) \quad (1.1)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.2)$$

where $\phi_1(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi_1(s_1,\cdots,s_r) = \frac{1}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=m+1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \quad (1.3)$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \quad (1.4)$$

$i = 1, \cdots, r$

Serie representation

If $z_i \neq 0; i = 1, \cdots, r$

$\delta_{h_i}^{(i)}(d_j^{(i)} + k_i) \neq \delta_j^{(i)}(\delta_{h_i}^{(i)} + \eta_i) for j \neq h_i, j, h_i = 1, \cdots, m_i (i = 1, \cdots, r), k_i, \eta_i = 0, 1, 2, \cdots (i = 1, \cdots, r)$, then

$$\bar{I}(z_1, \cdots, z_r) = \sum_{h_1=1}^{m_1} \cdots \sum_{h_r=1}^{m_r} \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \left[\phi_1 \left(\frac{dh_1^{(1)} + k_1}{\delta h_1^{(1)}}, \cdots, \frac{dh_r^{(r)} + k_r}{\delta h_r^{(r)}} \right) \right]_{j \neq h_i} \prod_{i=1}^{r} \frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i^{\frac{dh_i + k_i}{\delta h_i}} \tag{1.5}$$

This result can be proved on computing the residues at the poles :

$$s_i = \frac{dh_i^{(i)} + k_i}{\delta h_i^{(i)}}, (h_i = 1, \cdots, m_i, k_i = 0, 1, 2, \cdots) for i = 1, \cdots, r \tag{1.6}$$

We may establish the the asymptotic expansion in the following convenient form :

$$\bar{I}(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \to 0$$

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \to \infty$$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will note $\eta_{h_i,k_i} = \dfrac{dh_i^{(i)} + k_i}{\delta h_i}$, $(h_i = 1, \cdots, m_i, k_i = 0, 1, 2, \cdots) for i = 1, \cdots, r$ (1.7)

The I-function is defined and represented in the following manner.

$$I(z'_1, \cdots, z'_s) = I_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s} \left(\begin{array}{c|c} \mathrm{z'}_1 & (\mathrm{a'}_j; \alpha'^{(1)}_j, \cdots, \alpha'^{(s)}_j; A'_j)_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z'}_s & (\mathrm{b'}_j; \beta'^{(1)}_j, \cdots, \beta'^{(s)}_j; B'_j)_{1,q'} : \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{c'}^{(1)}_j, \gamma'^{(1)}_j; C'^{(1)}_j)_{1,p'_1}; \cdots; (c'^{(s)}_j, \gamma'^{(s)}_j; C'^{(r)}_j)_{1,p'_r} \\ \\ (\mathrm{d'}^{(1)}_j, \delta'^{(1)}_j; D'^{(1)}_j)_{1,q'_1}; \cdots; (d'^{(s)}_j, \delta'^{(s)}_j; D'^{(s)}_j)_{1,q'_r} \end{array} \right) \tag{1.8}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \phi(t_1, \cdots, t_s) \prod_{i=1}^{s} \theta_i(t_i) z_i'^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.9}$$

where $\phi(t_1, \cdots, t_s), \theta_i(t_i), i = 1, \cdots, s$ are given by :

$$\phi(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'}\Gamma^{A'_j}\left(1-a'_j+\sum_{i=1}^{s}\alpha'^{(i)}_j t_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A'_j}\left(a'_j-\sum_{i=1}^{s}\alpha'^{(i)}_j t_j\right)\prod_{j=m'+1}^{q'}\Gamma^{B'_j}\left(1-b'_j+\sum_{i=1}^{s}\beta^{(i)}_j t_j\right)} \quad (1.10)$$

$$\theta_i(t_i)=\frac{\prod_{j=1}^{n'_i}\Gamma^{C'^{(i)}_j}\left(1-c'^{(i)}_j+\gamma'^{(i)}_j t_i\right)\prod_{j=1}^{m'_i}\Gamma^{D'^{(i)}_j}\left(d'^{(i)}_j-\delta'^{(i)}_j t_i\right)}{\prod_{j=n'_i+1}^{p'_i}\Gamma^{C'^{(i)}_j}\left(c'^{(i)}_j-\gamma'^{(i)}_j t_i\right)\prod_{j=m'_i+1}^{q'_i}\Gamma^{D'^{(i)}_j}\left(1-d'^{(i)}_j+\delta'^{(i)}_j t_i\right)} \quad (1.11)$$

For more details, see Nambisan et al [3].

