A Weaker Form of Urysohn Spaces

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Abstract: *The purpose of this paper is to introduce a new type of spaces called, weakly generalized Urysohn spaces and discussed some of its basic properties. Also, the conditions for a weakly generalized Urysohn space becomes Urysohn space and g-Urysohn space have been analysed.*

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1. Introduction

In 1970, Levine introduced a class of closed sets called generalized closed (briefly g-closed) sets[5] in topological spaces.In 1988, Dorsett. C studied the notion of weakly Urysohn space [3]. Likewise, In 1999, Bhattacharyya et al. also contributed his study to this field of topology by introducing Pre-Urysohn spaces [8]. In 1999, Nagaveni. et al. introduced and investigated the weakly generalized closed sets[6]. In 2002,Ganster et al. studied the characteristics of semi Urysohn space[4]. In 2011, Navalagi studied some more properties of Pre- Urysohn spaces and P Urysohn spaces [7].

In this paper, we introduced weakly generalized Urysohn space. The conditions under which a weakly generalized Urysohn space coincides with Urysohn space, g-Urysohn space and weakly generalized Hausdroff spaces have been generated. Subspaces of weakly generalized Urysohn spaces are investigated. Also, introduced and investigated the notion of wg-continuous, wg-irresolute, wg-open and quasi wg-irresolute function on weakly generalized Urysohn space.

Throughout the paper (X, τ) and (Y, σ) (or simply X and Y) will always denote topological spaces. The interior and the closure of a subset A of (X, τ) are denoted by Int(A) and Cl(A) respectively.

2. Preliminaries

In this section, we list some definitions and results which are used in this sequel.

Definition: 2.1 A subset A of a space (X, τ) is called

i. generalized closed (i.e. g-closed set) [5] if $cl(A) \subset U$ whenever $A \subset U$ and U is open set.
ii. weakly generalized closed set (i.e. wg-closed set) [6] if $Cl(Int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

The complement of g-closed set (resp. wg- closed set) is said to be g-open set (resp. wg- open set). The collection of all g-open sets (resp. wg- open set) is denoted by GO(X) (resp. WGO(X)).

Definition: 2.2 A topological space X is called

i. Urysohn space (T_2' space) if every pair of distinct points $x, y \in X$, $x \neq y$ there exist $U \in O(X, x)$, $V \in O(X, y)$ such that $cl(U) \cap cl(V) = \Phi$.
ii. generalized Urysohn space ($g - T_2'$ space)[11] if every pair of distinct points $x, y \in X$, $x \neq y$, there exist $U \in GO(X, x)$, $V \in GO(X, y)$ such that $g\text{-}cl(U) \cap g\text{-}cl(V) = \Phi$.

Definition: 2.3 if X is said to be

i. g-T_1 space[10] if for $x, y \in X$ such that $x \neq y$, there exist a generalized open set containing x but not y and a generalized open set containing y but not x.
ii. g-T_2 space[10] if for $x, y \in X$ such that $x \neq y$, there exist a generalized open sets $U \in GO(X, x)$, $V \in GO(X, y)$ with $U \cap V = \Phi$.
iii. wg-T_1 space if for $x, y \in X$ such that $x \neq y$, there exist a weakly generalized open set containing x but not y and a weakly generalized open set containing y but not x.
iv. wg-T_2 space if for $x, y \in X$ such that $x \neq y$, there exist a weakly generalized open sets $U \in WGO(X, x)$, $V \in WGO(X, y)$ with $U \cap V = \Phi$.

Definition: 2.4 A space X is said to be

i. g-regular[2] if and only if for each $x \in X$ and $U \subset O(X, x)$ there exist $V \in GO(X, x)$ such that $x \in V \subset g\text{-}cl(V) \subset U$.
ii. weakly g-regular[2] if for each point x and a regular open set U containing x, there is a g-open set V such that $x \in V \subset cl(V) \subset U$.

