Face Integer Cordial Labeling of Graphs

M. Mohamed Sheriff[#1], A. Farhana Abbas[*2] and P. Lawrence Rozario Raj[**3]

[#] *P.G. and Research Department of Mathematics, Hajee Karutha Rowther Howdia College, Uthamapalayam - 625 533, Tamil Nadu, India.*

[*] *Research Scholar, School of Mathematics, Madurai Kamaraj University, Madurai - 625 021, Tamil Nadu, India.*

[**] *P.G. and Research Department of Mathematics, St. Joseph's College, Tiruchirappalli - 620 002, Tamil Nadu, India.*

Abstract - *In this paper, we have introduced and investigated the face integer cordial labeling of wheel W_n, fan f_n, triangular snake T_n, double triangular snake DT_n, star of cycle C_n and $DS(B_{n,n})$.*

Keywords - *Integer cordial labeling, face integer cordial labeling, face integer cordial graph.*

I. Introduction

We begin with simple, finite, planar, undirected graph. A (p,q) planar graph G means a graph G=(V,E), where V is the set of vertices with |V| = p, E is the set of edges with |E| = q and F is the set of interior faces of G with |F| = number of interior faces of G. For standard terminology and notations related to graph theory we refer to Harary [3]. A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u)-f(v)|$. Then $v_f(i)$ = number of vertices having label i under f and $e_f(i)$ = number of edges having label i under f*. A binary vertex labeling f of a graph G is called a cordial labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling. In [1], Cahit introduced the concept of cordial labeling of graph.

A product cordial labeling of a graph G with vertex set V is a function f from V to {0,1} such that if each edge uv is assigned a label f(u)f(v) then (i) the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and (ii) the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph. The concept of product cordial labeling of a graph was introduced by Sundaram et al. [8].

For graph G, the edge labeling function is defined as $f : E(G) \to \{0,1\}$ and induced vertex labeling function $f^* : V(G) \to \{0,1\}$ is given as if $e_1,e_2,\ldots,e_n$ are the edges incident to vertex v then $f^*(v) = f(e_1)f(e_2)\ldots f(e_n)$. Let us denote $v_f(i)$ is the number of vertices of G having label i under f^* and $e_f(i)$ is the number of edges of G having label i under f for i = 0,1. f is called edge product cordial labeling of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph G is called edge product cordial if it admits edge product cordial labeling. In [9], Vaidya et al. introduced the concept of edge product cordial labeling of graph.

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ka. It is denoted by a|b. If a does not divide b, then we denote $a \nmid b$. Let G = (V(G), E(G)) be a simple graph and $f : V(G) \to \{1,2,...,|V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if f(u)|f(v) or f(v)|f(u) and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph. Varatharajan et al. [10] introduced the concept of divisor cordial labeling of graphs.

For a planar graph G, the vertex labeling function is defined as $g : V(G) \to \{0,1\}$ and g(v) is called the label of the vertex v of G under g, induced edge labeling function $g^* : E(G) \to \{0,1\}$ is given as if e = uv then $g^*(e) = g(u)g(v)$ and induced face labeling function $g^{**} : F(G) \to \{0,1\}$ is given as if $v_1,v_2,\ldots,v_n$ and $e_1,e_2,\ldots,e_m$ are the vertices and edges of face f, then $g^{**}(f) = g(v_1)g(v_2)\ldots g(v_n)g^*(e_1)g^*(e_2)\ldots g^*(e_m)$. $v_g(i)$ is the number of vertices of G having label i under g, $e_g(i)$ is the number of edges of G having label i under g^* and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for i = 0,1. g is called face product cordial labeling of graph G if $|v_g(0)-v_g(1)| \leq 1$, $|e_g(0)-e_g(1)| \leq 1$ and $|f_g(0)-f_g(1)| \leq 1$. A graph G is face product cordial if it admits face product cordial labeling. Lawrence et al. introduced the concept of face product cordial labeling of graphs in [5] and they proved fan, $M(P_n)$, $S'(P_n)$ except for odd n, $T(P_n)$, T_n, H_n, S_n except for even n and one vertex union of mC_n and C_{mn} are face product cordial graph.

For a planar graph G, the edge labeling function is defined as $g : E(G) \to \{0,1\}$ and g(e) is called the label of the edge e of G under g, induced vertex labeling function $g^* : V(G) \to \{0,1\}$ is given as if $e_1,e_2,\ldots,e_m$ are the edges incident to vertex v, then

$g^*(v) = g(e_1)g(e_2)\ldots g(e_m)$ and induced face labeling function $g^{**}:F(G)\rightarrow \{0,1\}$ is given as if $v_1,v_2,\ldots,v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f then $g^{**}(f) = g^*(v_1)g^*(v_2)\ldots g^*(v_n)\ g(e_1)g(e_2)\ldots g(e_m)$. $v_g(i)$ is the number of vertices of G having label i under g^*, $e_g(i)$ is the number of edges of G having label i under g^* and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for i = 0,1. g is called face edge product cordial labeling of graph G if $|v_g(0)-v_g(1)|\le 1$, $|e_g(0)-e_g(1)|\le 1$ and $|f_g(0)-f_g(1)|\le 1$. A graph G is face edge product cordial if it admits face edge product cordial labeling. The concept of face edge product cordial labeling was introduced by Lawrence et al. in [4] and they proved the face edge product cordial labeling of T_n for even n, $M(P_n)$ for odd n, the star of cycle C_n for odd n, the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length and the path union of k copies of cycle C_n except for odd k and even n, and the total face edge product cordial labeling of f_n, W_n and the star of cycle C_n and the face product cordial labeling of the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length and the path union of k copies of cycle C_n except for odd k and even n.

