Doubt cubic H-ideals of BG-algebra

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Abstract: In this article we introduce the notion of Doubt cubic H-ideals of BG-algebra and discuss some of their properties.

Key words: BG-algebra, Doubt fuzzy BG-subalgebra, Doubt fuzzy H- ideal of BG algebra, Cubic Set, Doubt Cubic H- ideals of BG-algebra

1 Introduction

the study of BCK- algebra and BCI algebra was initiated by Imai and Iseki [2] in 1966. B-algebra was introduced by Neggers and Kim [8], which is related to BCI/BCK- algebra in many aspects. Kim and Kim[7] generalised B-algebra as BG-algebra and this algebra was fuzzyfied by Ahn and Lee[1]. Khalid and Ahmad [6] introduced fuzzy H-ideals in BCI-algebra in 1999. In 1994, Jun [5] introduced the concept of doubt fuzzy ideals in BCK/BCI- algebras. The notion of doubt fuzzy H-ideals in BCK-algebra was introduced by Zhan and Tan [10]. The concept of interval valued fuzzy sets, an extension of fuzzy sets was due to Zadeh [9] and based upon it, Jun [3] developed the notion of cubic sets. In this approach, doubt cubic H-ideal of BG-algebra is defined and some of its properties, investigated.

2 Preliminaries

Definition 2.1. A BG-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following:
(i) $x * x = 0$ (ii) $x * 0 = x$. (iii) $(x * y) * (0 * y) = x \quad \forall x, y \in X$
In his case we say $(X, *, 0)$ is a BG-algebra and by X now onwards we shall mean a BG-algebra. We can define a partial ordering '$\leq$' by $x \leq y$ if and only if $x * y = 0$.

Example 2.2. The set $X = \{0, 1, 2, 3\}$ with the caley table

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

is a BG-algebra.

Definition 2.3. A non empty subset S of a BG-algebra X is called a sub-algebra of X if $x * y \in S, \forall x, y \in S$.

Definition 2.4. A non empty subset I of a BG-algebra X is called a BG-ideal or an ideal of X if

(i) $0 \in I$ and (ii) $x * y \in I, y \in I \Rightarrow x \in I$.

Definition 2.5. An ideal I of a BG-algebra X is said to be closed if,

$0 * x \in I, \forall x \in I.$

Definition 2.6. A non empty subset I of a BG-algebra X is called a H-ideal of X if

(i) $0 \in I$ and (ii) $x * (y * z) \in I, y \in I \Rightarrow x * z \in I.$

Definition 2.7. The fuzzy set A in X is defined as $A = \{(x, \mu_A(x))|x \in X\}$, where $\mu_A : X \to [0, 1]$ is known as the membership value of x in A. For brevity by $\mu_A(x)$ we mean the fuzzy set A in X.

Definition 2.8. The fuzzy set μ_A in X is said to be a fuzzy sub-algebra of X if

$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}, \forall x, y \in X.$

Definition 2.9. The fuzzy set μ_A in X is said to be a fuzzy ideal of X if

(i) $\mu_A(0) \geq \mu_A(x)$ and (ii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}, \forall x, y \in X.$

Definition 2.10. The fuzzy set μ_A in X is said to be a doubt fuzzy sub-algebra (DF sub-algebra, for brevity) of X if

$\mu_A(x * y) \leq \max\{\mu_A(x), \mu_A(y)\}, \forall x, y \in X.$

Definition 2.11. The fuzzy set μ_A in X is said to be a doubt fuzzy ideal (DF ideal, for brevity) of X if

(i) $\mu_A(0) \leq \mu_A(x)$ and (ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}, \forall x, y \in X.$

Definition 2.12. The fuzzy set μ_A in X is said to be a fuzzy H-ideal of X if

(i) $\mu_A(0) \geq \mu_A(x)$ and (ii) $\mu_A(x*y) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}, \forall x, y, z \in X.$

Definition 2.13. The fuzzy set μ_A in X is said to be a doubt fuzzy H-ideal(DF H-ideal, for brevity) of X if

(i) $\mu_A(0) \leq \mu_A(x)$ and (ii) $\mu_A(x*y) \leq \max\{\mu_A(x*(y*z)), \mu_A(y)\}, \forall x, y, z \in X.$

By an interval number we mean a closed subinterval given by $\tilde{a} = [a^-, a^+]$ of the interval $[0, 1]$, where $0 \leq a^- \leq a^+ \leq 1$. Let us denote the set of all interval numbers by $D[0, 1]$. Let us consider $\tilde{a}_1 = [a_1^-, a_1^+]$ and $\tilde{a}_2 = [a_2^-, a_2^+]$. Then refined minimum ($r\min$) and refined maximum ($r\max$) of $\tilde{a}_1$ and $\tilde{a}_2$ are defined as

$r\min\{\tilde{a}_1, \tilde{a}_2\} = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}]$

$r\max\{\tilde{a}_1, \tilde{a}_2\} = [\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}]$

For $a_i \in D[0, 1]; i = 1, 2, 3, ...,$ we define

$r\inf \tilde{a}_1 = [r\inf a_i^-, r\inf a_i^+]$ and $r\sup \tilde{a}_1 = [r\sup a_i^-, r\sup a_i^+]$

We also define the symbols $\succeq, \preceq$ and $=$ as follows:

$\tilde{a}_1 \succeq \tilde{a}_2 \Leftrightarrow a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$

Also $\tilde{a}_1 \succ \tilde{a}_2$ means $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$.