Definition: 2.5 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a

i. g-continuous [5] if $f^{-1}(V)$ is g-closed in X for each closed subset of V of Y.
ii. weakly g-continuous (i.e.wg- continuous)[6] if inverse image of every open set inY iswg-open in X.

iii. weakly g-irresolute (i.e. wg-irresolute)[6] if inverse image of every wg-open set in Yiswg-open in X.
iv. wg-open [6] if image of every open set in X is wg-open in Y.

Remark:2.6 [6]Every α – set is wg-open set

Remark: 2.7[6] The bijection function f: X ⟶ Y be wg-open, Then for any F ∈ WGC(X), f(F) ∈ WGC(Y).

3. WEAKLY GENERALIZED URYSOHN SPACES

In this section, we introduce weakly generalized Urysohn Space and investigate some weaker separation axioms with weakly generalized Urysohn Space.

Definition: 3.1 The wg-closure of a subset A of X is, denoted by wg-Cl(A), defined to be the intersection of all wg-closed sets containing A.

Definition: 3.2A topological space X is called weakly generalized Urysohn space (wg – T_2' space) if every pair of distinct points x, y ∈ X, x ≠ y there exist U ∈ WGO(X, x), V ∈ WGO(X, y) such that wg-cl(U) ⋂ wg-cl(V) = Φ.

Remark: 3.3Every Urysohn space is weakly generalized Urysohn space. But the converse need not be true as seen from the following example.

Example: 3.4Let X = {a, b, c, d} and τ = {X, Φ, {a,b}}. Then {X, τ} is called weakly generalized Urysohn space.But it is not Urysohn Space.

Theorem: 3.5 Every weakly generalized Urysohn space (wg – T_2' space) is weakly generalized Hausdroff space (wg-T_2 space).

Proof: Let x, y be two distinct points of X. Since X is weakly generalized Urysohn space, there exist U ∈ WGO(X, x), V ∈ WGO(X, y) such that wg-cl(U) ⋂ wg-cl(V) = Φ. Hence U ⋂ V = Φ. Thus, X is weakly generalized Hausdroff space (wg-T_2 space).

Theorem: 3.6 Every weakly generalized Urysohn space is weakly generalized T_1 space.

Proof:Let x, y be two distinct points of X. Since X is weakly generalized Urysohn space, there exist U ∈ WGO(X, x), V ∈ WGO(X, y) such that wg-cl(U) ⋂ wg-cl(V) = Φ. This indicates that x ∉ wg-cl(V) and y ∉ wg-cl(U). now wg-cl(U) , wg-cl(V) ∈ WGC(X). Therefore, X – wg-cl(U), X – wg-cl(V) ∈ WGO(X) such that x ∈ wg-cl(V) and y ∈ wg-cl(U). Therefore, x ∉ wg-cl(U) and y ∉ wg-cl(V). Thus, X is wg-T_1 space.

Theorem: 3.7Every generalized Uyrsohn space (g- T_2' space) is weakly generalized Urysohn space (wg-T_2' space).

Proof:Let x, y be two distinct points of X. Since X is generalized Urysohn space, there exist U ∈ GO(X, x), V ∈ GO(X, y) such that g-cl(U) ⋂ g-cl(V) = Φ. Since every generalized open set is weakly generalized open set. Therefore, U ∈ GO(X, x) ⊆ WGO (X, x) and V ∈ GO(X, y) ⊆WGO (X, y) such that wg-cl(U) ⋂ wg-cl(V) = Φ. Hence X is weakly generalized Urysohn space.

Theorem: 3.8Every generalized Hausdroff space (g-T_2 space) is weakly generalized Hausdroff space (wg-T_2 space).

Proof:Let x, y be two distinct points of X. since X is generalized Haussdroff space, there exist U ∈ GO(X, x), V ∈ GO(X, y) such that U ⋂ V= Φ. Since every generalized open set is weakly generalized open set. Therefore, U ∈ GO(X, x) ⊆ WGO (X, x) and V ∈ GO(X, y) ⊆ WGO (X, y) such that U ⋂ V= Φ.