Let G be a simple connected graph with p vertices. Let $f:V\rightarrow \left[-\frac{p}{2},\ldots,\frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd be an injective map, which induces an edge labeling f^* such that f(uv) = 1, if $f(u)+f(v) \ge 0$ and f(uv) = 0 otherwise. Let $e_f(i)$ = number of edges labeled with i, where i = 0 or 1. f is said to be integer cordial if $|e_f(0)-e_f(1)| \le 1$. A graph G is called integer cordial if it admits an integer cordial labeling. Here $[-x,\ldots,x] = \{t / t$ is an integer and $|t| \le x\}$ and $[-x,\ldots,x]^* = [-x,\ldots,x]-\{0\}$.

In [7], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as cycle C_n, Path P_n, Wheel graph W_n; n > 3, Star graph $K_{1,n}$, Helm graph H_n, Closed helm graph CH_n are integer cordial, K_n is not integer cordial, $K_{n,n}$ is integer cordial iff n is even and $K_{n,n}\backslash M$ is integer cordial for any n, where M is a perfect matching of $K_{n,n}$.

Motivated by the concept of face product cordial labeling, face edge product cordial labeling and integer cordial labeling, we introduce two new types of labeling such as face integer cordial labeling and face integer edge cordial labeling of graph. For a planar graph G, the vertex labeling function is defined as $g : V \rightarrow \left[-\frac{p}{2},\ldots,\frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd be an injective map, which induces an edge labeling function $g^* : E(G)\rightarrow\{0,1\}$ such that $g^*(uv) = 1$, if $g(u)+g(v) \ge 0$ and $g^*(uv) = 0$ otherwise and face labeling function $g^{**}:F(G)\rightarrow\{0,1\}$ such that $g^{**}(f) = 1$, if $g^{**}(f) = g(v_1)+g(v_2)+\ldots+ g(v_n) \ge 0$ and $g^{**}(f) = 0$ otherwise, where $v_1, v_2, \ldots, v_n$ are the vertices of face f. g is called face integer cordial labeling of graph G if $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0)-f_g(1)| \le 1$. $e_g(i)$ is the number of edges of G having label i under g^* and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for i = 1,2. A planar graph G is face integer cordial if it admits face integer cordial labeling.

For a planar graph G, an edge labeling function is defined as $g : E\rightarrow\left[-\frac{p}{2},\ldots,\frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd be an injective map, which induces vertex labeling function $g^* : V(G) \rightarrow \{0,1\}$ such that $g^*(v) = 1$, if $\sum_i g(e_i) \ge 0$ and $g^*(v) = 0$ otherwise, where $e_1, e_2, \ldots, e_n$ are the adjacent edges of the vertex v and face labeling function $g^{**} : F(G) \rightarrow \{0,1\}$ such that $g^{**}(f) = 1$, if $g^{**}(f) = g(e_1)+g(e_2)+\ldots+g(e_n) \ge 0$ and $g^{**}(f) = 0$ otherwise, where $e_1, e_2, \ldots, e_n$ are the edges of face f. g is called face integer edge cordial labeling of graph G if $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$. $v_g(i)$ is the number of vertices of G having label i under g^* and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for i = 1,2. A planar graph G is face integer cordial if it admits face integer edge cordial labeling.

In [6], Mohamed Sheriff et al proved wheel graph, fan graph, friendship graph, triangular snake, alternative triangular snake, star of cycle, degree splitting graph of bistar, vertex switching of cycle, pendent vertex switching of path, helm, closed helm, middle graph of path and total graph of path are face integer edge cordial graph.

The present work is focused only on face integer cordial labeling of some new families of graphs. The face integer cordial labeling of wheel W_n, fan f_n, triangular snake T_n, double triangular snake DT_n, star of cycle C_n and $DS(B_{n,n})$ is presented. The brief summaries of definition which are necessary for the present investigation are provided below.

Definition : 1.1

A wheel W_n is a graph with n+1 vertices, formed by connecting a single vertex to all the vertices of cycle C_n. It is denoted by $W_n = C_n + K_1$.

Definition : 1.2

A triangular snake T_n is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $i = 1,2,\ldots,n-1$.

Definition : 1.3

The friendship graph F_n is one-point union of n copies of cycles C_3.

Definition : 1.4

The join of two graphs G and H is a graph G + H with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G)\cup E(H) \cup \{ uv : u\in V(G)$ and $v\in V(H) \}$. The graph $P_n + K_1$ is called a fan of n vertices and is denoted by f_n.

Definition : 1.5

Let G be a graph with two or more vertices than the total graph T(G) of graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition : 1.6

Let G be a graph with vertex set $V = S_1 \cup S_2 \cup \ldots \cup S_i \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of S_i for $1 \le i \le t$.

Remark : 1.1

Any unicyclic integer cordial graphs are face integer cordial graphs.

Remark : 1.2

Every planar graph G is always a subgraph of the face integer cordial graph $G \cup G$.

II. MAIN THEOREMS

Theorem : 2.1

The wheel W_n is a face integer cordial graph for $n \ge 3$.

Proof.

Let v be the apex vertex, $v_1, v_2, \ldots, v_n$ be rim vertices, $e_1, e_2, \ldots, e_{2n}$ be edges and $f_1, f_2, \ldots, f_n$ be interior faces of the wheel W_n, where $e_i = vv_i$, for $i = 1,2,\ldots,n$, $e_{n+i} = v_iv_{i+1}$, for $i = 1,2,\ldots,n-1$, $e_{2n} = v_nv_1$, $f_i = vv_iv_{i+1}v$, for $i = 1,2,\ldots,n-1$ and $f_n = vv_nv_1v$.

Let G be the wheel graph W_n.

Then $|V(G)| = n+1$, $|E(G)| = 2n$ and $|F(G)| = n$.

Case (i) : n is odd.

Let $n = 2k+1$.