Similar;y we can define $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \prec \tilde{a}_2$.

Finally $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^-, a_1^+ = a_2^+$.

An interval valued fuzzy set (IVF set)A defined in X is given by

$A = \{(x, [\mu_A^-(x), \mu_A^+(x)])|x \in X$, where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x), \forall x \in X$. An IVF set A is briefly denoted by $\tilde{\mu} = [\mu_A^-, \mu_A^+]$. If in the IVF set $\tilde{\mu}(x) = [\mu_A^-(x), \mu_A^+(x)], \mu_A^-(x) = c = \mu_A^+(x)$, where $0 < c \leq 1$, then $\tilde{\mu}(x) = [c, c]$, which is for our convenience is assumed to be a member of $D[0,1]$. So $\tilde{\mu}(x) \in D[0,1], \forall x \in X$, where $\tilde{\mu} : X \to D[0,1]$. $\tilde{\mu}(x)$ is called the degree of the membership of the element x to $\tilde{\mu}$ and $\mu_A^-(x), \mu_A^+(x)$ are respectively called lower and upper degrees of membership of x to $\tilde{\mu}$. By complement of $\tilde{\mu}$ we mean $[1 - \mu_A^-, 1 - \mu_A^+]$, denoted by $(\tilde{\mu})^c$.

Definition 2.14. A cubic set A in a non empty X is a structure of the form $A = \{x, \tilde{\mu}_A(x), \nu_A(x) | x \in X\}$, where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and ν_A a fuzzy set in X. It is briefly denoted by $A =< \tilde{\mu}_A(x), \nu_A(x) >=< [\mu_A^-(x), \mu_A^+(x)], \nu_A(x) >$. .

For two cubic sets A and B in X, their intersection denoted by $A \sqcap B$ is another cubic set in X given by $A \sqcap B =< \tilde{\mu}_A \tilde{\cap} \tilde{\mu_B}, \nu_A \cup \nu_B >$, where $(\tilde{\mu}_A \tilde{\cap} \tilde{\mu_B})(x) = r\min\{\tilde{\mu}_A(x)\}, \tilde{\mu}_B(x)$ and $(\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$
Similarly the union of A and B denoted by $A \sqcup B$ is another cubic set in X given by $A \sqcup B =< \tilde{\mu}_A \tilde{\cup} \tilde{\mu_B}, \nu_A \cap \nu_B >$, where $(\tilde{\mu}_A \tilde{\cup} \tilde{\mu_B})(x) = r\max\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$.

3 Doubt Cubic H-ideals of BG-algebra

Definition 3.1. A cubic set $A =< \tilde{\mu}_A, \nu_A >$ of X is said to be a doubt cubic sub algebra of X if for all $x, y \in X$,
(i) $\tilde{\mu}_A(x * y) \preceq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ (ii) $\nu_A(x * y) \geq min\{\nu_A(x), \nu_A(y)\}$

Example 3.2. Consider the BG-algebra $X = \{0, 1, 2, 3\}$ with the caley table

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

We define $\tilde{\mu}_A(0) = [0.3, 0.4], \tilde{\mu}_A(1) = [0.4, 0.6], \tilde{\mu}_A(2) = [0.3, 0.5], \tilde{\mu}_A(3) = [0.5, 0.9]$ and $\nu_A(0) = 0.6, \nu_A(1) = 0.5, \nu_A(2) = 0.3, \nu_A(3) = 0.4$. Then $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic sub algebra.

Theorem 3.3. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic sub algebra of a BG-algebra X. Then (i) $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x)$ and $\nu_A(0) \geq \nu_A(x)$, for all $x \in X$.*

Proof. We have for any $x \in X, \tilde{\mu}_A(0) = \tilde{\mu}_A(x * x) \preceq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and $\nu_A(0) = \nu_A(x * x) \geq \min\{\nu_A(x), \nu_A(x)\} = x$.

Theorem 3.4. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic sub algebra of a BG-algebra X. Then for all $x \in X$*
(i) $\tilde{\mu}_A(x^n * x) \preceq \tilde{\mu}_A(x)$ and $\nu_A(x^n * x) \geq \nu_A(x)$, if n is odd.
(ii) $\tilde{\mu}_A(x^n * x) = \tilde{\mu}_A(x)$ and $\nu_A(x^n * x) = \nu_A(x)$, if n is even.
(iii) $\tilde{\mu}_A(x * x^n) = \tilde{\mu}_A(0)$ and $\nu_A(x * x^n) = \nu_A(0)$, for all $n \in \mathbb{N}$.

Proof. (i) Clearly $\tilde{\mu}_A(x * x) = \tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x)$, so that the result is true for $n = 1$. Let the result be true for $n = 2p - 1, p \in \mathbb{N}$.
Then $\tilde{\mu}_A(x^{2p-1} * x) \preceq \tilde{\mu}_A(x)$.
Now $\tilde{\mu}_A(x^{2(p+1)-1} * x) = \tilde{\mu}_A(x^{2p-1+2} * x) = \tilde{\mu}_A(x^{2p-1} * (x * (x * x)))$
$= \tilde{\mu}_A(x^{2p-1} * (x * 0)) = \tilde{\mu}_A(x^{2p-1} * x) \preceq \tilde{\mu}_A(x)$

So the result is true for $n = 2(p + 1) - 1$, whenever it is true for $n = 2p - 1$. Hence by induction the result is true for all odd numbers.