Following the result of Braaksma [2] the I-function of r variables is analytic if

$$U_i=\sum_{j=1}^{p'}A'_j\alpha'^{(i)}_j-\sum_{j=1}^{q'}B'_j\beta'^{(i)}_j+\sum_{j=1}^{p'_i}C'^{(i)}_j\gamma'^{(i)}_j-\sum_{j=1}^{q'_i}D'^{(i)}_j\delta'^{(i)}_j\leqslant 0, i=1,\cdots,s \quad (1.12)$$

The integral (2.1) converges absolutely if

where$|arg(z'_k)|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,s$

$$\Delta_k=-\sum_{j=n'+1}^{p'}A'_j\alpha'^{(k)}_j-\sum_{j=1}^{q'}B'_j\beta^{(k)}_j+\sum_{j=1}^{m'_k}D'^{(k)}_j\delta'^{(k)}_j-\sum_{j=m'_k+1}^{q'_k}D'^{(k)}_j\delta'^{(k)}_j+\sum_{j=1}^{n'_k}C'^{(k)}_j\gamma'^{(k)}_j-\sum_{j=n'_k+1}^{p'_k}C'^{(k)}_j\gamma'^{(k)}_j>0 \quad (1.13)$$

We will note :

$$X=m'_1,n'_1;\cdots;m'_s,n'_s \quad (1.14)$$

$$Y=p'_1,q'_1;\cdots;p'_s,q'_s \quad (1.15)$$

$$A=(a'_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(s)}_j;A'_j)_{1,p'} \quad (1.16)$$

$$B=(b'_j;\beta'^{(1)}_j,\cdots,\beta'^{(s)}_j;B'_j)_{1,q'} \quad (1.17)$$

$$C=(c'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,\gamma'^{(s)}_j;C'^{(s)}_j)_{1,p'_s} \quad (1.18)$$

$$D=(d'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)_{1,q_s} \quad (1.19)$$

The contracted form is :

$$I(z_1,\cdots,z_s)=I^{0,n';X}_{p',q';Y}\left(\begin{array}{c|c} z'_1 & A:C \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & B:D \end{array}\right) \quad (1.20)$$

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_t}[z_1,\cdots,z_t]=\sum_{R_1,\cdots,R_t=0}^{h_1R_1+\cdots h_tR_t\leqslant L}(-L)_{h_1R_1+\cdots+h_tR_t}B(E;R_1,\cdots,R_t)\frac{z_1^{R_1}\cdots z_t^{R_t}}{R_1!\cdots R_t!} \tag{1.21}$$

the coefficients $B(E;R_1,\cdots,R_t)$ are arbitrary constants, real or complex.

2. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z,\mathfrak{s},a)$ is introduced by Srivastava et al ([7],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z;\mathfrak{s},a)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\times\frac{z^n}{n!} \tag{2.1}$$

with : $p,q\in\mathbb{N}_0,\lambda_j\in\mathbb{C}(j=1,\cdots,p),a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^*\ \ (j=1,\cdots,q),\rho_j,\sigma_k\in\mathbb{R}^+$

$(j=1,\cdots,p;k=1,\cdots,q)$

where $\Delta>-1$ when $\mathfrak{s},z\in\mathbb{C};\Delta=-1$ and $\mathfrak{s}\in\mathbb{C},when|z|<\bigtriangledown^*$, $\Delta=-1$ and $Re(\chi)>\frac{1}{2}$ when $|z|=\bigtriangledown^*$

$$\bigtriangledown^*=\prod_{j=1}^{p}\rho_j^{\rho_j}\prod_{j=1}^{q}\sigma_j^{\sigma_j};\Delta=\sum_{j=1}^{q}\sigma_j-\sum_{j=1}^{p}\rho_j;\chi=\mathfrak{s}+\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j+\frac{p-q}{2}$$

We denote these conditions the conditions (f).