Define $g : V(G) \to [-(k+1),\ldots,(k+1)]^*$ as follows.

$g(v) = 1$

$g(v_i) = -i \quad$ for $1 \le i \le \frac{n+1}{2}$

$g(v_{\frac{n+1}{2}+i}) = i+1 \quad$ for $1 \le i \le \frac{n-1}{2}$

Then induced edge labels are

$g^*(e_1) = 1$

$g^*(e_i) = 0 \quad$ for $2 \le i \le \frac{n+1}{2}$

$g^*(e_i) = 1 \quad$ for $\frac{n+3}{2} \le i \le n$

$g^*(e_{n+i}) = 0 \quad$ for $1 \le i \le \frac{n+1}{2}$

$g^*(e_{n+i}) = 1 \quad$ for $\frac{n+3}{2} \le i \le n$

Also the induced face labels are

$g^{**}(f_i) = 0 \quad$ for $1 \le i \le \frac{n-1}{2}$

$g^{**}(f_i) = 1 \quad$ for $\frac{n+1}{2} \le i \le n$

In view of the above defined labeling pattern, we have

$e_f(0) = e_f(1) = n$ and $f_g(1) = f_g(0) + 1 = \frac{n+1}{2}$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus the wheel W_n is the face integer cordial for n is odd.

Case 2: n is even.

Let $n = 2k$.

Define $g : V(G) \to [-k,\ldots,k]$ as follows.

$g(v) = 0$

$g(v_i) = -i \quad$ for $1 \le i \le \frac{n}{2}$

$g(v_{\frac{n}{2}+i}) = i \quad$ for $1 \le i \le \frac{n}{2}$

Then induced edge labels are

$g^*(e_i) = 0 \quad$ for $1 \le i \le \frac{n}{2}$

$g^*(e_i) = 1 \quad$ for $\frac{n+2}{2} \le i \le n$

$g^*(e_{n+i}) = 0 \quad$ for $1 \le i \le \frac{n}{2}$

$g^*(e_{n+i}) = 1 \quad$ for $\frac{n+2}{2} \le i \le n$

Also the induced face labels are

$g^{**}(f_i) = 0 \quad$ for $1 \le i \le \frac{n}{2}$

$g^{**}(f_i) = 1 \quad$ for $\frac{n+2}{2} \le i \le n$

In view of the above defined labeling pattern, we have

$e_f(0) = e_f(1) = n$ and $f_g(1) = f_g(0)+1 = n$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus the wheel W_n is the face integer cordial for n is even.

Hence the wheel W_n is the face integer cordial graph for $n \ge 3$.

Example : 2.1

The wheel W_5 and its face integer cordial labeling is shown in figure 2.1.

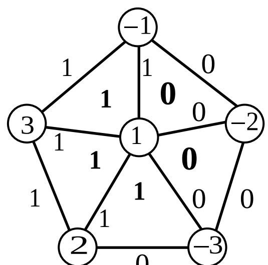

Figure 2.1

Theorem : 2.2

The fan f_n is face integer cordial graph for $n \geq 2$.

Proof.

Let $v, v_1, v_2, \ldots, v_n$, $e_1, e_2, \ldots, e_{2n-1}$ and $f_1, f_2, \ldots, f_{n-1}$ be the vertices, edges and an interior faces of f_n, where $e_i = vv_i$ for $i=1,2,\ldots,n$ and $e_{n+i} = v_i v_{i+1}$ for $i=1,2,\ldots,n-1$.

Let G be the fan graph f_n. Then $|V(G)| = n+1$, $|E(G)| = 2n-1$ and $|F(G)| = n-1$.

Case (i): n is odd and n = 2k+1.

Define $g : V(G) \to [-(k+1),\ldots,(k+1)]^*$ as follows.

$g(v) = 1$

$g(v_i) = 1+i$ for $1 \leq i \leq \frac{n-1}{2}$

$g(v_{\frac{n-1}{2}+i}) = -i$ for $1 \leq i \leq \frac{n+1}{2}$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \leq i \leq \frac{n+1}{2}$

$g^*(e_i) = 0$ for $\frac{n+3}{2} \leq i \leq n$

$g^*(e_{n+i}) = 1$ for $1 \leq i \leq \frac{n-1}{2}$

$g^*(e_{n+i}) = 0$ for $\frac{n+1}{2} \leq i \leq n-1$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \leq i \leq \frac{n-1}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+1}{2} \leq i \leq n-1$

In view of the above defined labeling pattern, we have $e_f(1) = e_f(0)+1 = n$ and $f_g(1) = f_g(0) = \frac{n-1}{2}$.

Thus $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$.

Therefore the fan f_n is the face integer cordial for n is odd.

Case (ii): n is even and n = 2k.

Define $g : V(G) \to [-k,\ldots, k]$ as follows.

$g(v) = 0$

$g(v_i) = i$ for $1 \leq i \leq \frac{n}{2}$

$g(v_{\frac{n}{2}+i}) = -i$ for $1 \leq i \leq \frac{n}{2}$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \leq i \leq \frac{n}{2}$

$g^*(e_i) = 0$ for $\frac{n+2}{2} \leq i \leq n$

$g^*(e_{n+i}) = 1$ for $1 \leq i \leq \frac{n}{2}$

$g^*(e_{n+i}) = 0$ for $\frac{n+2}{2} \leq i \leq n-1$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \leq i \leq \frac{n}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+2}{2} \leq i \leq n-1$

In view of the above defined labeling pattern, we have $e_f(1) = e_f(0)+1 = n$ and $f_g(1) = f_g(0)+1 = \frac{n}{2}$.

Thus $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$.

Therefore the fan f_n is the face integer cordial for n is even.

Hence the fan f_n is the face integer cordial graph for $n \geq 2$.

Example : 2.2

The fan f_5 and its face integer cordial labeling is shown in figure 2.2.