The second part follows similarly. The proofs of (ii) and (iii) are similar to the preceding one.

4 Doubt cubic H-ideals of BG-algebra

Definition 4.1. Let $A =< \tilde{\mu}_A, \nu_A >$ be a cubic set in a BG-algebra X. Then A is called a doubt cubic H-ideal of X if for all $x, y, z \in X$,
(i) $\tilde{\mu}_A(x) \succeq \tilde{\mu}_A(0)$
(ii) $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$
(iii) $\nu_A(x) \leq \nu_A(0)$
(iv) $\nu_A(x * z) \geq \min\{\nu_A(x * (y * z)), \nu_A(y)\}$

Remark 4.2. Given $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of a BG-algebra X, $\tilde{\mu}_A$ is a DF-ideal and ν_A is a fuzzy ideal of X.

Example 4.3. Consider the BG-algebra $X = \{0, 1, 2, 3\}$ with the caley table

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

We define $\tilde{\mu}_A(0) = [0.2, 0.4], \tilde{\mu}_A(2) = \tilde{\mu}_A(3) = [0.4, 0.8], \tilde{\mu}_A(1) = [0.5, 0.9]$ and $\nu_A(0) = 0.9, \nu_A(1) = 0.3, \nu_A(2) = \nu_A(3) = 0.7$. Then by routine calculation it can be shown that $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of X.

Definition 4.4. A cubic set $A =< \tilde{\mu}_A, \nu_A >$ in X is called a closed doubt cubic H-ideal of X if for all $x, y, z \in X$,
(i) $\tilde{\mu}_A(0 * x) \preceq \tilde{\mu}_A(x)$ (ii) $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$
(iii) $\nu_A(0 * x) \leq \nu_A(x)$ (iv) $\nu_A(x * z) \geq \min\{\nu_A(x * (y * z)), \nu_A(y)\}$

Theorem 4.5. *Every closed doubt cubic H-ideal of X is a doubt cubic H-ideal of X.*

Proof. Let $A =< \tilde{\mu}_A, \nu_A >$ be a closed doubt cubic H-ideal of X. Then $\tilde{\mu}_A(0 * x) \preceq \tilde{\mu}_A(x)$ and $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$ considering the first two conditions. In the second condition putting $z = 0$, we get $\forall x, y \in X$
$\tilde{\mu}_A(x * 0) = \tilde{\mu}_A(x) \preceq r\max\{\tilde{\mu}_A(x * (y * 0)), \tilde{\mu}_A(y)\} = r\max\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$.
Replacing x by 0,
$\tilde{\mu}_A(0) = \tilde{\mu}_A(x) \preceq r\max\{\tilde{\mu}_A(0 * y), \tilde{\mu}_A(y)\} \preceq r\max\{\tilde{\mu}_A(y), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$.
Using the remaining two conditions for ν_A it can be easily shown that
$\nu_A(0) \geq \nu_A(x)$. Hence the proof.

Remark 4.6. Every doubt cubic H-ideal of X is not necessarily a closed doubt cubic H-ideal of X. This prompts us to assert that "The class of closed doubt cubic H-ideal of X is a proper subclass of that of the doubt cubic H-ideals of x.

Theorem 4.7. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic H-ideal of X. Let $\{x_n\}$ be a sequence in X. Then*
(i) If $\lim_{n\to\infty} \tilde{\mu}_A(x_n) = [0, 0]$ then $\tilde{\mu}_A(0) = [0, 0]$
(ii) If $\lim_{n\to\infty} \nu_A(x_n) = 1$ then $\nu_A(0) = 1$.

Proof. Since $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x), \forall x \in X$, we have $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x_n), \forall n \in \mathbb{N}$. Clearly $[0, 0] \preceq \tilde{\mu}_A(0) \preceq \lim_{n\to\infty} \tilde{\mu}_A(x_n) = [0, 0]$. So $\tilde{\mu}_A(0) = [0, 0]$.
Similarly since $\nu_A(0) \geq \nu_A(x), \forall x \in X$, we have $1 \geq \nu_A(0) \geq \lim_{n\to\infty} \nu_A(x_n) = 1$, so that $\nu_A(0) = 1$.

Theorem 4.8. *For the doubt cubic H-ideal $A =< \tilde{\mu}_A, \nu_A >$ of X, $\tilde{\mu}_A$ is order preserving and ν_A is order reversing.*

Proof. Let $x \leq y$ in X, where $x \leq y$ means $x * y = 0$.
Then $\tilde{\mu}_A(x * 0) \preceq r\max\{\tilde{\mu}_A(x * (y * 0)), \tilde{\mu}_A(y)\}$
$\Rightarrow \tilde{\mu}_A(x) \preceq r\max\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\} = r\max\{\tilde{\mu}_A(0), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$.
Also $\nu_A(x * 0) \geq \min\{\nu_A(x * (y * 0)), \nu_A(y)\}$
$\Rightarrow \nu_A(x) \geq \min\{\nu_A(x * y), \nu_A(y)\} = \min\{\nu_A(), \nu_A(y)\} = \nu_A(y)$.