Theorem: 3.9A g-regular and T_2 space is generalized Urysohn space.

Proof:Let X be g-regular and T_2 space. Since X is T_2 space, for any pair of distinct points x, y ∈ X, x ≠ y there exist U ∈ O(X, x), V ∈ O(X, y) such that U ⋂ V = Φ. Now X – cl(U) ∈ O(y). the g-regularity of X gives the existance of a W ∈ GO(X, y), by the definition 2.4(i), such that y ∈ W ⊂gcl(W) ⊂ X – cl(U). This implies that gcl(W) ⋂ cl(W) = Φ which yields that gcl(U) ⋂gcl(W) = Φ. Since every open set is g-open set. Hence X is generalized Urysohn space.

Theorem: 3.10A weakly g-regular and T_2 space is Urysohn space.

Proof:Let x be weakly g-regular and T_2. Since X is T_2 for any pair of points x, y ∈ X, x ≠ y. There exist open sets U and V containing x and y such that U ⋂ V = Φ. Now X – cl(U) ∈ O(X, y). The weakly g-regularity of X gives the existence of a W ∈ GO(X, y), by the definition 2.4(ii), y ∈ W ⊂ cl(W) ⊂ U or X – cl(U). this implies cl(W) ⋂ cl(U) = Φ. Therefore, X is Urysohn space.

Corollary: 3.11A weakly g-regular and T_2 space is weakly generalized Urysohn space.

Proof: It readily follows from the statement every Uryshon space is weakly generalized Urysohn space.

Definition:3.12A space X is called **almost weakly g-regular space** if for each x∈X and each U ∈ O(X,x) there exists V ∈ WGO(X, x) such that x ∈ V ⊂ wg-cl(V) ⊂ U.

Theorem: 3.13Every almost weakly g-regular and T_2 space is weakly generalized urysohn space.

Proof: Let X be almost wg-regular and T_2 space. Let x, y ∈ X be any two distinct points. As X is T_2 space, there exists open sets U containing x and V containing y such that U ∩ V = Φ. Now X – Cl(U) ∈ O(X, y). The almost weakly g-regularity of X gives the existence of a M ∈ WGO(X, y), by the definition 3.12, such that y ∈ M ⊂wg-cl(M) ⊂X – Cl(U). This implies that cl(U) ∩ wg-cl(M) = Φ which yields that wg-cl(U) ∩ wg-cl(M) = Φ. Since every open set is weakly generalized open set, this shows that, X is weakly generalized Urysohn space.

4. SUBSPACES OF WEAKLY GENERALIZED URYSOHN SPACE

In this section, we investigate subspaces of weakly generalized Urysohn Space.

Theorem: 4.1Every open subspace of weakly generalized Urysohn space is weakly generalized Urysohn space.

Proof:Let Y be an open subspace of a weakly generalized Urysohn space X. Assume x and y in Y with x ≠ y. As X is weakly generalized Urysohn space, there exist wg-open sets U containing x and V containing y such that wg-cl(U) ∩ wg-cl(V) = Φ.

Lemma: 4.2In a topological space (X, τ) if A ∈ WGO(X) and B $\in\tau^{\alpha}$, then A ∩ B ∈ WGO(X).

Proof:Given A ∈ WGO(X) ie $A^c\in$ WGO(X) implies that $cl(int(A^c)) \subset U$ ans $A^c \subset U$ whenever U is open in X and B $\in\tau^{\alpha}$ie., B⊂int(cl(int(B))). Now $A^c \cup B^c \in$ WGC(X) then $(A \cap B)^c\in$ WGC (X). This implies A ∩B ∈WGO(X).

Lemma: 4.3If A ⊂ Y ⊂ X and Y ∈ WGO(X), then A ∈ WGO(X) if and only if A ∈ WGO(Y).