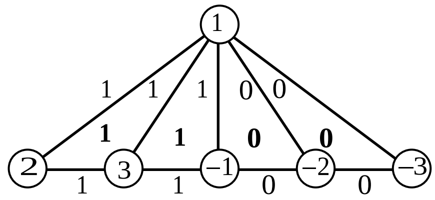

Figure 2.2

Theorem 2.3

Triangular snake T_n is face integer cordial graph for $n \geq 2$.

Proof :

Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}$ be vertices, $e_1, e_2, \ldots, e_{3n-3}$ be edges and f_1, f_2, ..., f_{n-1} interior faces of T_n, where $e_{2i-1} = v_i u_i$, $e_{2i} = u_i v_{i+1}$ and $e_{2n-2+i} = v_i v_{i+1}$ for $i = 1,2,\ldots,n-1$ and $f_i = v_i u_i v_{i+1} v_i$ for $i = 1,2,\ldots,n-1$.

Let G be the graph T_n. Then $|V(G)| = 2n-1$, $|E(G)| = 3n-3$ and $|F(G)| = n-1$.

Define $g : V(G) \to [-n,\ldots,n]$ as follows.

Case (i) : n is odd.

$g(u_i) = i$ for $1 \leq i \leq \frac{n-1}{2}$

$g(u_i) = \frac{n-1}{2} - i$ for $\frac{n+1}{2} \leq i \leq n-1$

$g(v_i) = \frac{n-1}{2} + i$ for $1 \leq i \leq \frac{n-1}{2}$

$g(v_i) = 0,$ for $i = \frac{n+1}{2}$

$g(v_i) = -i-1$ for $\frac{n+3}{2} \leq i \leq n$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \leq i \leq n-1$

$g^*(e_i) = 0$ for $n \leq i \leq 2n-2$

$g^*(e_{2n-2+i}) = 1$ for $1 \leq i \leq \frac{n-1}{2}$

$g^*(e_{2n-2+i}) = 0$ for $\frac{n+1}{2} \leq i \leq n-1$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \leq i \leq \frac{n-1}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+1}{2} \leq i \leq n-1$

In view of the above defined labeling pattern we have $e_f(0) = e_f(1) = \frac{3n-3}{2}$ and $f_g(0) = f_g(1) = \frac{n-1}{2}$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus T_n is face integer cordial graph for n is odd.

Case (ii) : n is even.

$g(u_i) = i$ for $1 \le i \le \frac{n-2}{2}$

$g(u_i) = 0$ for $i = \frac{n}{2}$

$g(u_i) = -i + \frac{n}{2}$ for $\frac{n+2}{2} \le i \le n-1$

$g(v_i) = i + \frac{n}{2}$ for $1 \le i \le \frac{n}{2}$

$g(v_i) = -i-1$ for $\frac{n+2}{2} \le i \le n$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \le i \le n-1$

$g^*(e_i) = 0$ for $n \le i \le 2n-2$

$g^*(e_{2n-2+i}) = 1$ for $1 \le i \le \frac{n}{2}$

$g^*(e_{2n-2+i}) = 0$ for $\frac{n+2}{2} \le i \le n-1$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \le i \le \frac{n}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+2}{2} \le i \le n-1$

In view of the above defined labeling pattern, we have

$e_f(1) = e_f(0)+1 = \frac{3n-2}{2}$ and $f_g(1) = f_g(0)+1 = \frac{n}{2}$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus T_n is face integer cordial graph for n is even.

Hence T_n is face integer cordial graph for $n \ge 2$.

Example 2.3

The graph T_5 and its face integer cordial labeling is shown in figure 2.3.

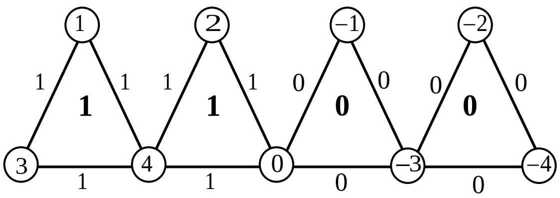

Figure 2.3

Theorem : 2.4

Double triangular snake DT_n is a face integer cordial graph for $n \ge 3$.

Proof.

Let $v_1,v_2,\ldots,v_n$, $u_1,u_2,\ldots,u_{n-1}$, $w_1,w_2,\ldots,u_{n-1}$ be vertices, $e_1,e_2,\ldots,e_{5n-5}$ be edges and $f_1,f_2,\ldots,f_{2n-2}$ be an interior faces of DT_n, where $e_{2i-1} = v_iu_i$, $e_{2i} = u_iv_{i+1}$, $e_{2n-2+i} = v_iv_{i+1}$, $e_{3n+2i-4} = v_iw_i$, and $e_{3n+2i-3} = w_iv_{i+1}$ for $i = 1,2,\ldots,n-1$, $f_i = v_iu_iv_{i+1}v_i$ for $i = 1,2,\ldots,n-1$ and $f_{i+n-1} = v_iw_iv_{i+1}v_i$ for $i = 1,2,\ldots,n-1$.

Let G be the double triangular snake DT_n. Then $|V(G)| = 3n-2$, $|E(G)| = 5n-5$ and $|F(G)| = 2n-2$.