Theorem 4.9. *The union of any two doubt cubic H-ideals of X, is again a doubt cubic H-ideal of X.*

Proof. Let $A =< \tilde{\mu}_A, \nu_A >$ and $B =< \tilde{\mu}_B, \nu_B >$ be two doubt cubic H-ideals of X. Then $A \sqcup B =< \tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B, \nu_A \cap \nu_B >$. We have
$(\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(0) = r\max\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} \preceq r\max\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\} = (\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x)$

And $(\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x * z) = r\max\{\tilde{\mu}_A(x * z), \tilde{\mu}_B(x * z)\}$
$\preceq r\max\{r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}, r\max\{\tilde{\mu}_B(x * (y * z)), \tilde{\mu}_B(y)\}\}$
$\preceq r\max\{r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_B(x * (y * z))\}, r\max\{\tilde{\mu}_A(y), \tilde{\mu}_B(y)\}\}$
$= r\max\{(\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x * (y * z)), (\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(y)\}$
Again $(\nu_A \cap \nu_B)(0) = \min\{\nu_A(0), \nu_B(0)\} \geq \min\{\nu_A(x), \nu_B(x)\} = (\nu_A \cap \nu_B)(x)$.
Finally $(\nu_A \cap \nu_B)(x * z) = \min\{\nu_A(x * z), \nu_A(x * z)\}$
$\geq \min\{\min\{\nu_A(x * (y * z)), \nu_A(y)\}, \min\{\nu_B(x * (y * z)), \nu_B(y)\}\}$
$= \min\{\min\{\nu_A(x * (y * z)), \nu_B(x * (y * z))\}, \min\{\nu_A(y), \nu_B(y)\}\}$
$= \min\{(\nu_A \cap \nu_B)(x * (y * z)), (\nu_A \cap \nu_B)(x)\}$.
Hence $A \sqcup B$ is a doubt cubic H-ideal of X.

Theorem 4.10. *(Generalisation) The union of any family of doubt cubic H-ideals of X, is again a doubt cubic H-ideal of X.*

Remark 4.11. The intersection of any two doubt cubic H-ideals of X will be a doubt cubic H-ideal of X if one is contained in the other.

Theorem 4.12. *(Generalisation) Let $A =< \tilde{\mu}_A, \nu_A >$ and $B =< \tilde{\mu}_B, \nu_B >$ be two doubt cubic H-ideals of X. Then $A \sqcap B$ is also doubt cubic H-ideal of X if $A \subseteq B$ or $B \subseteq A$.*

Proof. We have $(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(0) = r\min\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} \preceq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\} = \tilde{\mu}_A(x) = (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(x)[as\ \tilde{\mu}_A \preceq \tilde{\mu}_B]$
And $(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(x * z) = r\min\{\tilde{\mu}_A(x * z), \tilde{\mu}_B(x * z)\}$
$\preceq r\min\{r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}, r\max\{\tilde{\mu}_B(x * (y * z)), \tilde{\mu}_B(y)\}\}$
$\preceq r\min\{r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_B(x * (y * z))\}, r\max\{\tilde{\mu}_A(y), \tilde{\mu}_B(y)\}\}$
$= r\max\{r\min\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_B(x * (y * z))\}, r\min\{\tilde{\mu}_A(y), \tilde{\mu}_B(y)\}\}$
$= r\max\{(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(x * (y * z)), (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(y)\}$
Similarly it can be shown that $(\nu_A \cup \nu_B)(0) \geq (\nu_A \cup \nu_B)(x)$.
and $(\nu_A \cup \nu_B)(x * z) \geq \min\{(\nu_A \cup \nu_B)(x * (y * z)), (\nu_A \cup \nu_B)(x)\}$.
Hence $A \sqcap B$ is a doubt cubic H-ideal of X.

Theorem 4.13. *The union of any two closed doubt cubic H-ideals of X, is again a closed doubt cubic H-ideal of X.*

Proof. Let $A =< \tilde{\mu}_A, \nu_A >$ and $B =< \tilde{\mu}_B, \nu_B >$ be two closed doubt cubic H-ideals of X. Then $A \sqcup B =< \tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B, \nu_A \cap \nu_B >$. We have
$(\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(0 * x) = r\max\{\tilde{\mu}_A(0 * x), \tilde{\mu}_B(0 * x)\}$

$\preceq r\max\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\} = (\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x)$
And $(\nu_A \cap \nu_B)(0 * x) = \min\{\nu_A(0 * x), \nu_B(0 * x)\}$
$\geq \min\{\nu_A(x), \nu_B(x)\} = (\nu_A \cap \nu_B)(x)$.
The remaining part for $\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B$ and $\nu_A \cap \nu_B$ follows exactly as before. Hence $A \sqcup B$ is a closed doubt cubic H-ideal of X.

Theorem 4.14. *(Generalisation) The union of any family of closed doubt cubic H-ideals of X, is again a closed doubt cubic H-ideal of X.*

Remark 4.15. The intersection of any two closed doubt cubic H-ideals of X will be a closed doubt cubic H-ideal of X if one is contained in the other.