3. Required integral

We note $S(a,b,c)$, the Selberg integral , see Askey et al ([1], page 402) by :

$$S(a,b,c)=\int_0^1\cdots\int_0^1\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$=\prod_{j=0}^{n-1}\frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{3.1}$$

with $Re(a)>0,Re(b)>0,Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$

We consider the new integral , see Askey et al ([1], page 402) defined by :

Lemme

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$=\prod_{i=1}^{k}\frac{(a+(n-i)c)}{(a+b+(2n-i-1)c)}\;S(a,b,c) \tag{3.2}$$

with $Re(a)>0, Re(b)>0, Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$ and $k\leqslant n$

where $S(a,b,c)$ is defined by (3.1). In this paper, we will denote the modified Selberg integral

4. Main integral

Let $X_{u,v,w}=\prod_{i=1}^{n}x_i^u(1-x_i)^v\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2w}$ $\quad b_{n'}=\frac{\prod_{j=1}^{p}(\lambda_j)_{n'\rho_j}}{(a+n')^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n'\sigma_j}}$ and

$$B_t=\frac{(-L)_{h_1R_1+\cdots+h_tR_t}B(E;R_1,\cdots,R_t)}{R_1!\cdots R_t!}$$

we have the following formula

Theorem

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}\left(zX_{\alpha,\beta,\gamma};\mathfrak{s},a\right)$$

$$S_L^{h_1,\cdots,h_t}\left(\begin{array}{c}\mathrm{y}_1X_{\alpha_1,\beta_1,\gamma_1}\\ \cdot\;\cdot\;\cdot\\ \mathrm{y}_tX_{\alpha_t,\beta_t,\gamma_t}\end{array}\right)I\left(\begin{array}{c}\mathrm{z}_1X_{\delta_1,\psi_1,\phi_1}\\ \cdot\;\cdot\;\cdot\\ \mathrm{z}_rX_{\delta_r,\psi_r,\phi_r}\end{array}\right)I\left(\begin{array}{c}\mathrm{Z}_1X_{\epsilon_1,\eta_1,\zeta_1}\\ \cdot\;\cdot\;\cdot\\ \mathrm{Z}_sX_{\epsilon_s,\eta_s,\zeta_s}\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$\sum_{R_1,\cdots,R_t=0}^{h_1R_1+\cdots h_tR_t\leqslant L}\;\sum_{n'=0}^{\infty}\sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\left[\phi_1\left(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r}\right)\right]_{j\neq h_i}B_tb_{n'}\prod_{i=1}^{r}\frac{(-)^{k_i}}{\delta h_i^{(i)}k_i!}z_i^{\eta_{h_i,k_i}}$$

$$\frac{y^{n'}}{n'!}\,y_1^{R_1}\cdots y_t^{R_t}\,I^{0,n'+3n+2k;X}_{p'+3n+2k,q'+n+2k+1;Y}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdot\;\cdot\;\cdot\\ \cdot\;\cdot\;\cdot\\ \mathrm{Z}_s\end{array}\left|\begin{array}{c}\mathrm{A}\;,\\ \\ \\ \mathrm{B},\end{array}\right.\right.$$

$$[1\text{-a-n'}\alpha-\sum_{i=1}^{t}R_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-j(c+\gamma'R+\sum_{i=1}^{t}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});\epsilon_1+j\zeta_1,\cdots,\epsilon_s+j\zeta_s;1]_{0,n-1}$$

$$\cdots$$

$$(\text{-c-n'}\gamma-\sum_{i=1}^{t}R_i\gamma_i-\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s;n),$$

$$[1\text{-b-n'}\beta-\sum_{i=1}^{t}R_i\beta_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\psi_i-j(c+\gamma'R+\sum_{i=1}^{t}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});\eta_1+j\zeta_1,\cdots,\eta_s+j\zeta_s;1]_{0,n-1}$$

$$\cdots$$

$$\text{B}_1,B_2,B_3:$$

$$\left.\begin{matrix}[\text{ -(j+1)(c+n'}\gamma+\sum_{i=1}^{t}\gamma_iR_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});(j+1)\zeta_1,\cdots,(j+1)\zeta_s;1)]_{0,n-1},A_2,A_3:C\\ \cdots\\ \text{D}\end{matrix}\right)\quad(4.1)$$

where $B_1=[1-a-b-(\alpha+\beta)n'-\sum_{i=1}^{t}R_i(\alpha_i+\beta_i)-\sum_{i=1}^{r}(\delta_i+\phi_i)\eta_{G_i,g_i}-(n-1+j)\times$

$$(c+n'\gamma+\sum_{i=1}^{t}R_i\gamma_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});\epsilon_1+\eta_1+j\zeta_1,\cdots,\epsilon_s+\eta_s+j\zeta_s;1]_{0,n-1}\quad(4.2)$$