Case (i) : n is odd and $k = \frac{3n-3}{2}$

Define $g : V(G) \to [-k,\ldots,k]$ as follows

$g(u_i) = i$ for $1 \le i \le \frac{n-1}{2}$

$g(u_i) = \frac{n-1}{2} - i$ for $\frac{n+1}{2} \le i \le n-1$

$g(v_i) = \frac{n-1}{2} + i$ for $1 \le i \le \frac{n-1}{2}$

$g(v_i) = 0$ for $i = \frac{n+1}{2}$

$g(v_i) = -i-1$ for $\frac{n+3}{2} \le i \le n$

$g(w_i) = n-1+i$ for $1 \le i \le \frac{n-1}{2}$

$g(w_i) = -\left(\frac{n-1}{2}\right) - i$ for $\frac{n+1}{2} \le i \le n-1$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \le i \le n-1$

$g^*(e_i) = 0$ for $n \le i \le 2n-2$

$g^*(e_{2n-2+i}) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^*(e_{2n-2+i}) = 0$ for $\frac{n+1}{2} \le i \le n-1$

$g^*(e_{3n-3+i}) = 1$ for $1 \le i \le n-1$

$g^*(e_{3n-3+i}) = 0$ for $n \le i \le 2n-2$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+1}{2} \le i \le n-1$

$g^{**}(f_{n-1+i}) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^{**}(f_{n-1+i}) = 0$ for $\frac{n+1}{2} \le i \le n-1$

In view of the above defined labeling pattern, we have

$e_f(0) = e_f(1) = \frac{5n-5}{2}$ and $f_g(0) = f_g(1) = n-1$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus the graph DT_n is face integer cordial graph for n is odd.

Case 2: n is even and $k = \frac{3n-2}{2}$

Define $g : V(G) \to [-k,\ldots,k]^*$ as follows

$g(u_i) = i - 1 - \frac{3n-2}{2}$ for $1 \le i \le \frac{n-2}{2}$

$g(u_{\frac{n-2}{2}+i}) = i$ for $1 \le i \le \frac{n}{2}$

$g(v_i) = -n + i - 1$ for $1 \le i \le \frac{n}{2}$

$g(v_{\frac{n}{2}+i}) = \frac{n}{2} + i$ for $1 \le i \le \frac{n}{2}$

$g(w_i) = -\frac{n}{2} + i - 1$ for $1 \le i \le \frac{n}{2}$

$g(w_i) = n+i$ for $1 \le i \le \frac{n-2}{2}$

Then induced edge labels are

$g^*(e_i) = 0$ for $1 \le i \le n - 1$

$g^*(e_i) = 1$ for $n \le i \le 2n - 2$

$g^*(e_{2n-2+i}) = 0$ for $1 \le i \le \frac{n-2}{2}$

$g^*(e_{2n-2+i}) = 1$ for $\frac{n}{2} \le i \le n-1$

$g^*(e_{3n-3+i}) = 0$ for $1 \le i \le n - 1$

$g^*(e_{3n-3+i}) = 1$ for $n \le i \le 2n - 2$

Also the induced face labels are

$g^{**}(f_i) = 0$ for $1 \le i \le \frac{n-2}{2}$

$g^{**}(f_i) = 1$ for $\frac{n}{2} \le i \le n-1$

$g^{**}(f_{n-1+i}) = 0$ for $1 \le i \le \frac{n}{2}$

$g^{**}(f_{n-1+i}) = 1$ for $\frac{n+2}{2} \le i \le n-1$

In view of the above defined labeling pattern, we have $e_f(0) = e_f(1) = \frac{5n-4}{2}$ and $f_g(0) = f_g(1) = n-1$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus the graph DT_n is face integer cordial graph for n is even.

Hence the graph DT_n is face integer cordial graph for $n \ge 3$.

Example 2.4

The graph DT_4 and its face integer cordial labeling is shown in figure 2.4.

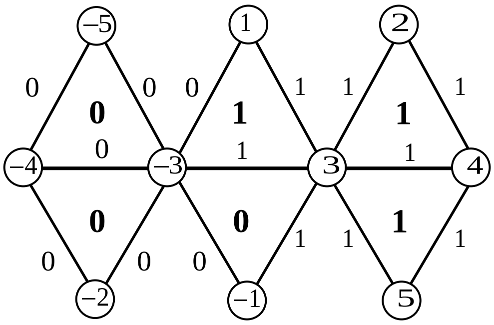

Figure 2.4

Theorem 2.5

The friendship graph F_n is face integer cordial graph for $n \ge 3$.

Proof :

Let $v, v_1, v_2, \ldots, v_{2n}$, $e_1, e_2, \ldots, e_{3n}$, $f_1, f_2, \ldots, f_n$ be the vertices, edges and an interior faces of F_n, where $e_{3i-2} = vv_{2i-1}$, $e_{3i-1} = v_{2i-1}v_{2i}$, $e_{3i} = v_{2i}v$ and $f_i = vv_{2i-1}v_{2i}v$, for $1 \le i \le n$. Let G be the friendship graph F_n. Then $|V(G)| = 2n+1$, $|E(G)| = 3n$ and $|F(G)| = n$.

Define $g : V(G) \to [-n,\ldots,n]$ as follows

Case (i) : n is odd

$g(v) = 0$

$g(v_i) = i$ for $1 \le i \le n$

$g(v_{n+i}) = -i$ for $1 \le i \le n$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \le i \le \frac{3n+1}{2}$

$g^*(e_i) = 0$ for $\frac{3n+3}{2} \le i \le 3n$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \le i \le \frac{n+1}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+3}{2} \le i \le n$

In view of the above defined labeling pattern, we have $e_f(1) = e_f(0) + 1 = \frac{3n+1}{2}$ and

$f_g(1) = f_g(0) + 1 = \frac{n+1}{2}$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Hence F_n is face integer cordial graph for n is odd.

Case (ii) : n is even

$g(v) = 0$

$g(v_i) = i$ for $1 \le i \le n$

$g(v_{n+i}) = -i$ for $1 \le i \le n$

Then induced edge labels are

$g^*(e_i) = 1,$ for $1 \le i \le \frac{3n}{2}$

$g^*(e_i) = 0,$ for $\frac{3n+2}{2} \le i \le 3n$

Also the induced face labels are

$g^{**}(f_i) = 1,$ for $1 \le i \le \frac{n}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+2}{2} \le i \le n$

In view of the above defined labeling pattern, we have $e_f(0) = e_f(1) = \frac{3n}{2}$ and $f_g(1) = f_g(0) = \frac{n}{2}$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Thus F_n is face integer cordial graph for n is even.