Theorem 4.16. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic H-ideal of X. Then the sets $X_{\tilde{\mu}_A} = \{x \in X | \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$ and $X_{\nu_A} = \{x \in X | \nu_A(x) = \nu_A(0)\}$ are H-ideals of X.*

Proof. Clearly $0 \in X_{\tilde{\mu}_A}$. Next for $x*(y*z), y \in X_{\tilde{\mu}_A}$, we have $\tilde{\mu}_A(x*(y*z)) = \tilde{\mu}_A(0) = \tilde{\mu}_A(y)$. And $\forall x, z \in X, \tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x * z)$.
Again $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\} = r\max\{\tilde{\mu}_A(0), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(0)$.
So $\tilde{\mu}_A(x * z) = \tilde{\mu}_A(0)$ and hence $x * z \in X_{\tilde{\mu}_A}$.
Thus for $x * (y * z), y \in X_{\tilde{\mu}_A}$, we get $x * z \in X_{\tilde{\mu}_A}$. So $X_{\tilde{\mu}_A}$ is a H-ideal of X. Similarly it can be shown that X_{ν_A} is also a H-ideal of X.

Theorem 4.17. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a cubic set in X. Then A is a doubt cubic H-ideal of X if and only if $\tilde{\mu}_A^c = [1 - \mu_A^+, 1 - \mu_A^-]$ and ν_A are fuzzy H-ideals of X.*

Proof. Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic H-ideal of X. Then it follows immediately that ν_A is a fuzzy ideal of X. Now
$\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x) \Rightarrow [\mu_A^-(0), \mu_A^+(0)] \preceq [\mu_A^-(x), \mu_A^+(x)]$
$\Rightarrow \mu_A^-(0) \leq \mu_A^-(x); \mu_A^+(0) \leq \mu_A^+(x)$
$\Rightarrow 1 - \mu_A^-(0) \geq 1 - \mu_A^-(x); 1 - \mu_A^+(0) \geq 1 - \mu_A^+(x)$
$\Rightarrow [1 - \mu_A^+(0), 1 - \mu_A^-(0)] \succeq [1 - \mu_A^+(x), 1 - \mu_A^-(x)] \Rightarrow \tilde{\mu}_A^c(0) \succeq \tilde{\mu}_A^c(x)$
Secondly $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$
$\Rightarrow \mu_A^-(x * z) \leq \max\{\mu_A^-(x * (y * z)), \mu_A^-(y)\}$
$\Rightarrow 1 - \mu_A^-(x * z) \geq 1 - \max\{\mu_A^-(x * (y * z)), \mu_A^-(y)\}$
$= \min\{1 - \mu_A^-(x * (y * z)), 1 - \mu_A^-(y)\}$
Similarly $1 - \mu_A^+(x * z) \geq \min\{1 - \mu_A^+(x * (y * z)), 1 - \mu_A^+(y)\}$
So $\tilde{\mu}_A^c(x * z) \succeq r\min\{\tilde{\mu}_A^c(x * (y * z)), \tilde{\mu}_A^c(y)\}$
Thus $\tilde{\mu}_A^c$ is a fuzzy H-ideal of X.
Conversely let $\tilde{\mu}_A^c$ and ν_A be fuzzy H-ideals of X. Then
$\tilde{\mu}_A^c(0) \succeq \tilde{\mu}_A^c(x)$
$\Rightarrow [1 - \mu_A^+(0), 1 - \mu_A^-(0)] \succeq [1 - \mu_A^+(x), 1 - \mu_A^-(x)]$

$\Rightarrow [\mu_A^-(0), \mu_A^+(0)] \preceq [\mu_A^-(x), \mu_A^+(x)]$
$\Rightarrow \tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x)$
And $\tilde{\mu}_A^c(x * z) \succeq r\min\{\tilde{\mu}_A^c(x * (y * z)), \tilde{\mu}_A^c(y)\}$
$\Rightarrow 1 - \mu_A^+(x * z) \geq \min\{1 - \mu_A^+(x * (y * z)), 1 - \mu_A^+(y)\}$
$= 1 - \max\{\mu_A^+(x * (y * z)), \mu_A^+(y)\}$
$\Rightarrow \tilde{\mu}_A^+(x * z) \leq r\max\{\tilde{\mu}_A^+(x * (y * z)), \tilde{\mu}_A^+(y)\}$
Likewise $\tilde{\mu}_A^-(x * z) \leq r\max\{\tilde{\mu}_A^-(x * (y * z)), \tilde{\mu}_A^-(y)\}$
Hence $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$
Secondly ν_A being a fuzzy H-ideal of X,
$\nu_A(0) \geq \nu_A(x)$ and $\nu_A(x * z) \geq \min\{\nu_A(x * (y * z)), \nu_A(y)\}$
Hence $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of X.

Definition 4.18. Let $A =< \tilde{\mu}_A, \nu_A >$ be a cubic set in X. Let $\tilde{q} = [q^-, q^+]$ be an interval number and $r \in [0, 1]$. Then the set $L_{\tilde{q}} = \{x \in X | \tilde{\mu}_A(x) \preceq \tilde{q}\}$ and $U_r\{x \in X | \nu_A(x) \geq r\}$ are respectively called lower $\tilde{q}$-level set of A and upper r-level cut of A. The cubic level set of A is the set given by $(U, L) = \{x \in X | \tilde{\mu}_A(x) \succeq \tilde{q}, \nu_A(x) \geq r\}$.

