MHD Flow of an Unsteady Dusty Fluid Through an Inclined Channel in Anholonomic Co-ordinate System

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Abstract: An analytical study of the MHD flow of a conducting dusty fluid through an inclined channel under gravity in Anholonomic co-ordinate system has been studied. The flow is due to the influence of time dependent pressure gradient along with the effect of the movement of the plates and uniform magnetic field. Flow analysis is carried out using differential geometry techniques and exact solutions of the problem are obtained using Laplace transform and variable separable methods, also which are discussed with the help of graphs drawn for different values of inclined angle, time parameter and number density of dust particles. Further the expressions for skin-friction are obtained at the boundaries.

Key Words: Frenet frame field system; inclined angle, channel, dusty fluid; velocity of dust phase and fluid phase, conducting dusty fluid, magnetic field.

AMS Subject Classification (2000): 76T10, 76T15;

1. Introduction

The inclined channel with a free surface has many important applications in the designs of drainage, irrigation canals, flood discharge channels, coating to paper rolls and in the field of hydraulic engineering. Hence the flow of a liquid in an inclined channel with a free surface under gravity has long been studied experimentally and several empirical results have been reported by many investigators like Vanoni [26] and Johnson [10].

The analytical results for velocity distribution and shearing stress of an unsteady laminar viscous fluid flow down an open channel with permeable bottom under the action of gravity has been discussed by Satya Prakash [21]. Ghosh and Debnath [8] have studied the hydromagnetic flow of a visco-elastic fluid between infinite parellel plate. Verna and Vyass [25] derived analytically, the velocity distribution and flux under steady laminar flow down an open inclined channel. Also, Singh [19] investigated the unsteady laminar flow of an incompressible viscous fluid between porous parallel plates.

The basic equations for the flow of dusty fluid were formulated by Saffman [18]. Since then many researchers have discussed the problem of dusty fluid. Some researchers like Micheal and Miller [14], Mitra [15], Norey [16], Amos [1], Liu [12], Saxena and Sharma [22], Kaur and Sharma [13], Ghosh [8], Agrawal and Varshney [2] have studied various problems under different intial and boundary conditions. Shri Ram, B.K.Gupta and N.P.Singh [20] have studied unsteady flow of a dusty viscous stratified fluid through an inclined open rectangular channel.

Frenet frames are a central construction in modern differential geometry, in which structure is described with respect to an object of interest rather than with respect to external coordinate systems. To investigate the kinematical properties of fluid flows in the field of fluid mechanics some researchers like Kanwal [11], Truesdell [23], Indrasena [9], Purushotham [17], Bagewadi and Gireesha [3], [4] have applied differential geometry techniques. Further, recently the authors [3], [4] have studied dusty fluid flow in Frenet frame field system, with appropriate boundary conditions.

In the present paper we have discussed an unsteady flow of a conducting dusty fluid through an inclined rectangular channel with naturally permeable bed, impermeable vertical walls under the influence of uniform magnetic field and an arbitrary time varying pressure gradient in Frenet frame field system. Here we have taken, the boundary conditions at the interface of the free flow region and porous medium is as suggested by Beaver's and Joseph [7]. Also we considered fluid flow in porous medium governed by Darcy's law and fluid in free flow region governed by Navier-Stoke's equations. Further by considering the fluid and dust particles are at rest initially, the analytical expressions are obtained for velocities of fluid and dust particles using Laplace transform and variable separable techniques. The changes in the velocity profiles for different values of inclined angle γ, time parameter t and number density of dust particles N are shown graphically.

2. Equations of Motion

The equations of motion of an unsteady viscous incompressible fluid with uniform distribution of dust particles are given by [18]:

For fluid phase

$$\nabla \cdot \overrightarrow{u} = 0, \qquad \text{(Continuity)} \tag{2.1}$$

$$\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u} \cdot \nabla)\overrightarrow{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \overrightarrow{u} + \frac{kN}{\rho}(\overrightarrow{v} - \overrightarrow{u}) + g\sin\gamma - \frac{\sigma B^2}{\rho}\overrightarrow{u} \qquad \text{(LinearMomentum)} \tag{2.2}$$

For dust phase

$$\nabla \cdot \overrightarrow{v} = 0, \qquad \text{(Continuity)} \tag{2.3}$$

$$\frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v} \cdot \nabla)\overrightarrow{v} = \frac{k}{m}(\overrightarrow{u} - \overrightarrow{v}) \qquad \text{(Linear Momentum)} \tag{2.4}$$

We have following nomenclature:

$\overrightarrow{u}-$velocity of the fluid phase, $\overrightarrow{v}-$velocity of dust phase, $\rho-$density of the gas, $p-$pressure of the fluid, $N-$number density of dust particles, $\nu-$kinematic viscosity, $k=6\pi a\mu-$ Stoke's resistance (drag coefficient), $a-$spherical radius of dust particle, $m-$mass of the dust particle, $\mu-$the co-efficient of viscosity of fluid particles, $t-$time, $g-$ the acceleration due to gravity, $\sigma-$ is the electrical conductivity, $B-$ is variable electromagnetic induction, $\gamma-$ is an inclined angle.

