A new strong convergence Theorem for Lipschitzian strongly pseudocontractive Mappings in a real Banach space

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Abstract
In this paper we prove the strong convergence of a new iteration scheme to a common fixed point of a finite family of Lipschitz strongly pseudocontractive mappings in a real Banach space. Result proved in this paper represents an extension and refinement of the previously known results in this area.

Keywords: *Iteration scheme, strongly pseudocontractive maps, L- Lipschitzian maps, Banach spaces*

I. Introduction

Let E be a real Banach space and C $\subseteq$ E a non empty subset. A mapping T: C $\rightarrow$ C is said to be strongly pseudo-contractive if there exists $t > 1$ such that the inequality

$$\|x-y\| \leq \|(1+t)(x-y)-tr(Tx-Ty)\| \quad (1.1)$$

holds for all x, y $\in$ C and r > 0. If t=1, then T is called pseudocontractive.
A mapping T: C $\rightarrow$ C is called L- Lipschitzian , if there exists L > 0 such that

$$\|Tx-Ty\| \leq L\|x-y\| \quad (1.2)$$

for all x, y$\in$C.
In 2000, Noor [4] gave the following three-step iterative scheme (or Noor iteration) for solving nonlinear operator equations in uniformly smooth Banach spaces.
Let C be a nonempty convex subset of E and let T: C $\rightarrow$ C be a mapping. For a given $x_0 \in$ C, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n Ty_n, \\ y_n = (1-b_n)x_n + b_n Tz_n, \\ z_n = (1-c_n)x_n + c_n Tx_n, \end{cases} \quad n \geq 0 \quad (1.3)$$

where $\{a_n\}_{n=0}^{\infty}$,$\{b_n\}_{n=o}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are three real sequences in [0,1] satisfying some conditions.
If c_n =0, then (1.3) reduces to:

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n Ty_n, \\ y_n = (1-b_n)x_n + b_n Tx_n, \end{cases} \quad n \geq 0 \quad (1.4)$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=o}^{\infty}$ are two real sequence in [0,1] satisfying some conditions. Equation (1.4) is called the two-step (or Ishikawa iterative process) introduced by Ishikawa [8].
If c_n =0 and b_n =0, then (1.3) reduces to:

$$\{x_{n+1} = (1-a_n)x_n + a_n Tx_n, \quad n \geq 0 \quad (1.5)$$

which is called Mann iterative scheme introduced by Mann [5].
In 2004, Rhoades and Şoltuz [9], introduced a multistep iterative algorithm by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \end{cases} \quad n \geq 0 \tag{1.6}$$

By taking k = 3 and k = 2 in (1.6) we obtain the well-known Noor [4] and Ishikawa[8] iterative schemes, respectively.

In 2006, Rafiq [3] introduced the following modified three-step iterative scheme and used it to approximate the unique common fixed point of a family of strongly pseudo contractive operators.

Let T_1, T_2, T_3: C→ C be three given mappings. For a given $x_0 \in$ C, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n T_1 y_n, \\ y_n = (1-b_n)x_n + b_n T_2 z_n, \\ z_n = (1-c_n)x_n + c_n T_3 x_n, \end{cases} \quad n \geq 0 \tag{1.7}$$

Observe that if $T_1 = T_2 = T_3 = T$,then (1.7) reduces to (1.3).

In this paper, we shall employ the following iterative scheme, which is called a modified multi-step iterative scheme:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_{1n})x_n + \alpha_{1n} T_1 y_{1n}, \\ y_{jn} = (1-\alpha_{(j+1)n})x_n + \alpha_{(j+1)n} T_{j+1} y_{(j+1)n}, \\ y_{(k-1)n} = (1-\alpha_{kn})x_n + \alpha_{kn} T_k x_n, \end{cases} \quad n \geq 0 \tag{1.8}$$

where j=1,2,3,….,k-2 and $\{\alpha_{in}\}_{n=1}^{\infty}$ is a sequence in [0,1] for each i=1,2,…,k.

Observe that if, $T_1 = T_2 = T_3 = \ldots = T_k = T$ then (1.8) reduces to (1.6).

It may be noted that the iteration schemes (1.3)-(1.7) may be viewed as special cases of (1.8).

II. Preliminaries

We shall make use of following lemma

Lemma 1. [6] If σ is a real number such that $\sigma \in [0, 1)$, and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of nonnegative numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then, for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying

$$u_{n+1} = \sigma u_n + \varepsilon_n, \forall n \in \mathbb{N},$$

we have $\lim_{n\to\infty} u_n = 0$.

Remark 1 We observe that inequality (1.1) is equivalent to the following one:

T is strongly pseudocontractive with constant r ∈ (0,1) if and only if for all x, y ∈ C, the following inequality holds:

$$\|x-y\| \leq \|x-y+s[(I-T-rI)x-(I-T-rI)y]\| \text{ for all } s>0.$$

For rest of the paper, $L_i \geq 1$ denotes the Lipschitz constant of T_i and L= $\max_{1\leq i\leq k}\{L_i\}$.