Theorem 4.19. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic H-ideal of X. Then $L_{\tilde{q}}$ and U_r are H-ideals of X.*

Proof. We have $L_{\tilde{q}} = \{x \in X | \tilde{\mu}_A(x) \preceq \tilde{q}\}$.
Clearly $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x * z), \forall x, z \in X$. So $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x * z) \preceq \tilde{q}$, whenever $x * z \in L_{\tilde{q}}$ i.e., $0 \in L_{\tilde{q}}$.
Secondly for $x * (y * z), y \in L_{\tilde{q}}$ we have $\tilde{\mu}_A(x * (y * z)) \preceq \tilde{q}$ and $\tilde{\mu}_A(y) \preceq \tilde{q}$. So $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y))\} = r\max\{\tilde{q}, \tilde{q}\} = \tilde{q}$ i.e., $x * z \in L_{\tilde{q}}$.
Therefore for $x * (y * z) \in L_{\tilde{q}}, x * z \in L_{\tilde{q}}$. Thus $L_{\tilde{q}}$ is a H-ideal of X.
Next we have $U_r\{x \in X | \nu_A(x) \geq r\}$.
Since $\nu_A(0) \geq \nu_A(x * z) \forall x, z \in X$, so $\nu_A(0) \geq \nu_A(x * z) \geq r$, whenever $x * z \in U_r$ i.e., $0 \in U_r$.
Secondly for $x * (y * z), y \in U_r$ we have $\nu_A(x * (y * z)) \geq r$ and $\nu_A(y) \geq r$. So $\nu_A(x * z) \geq \min\{\nu_A(x * (y * z)), \nu_A(y))\} = \min\{r, r\} = r$ i.e., $x * z \in U_r$.
Therefore for $x * (y * z) \in U_r, x * z \in U_r$. Thus U_r is a H-ideal of X.

Theorem 4.20. *Let $A =< \tilde{\mu}_A, \nu_A >$ be a cubic set in X such that the lower $\tilde{q}$-level set $L_{\tilde{q}}$ and upper r-level cut U_r of A are H-ideals of X. Then A is a doubt cubic H-ideal of X.*

Proof. Let us assume that A is not a doubt cubic H-ideal of X. Then we will find that $L_{\tilde{q}}$ and U_r are not H-ideals of X. By assumption there exist $x, y, z \in X$ such that $\tilde{\mu}_A(x * z) \succ r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y))\}$. Then there exists an interval

number $\tilde{q}$ (say) such that
$\tilde{\mu}_A(x*z) \succ \tilde{q} \succ r\max\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y))\}$.
Thus $x*(y*z), y \in L_{\tilde{q}}$. But $\tilde{\mu}_A(x*z) \succ \tilde{q}$ i.e., $x*z \notin L_{\tilde{q}}$.
So $L_{\tilde{q}}$ is not a H-ideals of X.
Secondly by our assumption there exist $x, y, z \in X$ such that $\nu_A(x*z) < \min\{\nu_A(x*(y*z)), \nu_A(y))\}$. Then there exists a member $r \in [0,1]$ such that $\nu_A(x*z) < r < \min\{\nu_A(C), \nu_A(y))\}$. Thus $x*(y*z), y \in U_r$. But $\nu_A(x*z) < r$ i.e., $x*z \notin U_r$.
So U_r is not a H-ideals of X. Hence the assertion.

Theorem 4.21. *Any H-ideal of X can be realised as a $\tilde{\mu}$-level doubt fuzzy H-ideal and ν-level doubt fuzzy H-ideal for some doubt cubic H-ideal of X.*

Proof. Let I be a H-ideal of X and $A =< \tilde{\mu}_A, \nu_A >$ a cubic set in X defined as

$$\tilde{\mu}_A(x) = \begin{cases} \tilde{u}, & x \in I \\ \tilde{v}, & otherwise \end{cases}$$

$$\nu_A(x) = \begin{cases} w, & x \in I \\ r, & otherwise \end{cases}$$

Where $\tilde{u}, \tilde{v} \in D[0,1]$, $\tilde{0} = [0,0] \preceq \tilde{u} \preceq \tilde{v}$ and $w, r \in [0,1], w \geq r$. By hypothesis $x*(y*z), y \in I, \forall x, y, z \in X$. Then $x*z \in I$.
Now $\tilde{\mu}_A(x*z) = \tilde{u} = r\max\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}$ and
$\nu_A(x*z) = w = \min\{\nu_A(x*(y*z)), \nu_A(y))\}$
If at least one $x*(y*z)$ and y is not in I, then at least one of $\tilde{\mu}_A(x*(y*z))$ and $\tilde{\mu}_A(y)$ is equal to $\tilde{v}$ and at least one of $\nu_A(x*(y*z))$ and $\nu_A(y))$ is equal to r so that
$\tilde{\mu}_A(x*z) \preceq \tilde{v} = r\max\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}$ and
$\nu_A(x*z) \geq w = \min\{\nu_A(x*(y*z)), \nu_A(y))\}$.
Since $0 \in I$, by definition of $\tilde{\mu}_A$ and ν_A, we have $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x), \forall x \in X$ and $\nu_(0) \geq \nu_A(x), \forall x \in X$. Therefore $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of X.