3. Frenet Frame Field System

Let $\overrightarrow{s}, \overrightarrow{n}, \overrightarrow{b}$ be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity and dusty phase velocity lines respectively as shown in the figure-1.

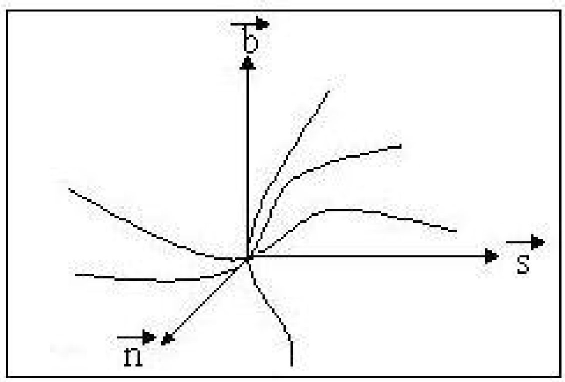

Figure-1: Frenet Frame Field System

Geometrical relations are given by Frenet formulae [6]

$$i)\quad \frac{\partial \overrightarrow{s}}{\partial s}=k_s\overrightarrow{n},\ \frac{\partial \overrightarrow{n}}{\partial s}=\tau_s\overrightarrow{b}-k_s\overrightarrow{s},\ \frac{\partial \overrightarrow{b}}{\partial s}=-\tau_s\overrightarrow{n}$$

$$ii)\quad \frac{\partial \overrightarrow{n}}{\partial n}=k'_n\overrightarrow{s},\ \frac{\partial \overrightarrow{b}}{\partial n}=-\sigma'_n\overrightarrow{s},\ \frac{\partial \overrightarrow{s}}{\partial n}=\sigma'_n\overrightarrow{b}-k'_n\overrightarrow{n}$$

$$iii)\quad \frac{\partial \overrightarrow{b}}{\partial b}=k''_b\overrightarrow{s},\ \frac{\partial \overrightarrow{n}}{\partial b}=-\sigma''_b\overrightarrow{s},\ \frac{\partial \overrightarrow{s}}{\partial b}=\sigma''_b\overrightarrow{n}-k''_b\overrightarrow{b}$$

$$iv)\quad \nabla.\overrightarrow{s}=\theta_{ns}+\theta_{bs};\ \nabla.\overrightarrow{n}=\theta_{bn}-k_s;\ \nabla.\overrightarrow{b}=\theta_{nb} \tag{3.1}$$

where $\partial/\partial s$, $\partial/\partial n$ and $\partial/\partial b$ are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, principal normal and binormal. The functions (k_s, k'_n, k''_b) and $(\tau_s, \sigma'_n, \sigma''_b)$ are the curvatures and torsions of the above curves and θ_{ns} and θ_{bs} are normal deformations of these spatial curves along their principal normal and binormal respectively.

4. Formulation and Solution of the Problem

Consider an unsteady laminar flow of an incompressible electrically conducting dusty fluid down in an inclined channel. The walls of the channel are taken to be normal to the surface of the bottom. The bottom is assumed to be inclined at an angle γ $(0 < \gamma < \pi/2)$ to the horizontal. The flow is assumed to along the binormal direction $\overrightarrow{b}$, and $\overrightarrow{n}$ is perpendicular to it as shown in the figure-2.

Both the fluid and the dust particles are supposed to be static in the beginning. The shape and size of the dust particles are assumed to be spherical and uniform respectively. Under the influence of time dependent pressure gradient, the flow is flowing down an inclined channel with permeable bottom under the action of gravity. For the above described flow the velocity components of both fluid and dust particles are respectively given by:

$$\left\{ \begin{array}{l} u_s = 0; \;\; u_n = 0; \\ v_s = 0; \;\; v_n = 0; \end{array} \right\}$$

$$i.e. \quad \overrightarrow{u} = u_b \overrightarrow{b}, \quad \overrightarrow{v} = v_b \overrightarrow{b} \tag{4.1}$$

where (u_s, u_n, u_b) and (v_s, v_n, v_b) are velocity components of fluid and dust particles respectively.