III. Main Results

Theorem 3.1. Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i$: i=1,2…,k$\}$: C→ C be a finite family of Lipschitzian strongly pseudocontractive mappings with Lipschitz constant $\{L_i: i=1,2,\ldots,k\}$. Suppose that F= $\bigcap_{i=1}^{k} Fix(T_i) \neq \Phi$. Let the sequence defined by (1.8) satisfying the condition:

$$0 < a \leq \alpha_{2n} \leq \alpha_{1n} \leq r\left[2\left\{\left(1+L^k\right)\left(3-r+L\right)+L\right\}\right]^{-1}$$

Where L= $\max_{1\le i\le k}\{L_i\}$ and a>0 is a constant. Then $\{x_n\}$ converges strongly to unique common fixed point of $\{T_i: i=1,2,\dots,k\}$.

Proof. Assume that p $\in F=\bigcap_{i=1}^{k} Fix(T_i)$, using the fact the T_i is strongly pseudocontractive for each i=1,2,…,k, we obtain $F=\bigcap_{i=1}^{k} Fix(T_i)=p\neq\Phi$.

By (1.8), we have

$$\begin{aligned}\|y_{(k-1)n}-p\| &= \|(1-\alpha_{kn})(x_n-p)+\alpha_{kn}(T_kx_n-p)\|\\ &\le(1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}\|T_kx_n-p\|\\ &\le[1+\alpha_{kn}(L-1)]\|x_n-p\|\\ &\le L\|x_n-p\|\end{aligned}$$

and

$$\begin{aligned}\|y_{(k-2)n}-p\| &= \|(1-\alpha_{(k-1)n})(x_n-p)+\alpha_{(k-1)n}(T_{(k-1)}y_{(k-1)n}-p)\|\\ &\le(1-\alpha_{(k-1)n})\|x_n-p\|+\alpha_{(k-1)n}L\|y_{(k-1)n}-p\|\\ &\le(1-\alpha_{(k-1)n})\|x_n-p\|+\alpha_{(k-1)n}L^2\|x_n-p\|\\ &\le L^2\|x_n-p\|\end{aligned}$$

By induction, we obtain

$$\begin{aligned}\|y_{1n}-p\| &\le(1-\alpha_{2n})\|x_n-p\|+\alpha_{2n}\|T_2y_{2n}-p\|\\ &\le L^{k-1}\|x_n-p\| \qquad (3.1)\end{aligned}$$

It follows from (3.1) that

$$\begin{aligned}\|x_n-T_1y_{1n}\| &\le\|x_n-p\|+\|p-T_1y_{1n}\|\le\|x_n-p\|+L\|y_{1n}-p\|\\ &\le(1+L^k)\|x_n-p\| \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\|T_1x_{n+1}-T_1y_{1n}\| &\le L\|x_{n+1}-y_{1n}\|\\ &\le L\|x_n-y_{1n}+\alpha_{1n}(T_1y_{1n}-x_n)\|\\ &\le L\|x_n-y_{1n}\|+L\alpha_{1n}\|T_1y_{1n}-x_n\|\\ &\le L\|\alpha_{2n}(x_n-T_2y_{2n})\|+L\alpha_{1n}(1+L^k)\|x_n-p\| \qquad (3.3)\\ &\le\alpha_{2n}L\|x_n-T_2y_{2n}\|+\alpha_{1n}L(1+L^k)\|x_n-p\|\\ &\le\alpha_{2n}L(1+L^{k-1})\|x_n-p\|+\alpha_{1n}L(1+L^k)\|x_n-p\|\end{aligned}$$

By (1.8), we have

$$\begin{aligned}x_n &= x_{n+1}+\alpha_{1n}x_n-\alpha_{1n}T_1y_{1n}\\ &= x_{n+1}+2\alpha_{1n}x_n-\alpha_{1n}x_n+r\alpha_{1n}x_n-r\alpha_{1n}x_n-\alpha_{1n}T_1y_{1n}\\ &\quad+\alpha_{1n}T_1x_{n+1}-\alpha_{1n}T_1x_{n+1}\\ &= x_{n+1}+2\alpha_{1n}[x_{n+1}+\alpha_{1n}x_n-\alpha_{1n}T_1y_{1n}]-\alpha_{1n}x_n+r\alpha_{1n}x_n\\ &\quad-r\alpha_{1n}[x_{n+1}+\alpha_{1n}-\alpha_{1n}T_1y_{1n}]\\ &\quad-\alpha_{1n}T_1y_{1n}+\alpha_{1n}T_1x_{n+1}-\alpha_{1n}T_1x_{n+1}\\ &= (1+\alpha_{1n})x_{n+1}+\alpha_{1n}(I-T_1-rI)x_{n+1}-(1-r)\alpha_{1n}x_n\\ &\quad+(2-r)\alpha_{1n}^2(x_n-T_1y_{1n})+\alpha_{1n}(T_1x_{n+1}-T_1y_{1n}) \qquad (3.4)\end{aligned}$$