The proof is obvious.

Lemma: 4.4Let x ∈X, then x ∈wg-cl(A) if and only if A ∩V ≠ Φ, for all V∈ WGO(X, x).

The proof is obvious.

Lemma: 4.5 If B ⊂ Y ⊂ X and Y $\in\tau^{\alpha}$, then wg-cl_Y(B) = wg-cl_X(B) ∩Y.

Proof:Let y ∈wg-cl_Y(B), so that y ∈ Y. let V ∈ WGO(X, y), by Lemma 4.2 V ∩ Y ∈ WGO(X, y). since every α- set is a wg-open sets and V ∩ Y ⊂ Y ⊂ X, Lemma 4.3 gives that V ∩ Y ∈ WGO(X, y). consequently, (V ∩ Y) ∩ B ≠ Φ whence V ∩ B ≠ Φ. So, by Lemma 4.4, y ∈wg-cl_X(B) which implies that y ∈wg-cl_X(B) ∩ Y. so, wg-cl_Y(B) ⊂wg-cl_Y(B) ∩ Y.

To establish the reverse inclusion, let y ∈wg-cl_X(B) ∩ Y, then y ∈wg-cl_Y(B), y ∈ Y. Take any V_0∈ WGO(Y, y) pursuing the same reasoning as above we obtain V_0∈ WGO(X, y). hence by the Lemma 4.4, V_0∩ B ≠ Φ. Hence y ∈wg-cl_Y(B). Therefore, wg-cl_X(B) ∩ Y ⊂wg-cl_Y(B). Hence wg-cl_Y(B) = wg-cl_X(B) ∩Y.

Theorem: 4.6Every α- subspace of a weakly generalized Urysohn space (X, τ) is weakly generalized Urysohn space.

Proof:Let Y ⊂ X and Y $\in\tau^{\alpha}$. Let x, y ∈ Y and x ≠ y. since Y ⊂ X, x, y are also distinct points of X. since X is weakly generalized Urysohn space, there exist U ∈ WGO(X, x), V ∈ WGO(X, y) such that wg-cl_X(U) ∩wg-cl_X(V) =Φ. Since Y $\in\tau^{\alpha}$, by Lemma 4.2, U ∩ Y ∈ WGO(X, x) and V ∩ Y ∈ WGO(X, y). also by Lemma 4.5, wg-cl_Y(U∩ Y) ∩wg-cl_Y(V∩ Y) = (wg-cl_X(U) ∩ Y) ∩ (wg-cl_X(V ∩ Y) ∩ Y) = ((wg-cl_X(U ∩Y) ∩wg-cl_X(V ∩ Y)) ∩ Y) ⊂ (wg-cl_X(U ∩ Y) ∩wg-cl_X(V ∩ Y)) ⊂ (wg-cl_X(U) ∩wg-cl_X(V)) ∩ Y ⊂wg-cl_X(U) ∩wg-cl_X(V) = Φ. Therefore, Y is weakly generalized Urysohn space.

Remark: 4.7 The property being weakly generalized Urysohn space is not hereditary as shown by the following example.

Example: 4.8 Let X be the same topological space of Example 3.4. Then {X, τ} is called weakly generalized Urysohn space but the subspace {b, c} of X is not weakly generalized Urysohn space.

5. FUNCTIONS WITH WEAKLY GENERALIZED URYSOHN SPACE

In this section, we investigate some functions with weakly generalized Urysohn Space.

Theorem: 5.1IfY is aweakly generalized Urysohn space and f: X ⟶ Y is a wg- irresolute injective, then X is weakly generalized Urysohn space.

Proof: Let x and y be two distinct points of X. Since f is injective, f(x) ≠ f(y) in Y. Since Y is weakly generalized Urysohn space, there exists weakly generalized open sets U and V containing f(x) and f(y) respectively. Since f is wg-irresolute, there exist f^{-1}(U) and f^{-1}(V) are weakly generalized open sets in X containing x and y respectively. Thus X is weakly generalized Urysohn space.