Hence F_n is face integer cordial graph for $n \ge 3$.

Example 2.5

The graph F_3 and its face integer cordial labeling is shown in figure 2.5.

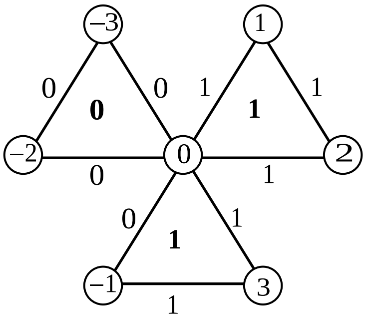

Figure 2.5

Theorem 2.6

$DS(B_{n,n})$ is face integer cordial graph for $n \geq 2$.

Proof.

Let $u,v,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n$ and $e_1,e_2,\ldots,e_{2n+1}$ be the vertices and edges of $B_{n,n}$.

Now $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u,v\}$ and $V_2 = \{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$. In order to obtain $DS(B_{n,n})$ is obtained from $B_{n,n}$ by adding the vertex w_1 to V_1 and w_2 to V_2.

$u,v, u_1,u_2,\ldots,u_n, v_1,v_2,\ldots,v_n,w_1,w_2, e_1,e_2,\ldots,e_{4n+3}$ and $f_1,f_2,\ldots,f_{2n}$ be the vertices, edges and an interior faces of $DS(B_{n,n})$, where $e_i = uu_i$, for $i = 1,2,\ldots,n$, $e_{n+1} = uv$, $e_{n+1+i} = vv_i$, $e_{2n+1+i} = w_1u_i$, $e_{3n+1+i} = w_1v_i$ for $i = 1,2,\ldots,n$, $e_{4n+2} = w_2u$, $e_{4n+3} = w_2v$ and $f_i = uu_iw_1u_{i+1}u$, $f_{n-1+i} = vv_iw_1v_{i+1}v$, for $i = 1,2,\ldots,n-1$, $f_{2n-1} = uvv_1w_1u_1u$ and $f_{2n} = uvw_2u$.

Let G be a graph $DS(B_{n,n})$. Then $|V(G)| = 2n+4$, $|E(G)| = 4n+3$ and $|F(G)| = 2n$.

Define $g : V(G) \to [-(n+2),..,(n+2)]^*$ as follows.

$f(u) = 2$

$f(v) = -1$

$f(w_1) = 1$

$f(w_2) = -(n+2)$

$f(u_i) = n+3 - i$ for $1 \leq i \leq n$

$f(v_i) = -(i+1)$ for $1 \leq i \leq n$.

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \leq i \leq n$

$g^*(e_{n+1}) = 1$

$g^*(e_{2n+1+i}) = 1$ for $1 \leq i \leq n$

$g^*(e_{n+1+i}) = 0$ for $1 \leq i \leq n$

$g^*(e_{3n+1+i}) = 0$ for $1 \leq i \leq n+2$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \leq i \leq n-1$

$g^{**}(f_{n-1+i}) = 0$ for $1 \leq i \leq n+1$

In view of the above defined labeling pattern, we have $e_f(0) = e_f(1)+1 = 2n+2$ and $f_g(1) = f_g(0) = n$.

Then $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$.

Hence $DS(B_{n,n})$ is the face integer cordial for $n \geq 3$.

Example 2.6

The graph $DS(B_{3,3})$ and its face integer cordial labeling is shown in figure 2.6.

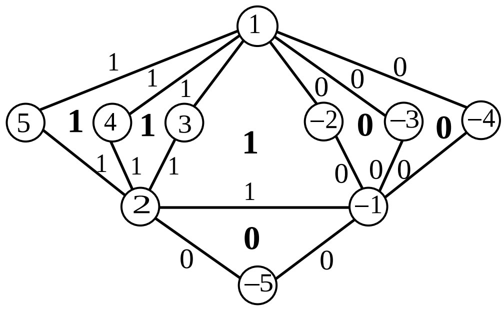

Figure 2.6

Theorem : 2.7

The star of cycle C_n is face integer cordial graph for $n \geq 3$.

Proof.

Let $v_1,v_2,\ldots,v_n$, $v_{11},v_{12},\ldots,v_{1n}$, $v_{21},v_{22},\ldots,v_{2n},\ldots$, $v_{n1},v_{n2},\ldots,v_{nn},e_1,e_2,\ldots,e_{2n},e_{11},e_{12},\ldots,e_{1n},e_{21},e_{22},\ldots,e_{2n},\ldots$, $e_{n1}, e_{n2}, \ldots,e_{nn}$ and $f_1, f_2, \ldots, f_{n+1}$ be vertices, edges and an interior faces of the star of cycle C_n. $v_1,v_2,\ldots,v_n$ be the vertices of central cycle C_n, $v_{i1},v_{i2},\ldots,v_{in}$ be the vertices of the cycle C_n^i, where $1 \leq i \leq n$ and v_{i1} be adjacent to the i^{th} vertex of the central cycle C_n. $e_i = v_i v_{i+1}$, for $1 \leq i \leq n-1$, $e_n = v_nv_1$, $e_{n+i} = v_i v_{i1}$, for $1 \leq i \leq n$, $e_{ij} = v_{ij} v_{(i+1)j}$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, $e_{in} = v_{in}v_{i1}$, for $1 \leq i \leq n$, $f_1 = v_1v_2\ldots v_n$ and $f_{i+1} = v_{i1}v_{i2}\ldots v_{in}$ for $1 \leq i \leq n$.