Observe that

$$p=(1+\alpha_{1n})p+\alpha_{1n}(I-T_1-rI)p-(1-r)\alpha_{1n}p \qquad (3.5)$$

Together with (3.4) and (3.5), we obtain

$$\begin{aligned} x_n - p = (1+\alpha_{1n})(x_{n+1} - p) + \alpha_{1n}[(I - T_1 - rI)x_{n+1} \\ -(I - T_1 - rI)p] - (1-r)\alpha_{1n}(x_n - p) \\ +(2-r)\alpha_{1n}^2(x_n - T_1 y_{1n}) + \alpha_{1n}(T_1 x_{n+1} - T_1 y_{1n}) \end{aligned} \tag{3.6}$$

It follows from Remark 1 and (3.6) that

$$\begin{aligned} \| x_n - p \| &\geq (1+\alpha_{1n}) \| (x_{n+1} - p) + \frac{\alpha_{1n}}{1+\alpha_{1n}}[(I - T_1 - rI)x_{n+1} \\ &\quad -(I - T_1 - rI)p] \| - (1-r)\alpha_{1n} \| x_n - p \| \\ &\quad -(2-r)\alpha_{1n}^2 \| x_n - T_1 y_{1n} \| - \alpha_{1n} \| T_1 x_{n+1} - T_1 y_{1n} \| \\ &\geq (1+\alpha_{1n}) \| x_{n+1} - p \| - (1-r)\alpha_{1n} \| x_n - p \| \\ &\quad -(2-r)\alpha_{1n}^2 \| x_n - T_1 y_{1n} \| - \alpha_{1n} \| T_1 x_{n+1} - T_1 y_{1n} \| \end{aligned}$$

This implies that

$$\begin{aligned} \| x_{n+1} - p \| &\leq \left(\frac{1}{1+\alpha_{1n}}\right) \| x_{n+1} - p \| + \frac{(1-r)\alpha_{1n}}{1+\alpha_{1n}} \| x_n - p \| \\ &\quad + \frac{(2-r)\alpha_{1n}^2}{1+\alpha_{1n}} \| x_n - T_1 y_{1n} \| + \frac{\alpha_{1n}}{1+\alpha_{1n}} \| T_1 x_{n+1} - T_1 y_{1n} \| \end{aligned} \tag{3.7}$$

We observe that

$$1 \leq 1 + \alpha_{1n}^3 = 1 + \alpha_{1n}^3 + \alpha_{1n}^2 - \alpha_{1n}^2 = (1+\alpha_{1n})(1 - \alpha_{1n} + \alpha_{1n}^2) \tag{3.8}$$

Using (3.8) in (3.7), we obtain

$$\begin{aligned} \| x_{n+1} - p \| &\leq (1 - \alpha_{1n} + \alpha_{1n}^2) \| x_n - p \| + (1-r)\alpha_{1n} \| x_n - p \| \\ &\quad + (2-r)\alpha_{1n}^2 \| x_n - T_1 y_{1n} \| + \alpha_{1n} \| T_1 x_{n+1} - T_1 y_{1n} \| \end{aligned} \tag{3.9}$$

Using (3.2) and (3.3) in (3.9), we obtain

$$\begin{aligned} \| x_{n+1} - p \| &\leq (1 - \alpha_{1n} + \alpha_{1n}^2) \| x_n - p \| + (1-r)\alpha_{1n} \| x_n - p \| \\ &\quad + (2-r)\alpha_{1n}^2(1+L^k) \| x_n - p \| + \alpha_{1n}\alpha_{2n}L(1+L^{k-1}) \| x_n - p \| \\ &\quad + \alpha_{1n}^2 L(1+L^k) \| x_n - p \| \\ &\leq (1 - r\alpha_{1n}) \| x_n - p \| + \alpha_{1n}^2[(1+L^k)(2-r) \\ &\quad + 1 + L(1+L^k)] \| x_n - p \| + \alpha_{1n}\alpha_{2n}L(1+L^{k-1}) \| x_n - p \| \\ &\leq \left(1 - \frac{r\alpha_{1n}}{2}\right) \| x_n - p \| + \alpha_{1n}^2[(1+L^k)(L+2-r) \\ &\quad + 1] \| x_n - p \| + \alpha_{1n}\alpha_{2n}L(1+L^{k-1}) \| x_n - p \| \\ &\quad - \frac{r\alpha_{1n}}{2} \| x_n - p \| \\ &\leq \left(1 - \frac{ra}{2}\right) \| x_n - p \| + \alpha_{1n}[\alpha_{1n}\{(1+L^k)(3-r+L) + L\} \\ &\quad - \frac{r}{2}] \| x_n - p \| \\ &\leq \left(1 - \frac{ra}{2}\right) \| x_n - p \| \end{aligned}$$

Set $u_{n+1} = \| x_n - p \|$, $\varepsilon_n = 0$ and $\sigma = 1 - \frac{ra}{2}$

Applying lemma 1, we obtain $\| x_n - p \| \to 0$ as $n \to \infty$.

This completes the proof.

Acknowledgement

The work is financially supported by the Council for Scientific and Industrial Research (CSIR), New Delhi.

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