Theorem:5.2 IfY is aHausdroff space and f: X ⟶ Y is bothwg- continuous and injective, then X is weakly generalized Urysohn space.

Proof: Let x and y be two distinct points of X. Since f is injective, f(x) ≠ f(y) in Y. Since Y is Hausdroff space, there exists open sets U and V containing f(x) and f(y) respectively. Since f is wg- continuous, there exist f^{-1}(U) and f^{-1}(V) are weakly generalized open sets in X containing x and y respectively. Thus X is weakly generalized Urysohn space.

Theorem: 5.3If the bijection f: X ⟶ Y is wg-open and X is weakly generalized Urysohn space, then Y is weakly generalized Urysohn space.

Proof:Let y_1, $y_2 \in$ Y and $y_1 \neq y_2$. Since f is bijective $f^{-1}(y_1)$, $f^{-1}(y_2) \in$ X and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The weakly generalized Urysohn property of X gives the existence of sets U ∈ WGO ($f^{-1}(y_1)$), V ∈ WGO ($f^{-1}(y_2)$) with wg-cl_x(U) ⋂ wg-cl_x(V) = Φ. Here,wg-cl_x(U) is wg-closed sets in X. The bijectivity and wg-openess of f together then indicate, by Remark 2.7, that f (wg-cl_x(U)) ∈ WGC(Y). Again, from U ⊂ wg-cl_x(U) it follows that f(U) ⊂ f(wg-cl_x(U)). Since wg-clousure respects inclusion, wg-cl_y(f(U)) ⊂wgcl_y(f(wg-cl_x(U)) = f(wg-cl_x(U)). In like manner,wg-cl_y(f(V)) ⊂ f(wg-cl_x(V)). Therefore, by the injectivity of f, wg-cl_y(f(U)) ⋂wg-cl_y(f(V)) ⊂ f(wg-cl_x(U)) ⋂ f(wg-cl_x(V)) = f [(wg-cl_x(U)) ⋂(wg-cl_x(V))] = f(Φ) = Φ. Thus wg-openness of f gives existence of two sets f(U) ∈ WGO(Y, y_1), f(V) ∈ WGO(Y, y_2) with wg-cl_y(f(U)) ⋂wg-cl_y(f(V)) = Φ. Hence Y is weakly generalized urysohn space.

Definition: 5.4A function f: X ⟶ Y is quasi-weakly generalized irresolute(qwgi) if for eaxh x ∈ X and for each V ∈ WGO(f(x)) there exists U ∈ WGO(x) such that f(U) ⊂wg-cl_y(f(U)).

Theorem: 5.5If Y is weakly generalized Urysohn space and f: X ⟶ Y is quasi-weakly generalized irresolute, then X is wg- T_2 space.

Proof:Since f is injective, for any pair of distinct points x_1, $x_2 \in$ X, $f(x_1) \neq f(x_2)$. The weakly generalized Urysohn property indicates that there exist V_i ∈WGO (Y, $f(x_i)$), i = 1, 2 such that wg-$cl_y(V_1)$ ⋂wg-$cl_y(V_2)$ = Φ. Hence f^{-1}(wg-$cl_y(V_1)$) ⋂ f^{-1}(wg-$cl_y(V_2)$)= Φ. Since f is quasi-weakly generalized irresolute, there exists U_i∈WGO (X, x_i), i = 1, 2 such that $f(U_i)$ ⊂wg-$cl_y(V_i)$, i = 1,2. It, then, follows that $U_i \subset f^{-1}$(wg-$cl_y(V_i)$), i= 1, 2. Hence $U_1 \cap U_2 \subset f^{-1}$(wg-$cl_y(V_1)$) ⋂ f^{-1}(wg-$cl_y(V_2)$) = Φ. This implies that X is wg-T_2 space.

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