$$A_2=[-a-n'\alpha-\sum_{i=1}^{t}R_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-(n-j)(c+m\gamma+\sum_{i=1}^{t}\gamma_iR_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s;1]_{1,k}\quad(4.3)$$

$$B_2=[1-a-n'\alpha-\sum_{i=1}^{t}R_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-(n-j)(c+m\gamma+\sum_{i=1}^{t}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s;1]_{1,k}\quad(4.4)$$

$$B_3=[-a-n'\alpha-\sum_{i=1}^{t}R_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-b-m\beta-\sum_{i=1}^{t}R_i\beta_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\psi_i$$

$$\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots-(2n-j-1)(c+m\gamma+\sum_{i=1}^{t}\gamma_iR_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_s+\eta_s+(2n-j-1)\zeta_s;1]_{1,k}\quad(4.5)$$

$$A_3=[1-a-n'\alpha-\sum_{i=1}^{t}R_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-b-m\beta-\sum_{i=1}^{t}R_i\beta_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\psi_i$$

$$\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots-(2n-j-1)(c+m\gamma+\sum_{i=1}^{t}\gamma_iR_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_s+\eta_s+(2n-j-1)\zeta_s;1]_{1,k} \quad \textbf{(4.6)}$$

A,B,C and D are defined respectively by (1.16), (1.17), (1.18) and (1.19)

Provided that

a) $min\{\alpha,\beta,\gamma,\alpha_i,\beta_i,\gamma_i,\delta_j,\psi_j,\phi_j,\epsilon_l,\eta_l,\zeta_l\}>0, i=1,\cdots,t, j=1,\cdots,r, l=1,\cdots,s$,

b) $A=Re[a+n'\alpha+\sum_{i=1}^{r}\delta_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\epsilon_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}]>0$

c) $B=Re[b+n'\beta+\sum_{i=1}^{r}\psi_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\eta_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}]>0$

d) $C=Re[c+n'\gamma+\sum_{i=1}^{r}\phi_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\zeta_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}]>Max\left\{-\frac{1}{n},-\frac{A}{n-1},-\frac{B}{n-1}\right\}$

e) $|arg(Z_k)|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,s$

f) The conditions (f) are satisfied

g) The series occuring on the right-hand side of (3.1) are absolutely and uniformly convergent.

Proof

first, expressing the extension of the Hurwitz-Lerch Zeta function in serie with the help of equation (2.1), $\bar{I}$-function of r-variables in series with the help of equation (1.5), the general class of polynomial of several variables $S_L^{h_1,\cdots,h_t}[.]$ with the help of equation (1.21) and the I-function of s variables defined by Nambisan et al [2] in Mellin-Barnes contour integral with the help of equation (1.9), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now evaluating the resulting modified Selberg integral with the help of equation (3.2). Use the following relations $\Gamma(a)(a)_n=\Gamma(a+n)$ and $a=\frac{\Gamma(a+1)}{\Gamma(a)}$ several times with $Re(a)>0$. Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

5. Particular cases

1) If $A'_j=B'_j=C_j'^{(i)}=D_j'^{(i)}=1$, The multivariable I-functions defined by Nambisan reduces to multivariable H-function defined by Srivastava et al [6]. We have.

Corollary 1

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}(zX_{\alpha,\beta,\gamma};\mathfrak{s},a)$$

$$S_L^{h_1,\cdots,h_t}\left(\begin{array}{c} \mathrm{y}_1 X_{\alpha_1,\beta_1,\gamma_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_t X_{\alpha_t,\beta_t,\gamma_t} \end{array}\right) \bar{I}\left(\begin{array}{c} \mathrm{z}_1 X_{\delta_1,\psi_1,\phi_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r X_{\delta_r,\psi_r,\phi_r} \end{array}\right) H\left(\begin{array}{c} \mathrm{Z}_1 X_{\epsilon_1,\eta_1,\zeta_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s X_{\epsilon_s,\eta_s,\zeta_s} \end{array}\right) \mathrm{d}x_1\cdots\mathrm{d}x_n =$$

$$\sum_{R_1,\cdots,R_t=0}^{h_1R_1+\cdots h_tR_t\leqslant L} \sum_{n'=0}^{\infty}\sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}[\phi_1(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r})]_{j\neq h_i} B_t b_{n'}\prod_{i=1}^{r}\frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i^{\eta_{h_i,k_i}}$$