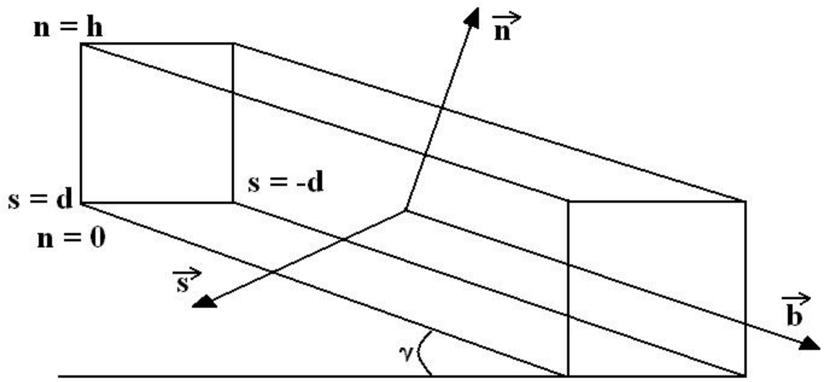

Figure 2: Schematic diagram of dusty fluid flow in a channel.

By virtue of system of equations (3.1) the intrinsic decomposition of equations (2.2) and (2.4) using equation (4.1) give the following forms:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \nu\left(\tau_s k_s u_b - 2\sigma'_n \frac{\partial u_b}{\partial n}\right) \tag{4.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} + \nu\left(\sigma'_n k'_n u_b + k''_b \sigma''_b u_b - 2\tau_s \frac{\partial u_b}{\partial s}\right) \tag{4.3}$$

$$\frac{\partial u_b}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial b} + \nu\left[\frac{\partial^2 u_b}{\partial s^2} + \frac{\partial^2 u_b}{\partial n^2} - C_r u_b\right] + \frac{kN}{\rho}(v_b - u_b) + g\sin\gamma - \frac{\sigma B^2}{\rho}u_b \tag{4.4}$$

$$\frac{\partial v_b}{\partial t} = \frac{k}{m}(u_b - v_b) \tag{4.5}$$

$$v_b^2 k''_b = 0 \tag{4.6}$$

where $C_r = (\tau_s^2 + \sigma_n'^2 + k''^2{}_b)$ is called curvature number [4].

From equation (4.6) we see that $v_b^2 k''_b = 0$ which implies either $v_b = 0$ or $k''_b = 0$. The choice $v_b = 0$ is impossible, since if it happens then $u_b = 0$, which shows that the flow doesn't exist. Hence $k''_b = 0$, it suggests that the curvature of the streamline along binormal direction is zero. Thus no radial flow exists.

The flow in the porous media is governed by the Darcy's equation,

$$Q = \frac{K_0}{\mu}\left(-\frac{\partial p}{\partial b} + \rho\, g\sin\gamma\right) \tag{4.7}$$

where Q is the velocity in the porous media and K_0 is the variable permeability of the medium.

The condition at the interface of the free flow region and porous medium, following Beavers and Joseph [7] is given by

$$\left(\frac{\partial u_s}{\partial n}\right)_{n=h} = \frac{-\alpha}{\sqrt{K}}(u_s - Q)$$
$$\left(\frac{\partial v_s}{\partial n}\right)_{n=h} = \frac{-\alpha}{\sqrt{K}}(v_s - Q)$$

From equations (4.4) and (4.7),we have

$$\frac{\partial u_b}{\partial t} = \frac{Q\mu}{\rho K_0} + \nu\left[\frac{\partial^2 u_b}{\partial s^2} + \frac{\partial^2 u_b}{\partial n^2} - C_r u_b\right] + \frac{kN}{\rho}(v_b - u_b) - \frac{\sigma B^2}{\rho}u_b \tag{4.8}$$

By defining the depth of the channel h as the characteristic length and the mean flow velocity U_0 as the characteristic velocity. We introduce the following non-dimensional quantities.

$$S^* = \frac{s}{h},\quad n^* = \frac{n}{h},\quad b^* = \frac{b}{h},\quad u_b^* = \frac{\mu M^2 u_b}{U_0},\quad v_b^* = \frac{\mu M^2 v_b}{U_0},\quad Q^* = \frac{\mu M^2 Q}{U_0}$$
$$k_0^* = \frac{k_0}{h^2},\quad t_0^* = \frac{\mu t}{\rho h^2},\quad u_1^* = \frac{\rho M^2 h^3 u_1}{U_0},\quad u_0^* = \frac{\rho M^2 h^2 u_0}{U_0},\quad T^* = \frac{\mu T}{\rho h^2}$$