Theorem 4.22. *A cubic set $A =< \tilde{\mu}_A, \nu_A >$ of X is a doubt cubic H-ideal of X if and only if $\tilde{\mu}_A$ is a doubt fuzzy H-ideal and $\bar{\nu_A} = 1 - \nu_A$ is a doubt fuzzy H-ideal of X.*

Proof. Let $A =< \tilde{\mu}_A, \nu_A >$ be a doubt cubic H-ideal of X. Then it immediately follows that $\tilde{\mu}_A$ s a doubt fuzzy H-ideal of X.
Secondly $\nu_A(0) \geq \nu_A(x) \Rightarrow 1 - \nu_A(0) \leq 1 - \nu_A(x) \Rightarrow \bar{\nu}_A(0) \leq \bar{\nu}_A(x), \forall x \in X$
And $\nu_A(x*z) \geq \min\{\nu_A(x*(y*z))), \nu_A(y)\}$
$\Rightarrow 1 - \nu_A(x*z) \leq 1 - \min\{\nu_A(x*(y*z))), \nu_A(y)\}$
$\Rightarrow \bar{\nu}_A(x*z) \leq \max\{1 - \nu_A(x*(y*z))), 1 - \nu_A(y)\} = \max\{\bar{\nu}_A(x*(y*z))), \bar{\nu}_A(y)\}$.
So $\bar{\nu}_A$ is a doubt fuzzy H-ideal of X.

Conversely let $\tilde{\mu}_A$ be a doubt fuzzy H-ideal of X and ν_A be a doubt fuzzy H-ideal of X. Since $\tilde{\mu}_A$ be a doubt fuzzy H-ideal of X, so $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x)$ and $\tilde{\mu}_A(x * z) \preceq r\max\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\} \forall x, y, z \in X$.
Secondly, since ν_A be a doubt fuzzy H-ideal of X, so
$\bar{\nu}_A(0) \leq \bar{\nu}_A(x) \Rightarrow 1 - \nu_A(0) \leq 1 - \nu_A(x) \Rightarrow \nu_A(0) \geq \nu_A(x), \forall x \in X$
And $\bar{\nu}_A(x * z) \leq \max\{\bar{\nu}_A(x * (y * z))), \bar{\nu}_A(y)\}$
$\Rightarrow 1 - \nu_A(x * z) \leq \max\{1 - \nu_A(x * (y * z))), 1 - \nu_A(y)\}$
$= 1 - \min\{\nu_A(x * (y * z)), \nu_A(y)\}$ i.e., $\nu_A(x * z) \geq \min\{\nu_A(x * (y * z)), \nu_A(y)\}$
So $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of X.

Theorem 4.23. *A cubic set $A =< \tilde{\mu}_A, \nu_A >$ of X is a closed doubt cubic H-ideal of X if and only if $\tilde{\mu}_A$ is a closed doubt fuzzy H-ideal and $\bar{\nu_A} = 1 - \nu_A$ is a closed doubt fuzzy H-ideal of X.*

Proof. Similar to the proof of the preceding theorem.

5 Homomorphism of doubt cubic H-ideals of BG-algebra

Let X and Y be two BG-algebra and $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ denote the family of cubic sets in X and Y respectively. A mapping $f : X \to Y$ induces two mappings $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ given by $A \mapsto \mathcal{C}_f(A)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ given by $B \mapsto \mathcal{C}_f^{-1}(B)$, where $\mathcal{C}_f(A)$ and $\mathcal{C}_f^{-1}(B)$ are given by,

$$\mathcal{C}_f(\tilde{\mu}_A)(y) \begin{cases} = r\inf\{\tilde{\mu}_A(x) | y = f(x)\}, & f^{-1} \neq \phi \\ = [1, 1], & otherwise \end{cases}$$

$$\mathcal{C}_f(\nu_A)(y) = \begin{cases} \sup\{\nu_A(x) | y = f(x)\}, & f^{-1} \neq \phi \\ 0, & otherwise \end{cases}$$

for all $y \in Y$.
And $\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x)), \mathcal{C}_f^{-1}(\nu_B)(x) = \mu_B(f(x))$ for all $x \in X$.
The mappings $\mathcal{C}_f$ and $\mathcal{C}_f^{-1}$ are respectively known as cubic and inverse cubic transformations induced by $f : X \to Y$. The cubic set $A =< \tilde{\mu}_A, \nu_A >$ in X is said to have cubic property if for any subset B of X there exists $x_0 \in B$ such that $\tilde{\mu}_A(x_0) = r\inf\{\tilde{\mu}_A(x) | x \in B\}$ and $\nu_A(x_0) = \sup\{\nu_A(x) | x \in B\}$

Theorem 5.1. *For a transformation $f : X \to Y$, where X and Y are BG-algebra, let $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation respectively induced by f.*

(i) If $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of X, then $\mathcal{C}_f(A)$ is a doubt cubic H-ideal of Y.
(ii) If $A =< \tilde{\mu}_A, \nu_A >$ is a doubt cubic H-ideal of Y then $\mathcal{C}_f^{-1}(A)$ is a doubt cubic H-ideal of X.