$$\frac{y^{n'}}{n'!} y_1^{R_1}\cdots y_t^{R_t} H^{0,n'+3n+2k;X}_{p'+3n+2k,q'+2n+2k;W}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s \end{array}\middle| \begin{array}{c} \mathrm{A}\ , \\ \\ \\ \mathrm{B}, \end{array}\right.$$

$$[\text{1-a-n'}\alpha - \sum_{i=1}^t R_i\alpha_i - \sum_{i=1}^r \eta_{G_i,g_i}\delta_i - j(c+\gamma' R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i\eta_{G_i,g_i}); \epsilon_1 + j\zeta_1,\cdots,\epsilon_s + j\zeta_s]_{0,n-1}$$
$$\cdot\ \cdot\ \cdot$$
$$(\text{-c-n'}\gamma - \sum_{i=1}^t R_i\gamma_i - \sum_{i=1}^r \phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s),\cdots,(-c-n'\gamma - \sum_{i=1}^t R_i\gamma_i - \sum_{i=1}^r \phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s)$$

$$[\text{1-b-n'}\beta - \sum_{i=1}^t R_i\beta_i - \sum_{i=1}^r \eta_{G_i,g_i}\psi_i - j(c+\gamma' R + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i\eta_{G_i,g_i}); \eta_1 + j\zeta_1,\cdots,\eta_s + j\zeta_s]_{0,n-1}$$
$$\cdot\ \cdot\ \cdot$$
$$\mathrm{B}_1, B_2, B_3 :$$

$$\left.\begin{array}{c} [\text{ -(j+1)(c+n'}\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i\eta_{G_i,g_i}); (j+1)\zeta_1,\cdots,(j+1)\zeta_s;1)]_{0,n-1}, A_2, A_3 : C \\ \cdot\ \cdot\ \cdot \\ \mathrm{D} \end{array}\right) \quad (5.1)$$

under the same notations and conditions that (4.1) with $A'_j = B'_j = C'^{(i)}_j = D'^{(i)}_j = 1$

2) If $B(L;R_1,\cdots,R_t) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_t\theta_j^{(t)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(t)}}(b_j^{(t)})_{R_t\phi_j^{(t)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi'_j+\cdots+m_t\psi_j^{(t)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(t)}}(d_j^{(t)})_{R_t\delta_j^{(t)}}}$ **(5.2)**

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_t}[z_1,\cdots,z_t]$ reduces to generalized Lauricella function defined by Srivastava et al [4].

$$F^{1+\bar{A}:B';\cdots;B^{(t)}}_{\bar{C}:D';\cdots;D^{(t)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_t \end{array}\middle| \begin{array}{c} [(\text{-L});\mathrm{R}_1,\cdots,R_t][(a);\theta',\cdots,\theta^{(t)}]:[(b');\phi'];\cdots;[(b^{(t)});\phi^{(t)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(t)}]:[(d');\delta'];\cdots;[(d^{(t)});\delta^{(t)}] \end{array}\right) \quad \textbf{(5.3)}$$

We have the following formula

Corollary 2

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} x_i \prod_{i=1}^{n} x_i^{a-1}(1-x_i)^{b-1} \prod_{1\leqslant j<k\leqslant n} |x_j - x_k|^{2c} \phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}(zX_{\alpha,\beta,\gamma};\mathfrak{s},a)$$

$$F^{1+\bar{A}:B';\cdots;B^{(t)}}_{\bar{C}:D';\cdots;D^{(t)}}\left(\begin{array}{c} \mathrm{y}_1 X_{\alpha_1,\beta_1,\gamma_1} \\ \cdots \\ \mathrm{y}_t X_{\alpha_t,\beta_t,\gamma_t} \end{array}\middle| \begin{array}{c} [(\text{-L});\mathrm{R}_1,\cdots,R_t][(a);\theta',\cdots,\theta^{(t)}]:[(b');\phi'];\cdots;[(b^{(t)});\phi^{(t)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(t)}]:[(d');\delta'];\cdots;[(d^{(t)});\delta^{(t)}] \end{array}\right)$$