Let G be the star of cycle C_n.

Then $|V(G)| = n(n+1)$, $|E(G)| = n(n+2)$ and $|F(G)| = n+1$.

Case (i) : n is even and $k = \frac{n(n+1)}{2}$

Define $g : V(G) \to [-k,\ldots,k]^*$ as follows

$g(v_i) = i$, for $1 \leq i \leq n$

$g(v_{ij}) = -[(i-1)n+j]$,

for $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq n$,

$g(v_{ij}) = \left(i - \frac{n+1}{2}\right) n + j$,

for $\frac{n+3}{2} \leq i \leq n$ and $1 \leq j \leq n$,

Then induced edge labels are

$g^*(e_i) = 1$, for $1 \leq i \leq n+1$

$g^*(e_{n+i}) = 0$, for $2 \leq i \leq \frac{n+1}{2}$

$g^*(e_{n+i}) = 1$ for $\frac{n+3}{2} \leq i \leq n$

$g(e_{ij}) = 0$ for $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq n$,

$g(e_{ij}) = 1$ for $\frac{n+3}{2} \leq i \leq n$ and $1 \leq j \leq n$,

Also the induced face labels are

$g^{**}(f_1) = 1$

$g^{**}(f_{1+i}) = 0$ for $1 \le i \le \frac{n+1}{2}$

$g^{**}(f_{1+i}) = 1$ for $\frac{n+3}{2} \le i \le n$

In view of the above defined labeling pattern, we have $e_f(1) = e_f(0)+1 = \frac{n(n+2)+1}{2}$ and $f_g(0) = f_g(1) = \frac{n+1}{2}$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Hence G is face integer cordial graph for n is odd.

Case (ii) : n is even and $k = \frac{n(n+1)}{2}$

Define $g : V(G) \to [-k,\dots,k]^*$ as follows

$g(v_i) = -i$ for $1 \le i \le \frac{n}{2}$

$g(v_i) = \left(i - \frac{n}{2}\right)$ for $\frac{n+2}{2} \le i \le n$

$g(v_{ij}) = -[\frac{n}{2} + (i-1)n + j]$

for $1 \le i \le \frac{n}{2}$ and $1 \le j \le n$,

$g(v_{ij}) = \frac{n}{2} + \left(i - \frac{n+2}{2}\right)n + j$

for $\frac{n+2}{2} \le i \le n$ and $1 \le j \le n$,

Then induced edge labels are

$g^*(e_i) = 0$ for $1 \le i \le \frac{n}{2}$

$g^*(e_i) = 1$ for $\frac{n+2}{2} \le i \le n$

$g^*(e_{n+i}) = 0$ for $1 \le i \le \frac{n}{2}$

$g^*(e_{n+i}) = 1$ for $\frac{n+2}{2} \le i \le n$

$g(e_{ij}) = 0$ for $1 \le i \le \frac{n}{2}$ and $1 \le j \le n$,

$g(e_{ij}) = 1$, for $\frac{n+2}{2} \le i \le n$ and $1 \le j \le n$,

Also the induced face labels are

$g^{**}(f_1) = 1$

$g^{**}(f_{1+i}) = 0$ for $1 \le i \le \frac{n}{2}$

$g^{**}(f_{1+i}) = 1$ for $\frac{n+2}{2} \le i \le n$

In view of the above defined labeling pattern , we have $e_f(0) = e_f(1) = \frac{n(n+2)}{2}$ and $f_g(1) = f_g(0)+1 = \frac{n+2}{2}$.

Then $|e_g(0)-e_g(1)| \le 1$ and $|f_g(0)-f_g(1)| \le 1$

Hence G is face integer cordial graph for n is even.

Therefore the star of cycle C_n is face integer cordial graph for $n \ge 3$.

Example : 2.7

The star of cycle C_5 and its face integer cordial labeling of graph is shown in figure 2.7.

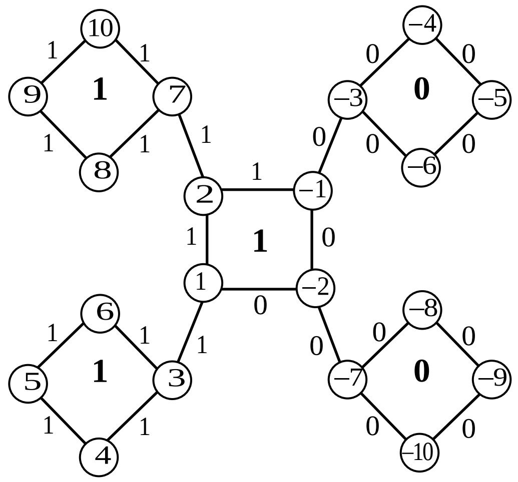

Figure 2.7

Theorem 2.8

$T(P_n)$ is face integer cordial graph for $n \ge 3$.

Proof :

Let $v_1, v_2, \dots, v_n$, $u_1, u_2, \dots, u_{n-1}$ be vertices, $e_1, e_2, \dots, e_{4n-5}$ be edges and $f_1, f_2, \dots, f_{2n-3}$ interior faces of $T(P_n)$, where $e_{2i-1} = v_iu_i$, $e_{2i} = u_iv_{i+1}$, $e_{2n-2+i} = v_iv_{i+1}$ for $i = 1,2,\dots,n-1$, $e_{3n-3+i} = u_iu_{i+1}$ for $i = 1,2,\dots,n-2$, $f_i = v_iu_iv_{i+1}v_i$ for $i = 1,2,\dots,n-1$ and $f_{n-1+I} = u_iv_{i+1}u_{i+1}u_i$ for $i = 1,2,\dots,n-2$.

Let G be the graph $T(P_n)$.

Then $|V(G)| = 2n-1$, $|E(G)| = 4n-5$ and $|F(G)| = 2n-3$.