Proof. Let $f(x) \in f(X)$. Then there exists $x_0 \in f^{-1}(f(x))$ such that
$\tilde{\mu}_A(x_0) = r\inf\{\tilde{\mu}_A(a)|a \in f^{-1}(f(x))\}$ and $\nu_A(x_0) = \sup\{\nu_A(a)|a \in f^{-1}(f(x))\}$.
Now $\mathcal{C}_f(\tilde{\mu}_A)(f(0)) = r\inf\{\tilde{\mu}_A(z)|z \in f^{-1}(f(0))\} = \tilde{\mu}_A(0_0)$,
where $0_0 \in f^{-1}(f(0))$. In particular when $0_0 = 0$, $\tilde{\mu}_A(0) \preceq \tilde{\mu}_A(x_0)$
$\Rightarrow \mathcal{C}_f(\tilde{\mu}_A)(f(0)) \preceq r\inf\{\tilde{\mu}_A(a)|a \in f^{-1}(f(x))\} = \mathcal{C}_f(\tilde{\mu}_A)(f(x))$
And $\mathcal{C}_f(\nu_A)(f(0)) = \sup\{\nu_A(z)|z \in f^{-1}(f(0))\}$, where $0_0 \in f^{-1}(f(0))$. In particular when $0_0 = 0$,$\nu_A(0) \geq \nu_A(x_0)$
$\Rightarrow \mathcal{C}_f(\nu_A)(f(0)) \geq \sup\{\nu_A(a)|a \in f^{-1}(f(x))\} = \mathcal{C}_f(\nu_A)(f(x))$
Again $\mathcal{C}_f(\tilde{\mu}_A)(f(x) * f(z)) = \mathcal{C}_f(\tilde{\mu}_A)(f(x * z))$
$= r\inf\{\tilde{\mu}_A(x' * z')|x' * z' \in f^{-1}(f(x * z))\} = \tilde{\mu}_A(x_0 * z_0)$
$\preceq r\max\{\tilde{\mu}_A(x_0 * (y_0 * z_0)), \tilde{\mu}_A(y_0)\}$
$= r\max\{r\inf\{\tilde{\mu}_A(a * (b * c))|a * (b * c) \in f^{-1}(f(x * (y * z)))\}, r\inf\{\tilde{\mu}_A(d)|d \in f^{-1}(f(y))\}\} = r\max\{\mathcal{C}_f(\tilde{\mu}_A)(f(x * (y * z)), \mathcal{C}_f(\tilde{\mu}_A)(f(y))\}$
$= r\max\{\mathcal{C}_f(\tilde{\mu}_A)(f(x) * f(y * z)), \mathcal{C}_f(\tilde{\mu}_A)(f(y))\}$
$= r\max\{\mathcal{C}_f(\tilde{\mu}_A)(f(x) * (f(y) * f(z))), \mathcal{C}_f(\tilde{\mu}_A)(f(y))\}$
And $\mathcal{C}_f(\nu_A)(f(x) * f(z)) = \mathcal{C}_f(\nu_A)(f(x * z))$
$= \sup\{\nu_A(x' * z')|x' * z' \in f^{-1}(f(x * z))\} = \nu_A(x_0 * z_0)$
$\geq \min\{\nu_A(x_0 * (y_0 * z_0)), \nu_A(y_0)\}$
$= \min\{\sup\{\nu_A(a*(b*c))|a*(b*c) \in f^{-1}(f(x*(y*z)))\}, \sup\{\nu_A(d)|d \in f^{-1}(f(y))\}\}$
$= \min\{\mathcal{C}_f(\nu_A)(f(x * (y * z))), \mathcal{C}_f(\nu_A)(f(y))\}$
$= \min\{\mathcal{C}_f(\nu_A)(f(x) * f(y * z))), \mathcal{C}_f(\nu_A)(f(y))\}$
$= \min\{\mathcal{C}_f(\nu_A)(f(x) * (f(y) * f(z))), \mathcal{C}_f(\nu_A)(f(y))\}$
Therefore $\mathcal{C}_f(A)$ is a doubt cubic H-ideal of Y.
(ii) Let $x \in X$ and $f(x) \in Y$. Now
$\mathcal{C}_f^{-1}(\tilde{\mu}_A)(0) = \tilde{\mu}_A(f(0)) \preceq \tilde{\mu}_A(f(x)) = \mathcal{C}_f^{-1}(\tilde{\mu}_A)(x)$ and
$\mathcal{C}_f^{-1}(\nu_A)(0) = \nu_A(f(0)) \geq \nu_A(f(x)) = \mathcal{C}_f^{-1}(\nu_A)(x)$
Again $\mathcal{C}_f^{-1}(\nu_A)(x * z) = \tilde{\mu}_A(f(x * z)) = \tilde{\mu}_A(f(x) * f(z))$
$\preceq r\max\{\tilde{\mu}_A(f(x) * (f(y) * f(z))), \tilde{\mu}_A(f(y))\}$
$= r\max\{\tilde{\mu}_A(f(x) * (f(y * z))), \tilde{\mu}_A(f(y))\}$
$= r\max\{\tilde{\mu}_A(f(x * (y * z)), \tilde{\mu}_A(f(y))\}$
$= r\max\{\mathcal{C}_f^{-1}(\tilde{\mu})(x * (y * z)), \mathcal{C}_f^{-1}(\tilde{\mu}_A)(y)\}$.
And $\mathcal{C}_f^{-1}(\nu_A)(x * z) = \nu_A(f(x * z)) = \nu_A(f(x) * f(z))$
$\geq \min\{\nu_A(f(x) * (f(y) * f(z))), \nu_A(f(y))\} = \min\{\nu_A(f(x) * (f(y * z)), \nu_A(f(y))\}$
$= \min\{\nu_A(f(x * (y * z)), \nu_A(f(y))\} = \min\{\mathcal{C}_f^{-1}(\nu_A)(x * (y * z)), \mathcal{C}_f^{-1}(\nu_A)(y)\}$.
So $\mathcal{C}_f^{-1}(A)$ is a doubt cubic H-ideal of X.

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