$$\bar{I}\left(\begin{array}{c} \mathrm{z}_1 X_{\delta_1,\psi_1,\phi_1} \\ \cdots \\ \mathrm{z}_r X_{\delta_r,\psi_r,\phi_r} \end{array}\right) I\left(\begin{array}{c} \mathrm{Z}_1 X_{\epsilon_1,\eta_1,\zeta_1} \\ \cdots \\ \mathrm{Z}_s X_{\epsilon_s,\eta_s,\zeta_s} \end{array}\right) \mathrm{d}x_1\cdots\mathrm{d}x_n = \sum_{R_1,\cdots,R_t=0}^{h_1R_1+\cdots h_tR_t\leqslant L}$$

$$\sum_{n'=0}^{\infty}\sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\left[\phi_1\left(\eta_{h_1,k_1},\cdots,\eta_{h_r,k_r}\right)\right]_{j\neq h_i} B'_t b_{n'} \prod_{i=1}^{r}\frac{(-)^{k_i}}{\delta h_i^{(i)} k_i!} z_i^{\eta_{h_i,k_i}}$$

$$\frac{y^{n'}}{n'!}\, y_1^{R_1}\cdots y_t^{R_t}\, I^{0,n'+3n+2k;X}_{p'+3n+2k,q'+n+2k+1;Y}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdots \\ \cdots \\ \mathrm{Z}_s \end{array}\middle| \begin{array}{c} \mathrm{A}\ , \\ \\ \\ \mathrm{B}, \end{array}\right.$$

$$[\text{1-a-n}'\alpha - \sum_{i=1}^t R_i\alpha_i - \sum_{i=1}^r \eta_{G_i,g_i}\delta_i - j(c+\gamma'R+\sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i\eta_{G_i,g_i});\epsilon_1 + j\zeta_1,\cdots,\epsilon_s + j\zeta_s;1]_{0,n-1}$$
$$\cdots$$
$$(\text{-c-n}'\gamma - \sum_{i=1}^t R_i\gamma_i - \sum_{i=1}^r \phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s;n),$$

$$[\text{1-b-n}'\beta - \sum_{i=1}^t R_i\beta_i - \sum_{i=1}^r \eta_{G_i,g_i}\psi_i - j(c+\gamma'R+\sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i\eta_{G_i,g_i});\eta_1 + j\zeta_1,\cdots,\eta_s + j\zeta_s;1]_{0,n-1}$$
$$\cdots$$
$$\mathrm{B}_1, B_2, B_3:$$

$$\left.[\ \text{-(j+1)(c+n}'\gamma + \sum_{i=1}^t \gamma_i R_i + \sum_{i=1}^r \phi_i\eta_{G_i,g_i});(j+1)\zeta_1,\cdots,(j+1)\zeta_s;1)]_{0,n-1}, A_2, A_3 : C \atop \cdots \atop \mathrm{D}\right) \quad (5.4)$$

under the same conditions and notations that (4.1)

and $B_t' = \dfrac{(-L)_{h_1R_1+\cdots+h_tR_t}B(E;R_1,\cdots,R_t)}{R_1!\cdots R_t!}$; $B(L;R_1,\cdots,R_t)$ is defined by (5.2)

6. Conclusion

In this paper we have evaluated a modified Selberg integral involving the product of the multivariable $\bar{I}$-function defined by nambisan et al [3], the multivariable I-function defined by Nambisan et al [3], a extension of the Hurwitz-Lerch Zeta function and a general class of polynomials of several variables. The integral established in this paper is of very general nature as it contains multivariable I-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

REFERENCES

[1] Andrew G.G and Askey R. Special function. Cambridge. University. Press 1999.

[2] B. L. J. Braaksma, “Asymptotic expansions and analytic continuations for a class of Barnes integrals,”Compositio Mathematical, vol. 15, pp. 239–341, 1964.

[3] Prathima J. Nambisan V. and Kurumujji S.K. A Study of I-function of Several Complex Variables, International Journalof Engineering Mathematics Vol(2014) , 2014 page 1-12

[4] Srivastava H.M. and Daoust M.C. Certain generalized Neumann expansions associated with Kampé de Fériet function. Nederl. Akad. Wetensch. Proc. Ser A72 = Indag Math 31(1969) page 449-457.

[5] Srivastava H.M. And Garg M. Some integral involving a general class of polynomials and multivariable H-function. Rev. Roumaine Phys. 32(1987), page 685-692.

[6] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

[7] H.M. Srivastava, R.K. Saxena, T.K. Pogány, R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch zeta function, Integr.Transf. Spec. Funct. 22 (2011) 487–506

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