Define $g : V(G) \to [-n,\dots,n]$ as follows.

Case (i) : n is odd.

$g(u_i) = i$ for $1 \le i \le \frac{n-1}{2}$

$g(u_i) = \frac{n-1}{2} - i$ for $\frac{n+1}{2} \le i \le n-1$

$g(v_i) = \frac{n-1}{2} + i$ for $1 \le i \le \frac{n-1}{2}$

$g(v_i) = 0$ for $i = \frac{n+1}{2}$

$g(v_i) = -i-1$ for $\frac{n+3}{2} \le i \le n$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \le i \le n-1$

$g^*(e_i) = 0$ for $n \le i \le 2n-2$

$g^*(e_{2n-2+i}) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^*(e_{2n-2+i}) = 0$ for $\frac{n+1}{2} \le i \le n-1$

$g^*(e_{3n-3+i}) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^*(e_{3n-3+i}) = 0$ for $\frac{n+1}{2} \le i \le n-2$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+1}{2} \le i \le n-1$

$g^{**}(f_{i+n}) = 1$ for $1 \le i \le \frac{n-1}{2}$

$g^{**}(f_{i+n}) = 0$ for $\frac{n+1}{2} \le i \le n-1$

In view of the above defined labeling pattern we have $e_f(1) = e_f(0)+1 = 2n-2$ and $f_g(1) = f_g(0)+1 = n-1$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus $T(P_n)$ is face integer cordial graph for n is odd.

Case (ii) : n is even.

$g(u_i) = i$ for $1 \le i \le \frac{n-2}{2}$

$g(u_i) = 0$ for $i = \frac{n}{2}$

$g(u_i) = -i+\frac{n}{2}$ for $\frac{n+2}{2} \le i \le n-1$

$g(v_i) = i+\frac{n}{2}$ for $1 \le i \le \frac{n}{2}$

$g(v_i) = -i-1$ for $\frac{n+2}{2} \le i \le n$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \le i \le n-1$

$g^*(e_i) = 0$ for $n \le i \le 2n-2$

$g^*(e_{2n-2+i}) = 1$ for $1 \le i \le \frac{n}{2}$

$g^*(e_{2n-2+i}) = 0$ for $\frac{n+2}{2} \le i \le n-1$

$g^*(e_{3n-3+i}) = 1$ for $1 \le i \le \frac{n-2}{2}$

$g^*(e_{3n-3+i}) = 0$ for $\frac{n}{2} \le i \le n-2$

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \le i \le \frac{n}{2}$

$g^{**}(f_i) = 0$ for $\frac{n+2}{2} \le i \le n-1$

$g^{**}(f_{i+n}) = 1$ for $1 \le i \le \frac{n-2}{2}$

$g^{**}(f_{i+n}) = 0$ for $\frac{n}{2} \le i \le n-2$

In view of the above defined labeling pattern we have $e_f(1) = e_f(0)+1 = 2n-2$ and $f_g(1) = f_g(0)+1 = n-1$.

Then $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus $T(P_n)$ is face product cordial graph for n is even.

Hence $T(P_n)$ is face product cordial graph for $n \ge 3$.

Example 2.8

The graph $T(P_4)$ and its face integer cordial labeling is shown in figure 2.8.

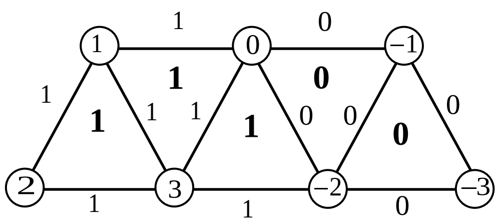

Figure 2.8

III. CONCLUSIONS

In this paper, we prove wheel W_n, fan f_n, triangular snake T_n, double triangular snake DT_n, star of cycle C_n and $DS(B_{n,n})$ are face integer cordial graph. In the subsequent paper, we will prove vertex switching of cycle, pendent vertex switching of path, helm, closed helm, middle graph of path, total graph of path and subdivision of rim edges of wheel.

REFERENCES

[1]. I. Cahit, Cordial graphs: A weaker version of graceful and harmonious graphs, Ars Combinatoria, Vol 23, pp. 201-207, 1987.

[2]. J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 16, # DS6, 2016.

[3]. F. Harary, Graph theory, Addison Wesley, Reading, Massachusetts, 1972.

[4]. P. Lawrence Rozario Raj and R. Lawrence Joseph Manoharan, Face and total face edge product cordial graphs, International Journal of Mathematics Trends and Technology, Vol. 19, No. 2, pp 136-149, 2015.

[5]. P. Lawrence Rozario Raj and R. Lawrence Joseph Manoharan, Face and Total Face Product Cordial Labeling of Graphs, International Journal of Innovative Science, Engineering & Technology, Vol.2, Issue 9, pp 93-102, 2015

[6]. M. Mohamed Sheriff and A. Farhana Abbas, Face Integer Edge Cordial Labeling of Graphs, communicated.

[7]. T.Nicholas and P.Maya, Some results on integer cordial graph, Journal of Progressive Research in Mathematics (JPRM), Vol. 8, Issue 1, pp 1183-1194, 2016.

[8]. M.Sundaram, M.Ponraj and S.Somsundaram, Product Cordial Labeling of Graphs, Bull. Pure and Applied Sciences (Mathematics and Statistics), 23E, pp 155-163, 2004.

[9]. S. K. Vaidya and C. M. Barasara, Edge Product Cordial Labeling of Graphs, J. Math. Comput. Sci. Vol 2, No. 5, pp 1436-1450, 2012.

[10]. R. Varatharajan, S. Navanaeethakrishnan and K. Nagarajan, Divisor cordial graphs, *International J. Math. Combin.*, Vol 4, 2011, pp. 15-25.