The equations (4.5) and (4.8)transformed to(after dropping the asterisks over them)

$$\frac{\partial u_b}{\partial t} = P + \nu\left[\frac{\partial^2 u_b}{\partial s^2} + \frac{\partial^2 u_b}{\partial n^2} - C_r u_b\right] + \frac{l}{\omega}(v_b - u_b) - M^2 u_b \tag{4.9}$$

$$\frac{\partial v_b}{\partial t} = \frac{1}{\omega}(u_b - v_b) \tag{4.10}$$

where $M^2 = \frac{\sigma B^2 h^2}{\mu}$, $P = \frac{M^2 h^2}{U_0}\left[-\frac{\partial p}{\partial b} + \rho g \sin\gamma\right]$, $l = \frac{mN}{\rho}$ and $\omega = \frac{m\mu}{kh^2\rho}$.

Since we have assumed that the constant pressure gradient to be impressed on the system for $t > 0$, we can write

$$-\frac{\partial p}{\partial b} = p_0 + p_1 t$$

The initial and boundary conditions are

$$\begin{aligned} u_b &= 0, \quad v_b = 0 && \text{at} \quad t = 0 \\ u_b &= u_1 e^{iw_1 t} + u_2 e^{-iw_2 t} && \text{at} \quad s = \pm d \\ u_b &= 0 && \text{at} \quad n = 0 \quad \text{and} \quad n = 1 \end{aligned}$$

where w_1 and w_2 are constants. Let U_b and V_b are given by

$$U_b = \int_0^\infty e^{-xt} \; u_b \; dt \quad \text{and} \quad V_b = \int_0^\infty e^{-xt} \; v_b \; dt \tag{4.11}$$

denote the Laplace transforms of u_b and v_b respectively.

Then (4.9) and (4.10) becomes,

$$xU_b = P(x) + \nu \left(\frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - C_r U_b \right) + \frac{l}{\omega}(V_b - U_b) - M^2 U_b \tag{4.12}$$

$$xV_b = \frac{1}{w}(U_b - V_b) \quad V_b = \frac{U_b}{(1 + x\omega)} \tag{4.13}$$

and boundary conditions are,

$$\begin{aligned} P(x) &= \frac{p_0}{x} + \frac{p_1}{x^2} \\ U_b &= \frac{u_1}{x - iw_1} + \frac{u_2}{x + iw_2} && \text{at} \quad s = \pm r \\ U_b &= 0 && \text{at} \quad n = 0 \quad \text{and} \quad n = 1 \end{aligned} \tag{4.14}$$

From equations (4.13) and (4.14) we obtain, the following equation

$$\begin{aligned} \frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - q^2 U_b &= \frac{-p(x)}{\nu} \\ \frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - q^2 U_b &= -\frac{M^2 h^2}{U_0 \nu} \left[\frac{p_0 x + p_1 + \rho g \sin \gamma x^2}{x^2} \right] \end{aligned}$$

$$\frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - q^2 U_b + R = 0 \tag{4.15}$$

where

$$q^2 = \left(C_r + M^2 + \frac{x}{\nu} + \frac{xl}{\nu(1+x\omega)}\right)$$

$$\text{and} \quad R = \frac{M^2h^2}{U_0\nu}\left[\frac{p_0x + p_1 + \rho g \sin\gamma x^2}{x^2}\right]$$

To solve equation (4.12) we assume the solution in the following form

$$U_b(s,n) = w_1(s,n) + w_2(s) \tag{4.16}$$

Substitution of $U_b(s,n)$ in equation (4.15) yields

$$\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_2}{\partial s^2} + \frac{\partial^2 w_1}{\partial n^2} - q^2(w_1 + w_2) + R = 0$$

so that if w_2 satisfies

$$\frac{\partial^2 w_2}{\partial s^2} - q^2 w_2 + R = 0$$

then

$$\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_1}{\partial n^2} - q^2 w_1 = 0 \tag{4.17}$$

In similar manner if $U_b(s,n)$ is inserted in no slip boundary conditions, one can obtain

$$\left\{\begin{array}{l} U_b(r,n) = w_1(r,n) + w_2(r) = \frac{u_1}{(x-iw_1)} + \frac{u_2}{(x+iw_2)} \\ U_b(-r,n) = w_1(-r,n) + w_2(-r) = \frac{u_1}{(x-iw_1)} + \frac{u_2}{(x+iw_2)} \\ U_b(s,0) = w_1(s,0) + w_2(s) = 0 \\ U_b(s,1) = w_1(s,1) + w_2(s) = 0 \end{array}\right\}$$

By solving the problem

$$\frac{\partial^2 w_2}{\partial s^2} - q^2 w_2 + R = 0$$

$$w_2(r) = \frac{u_1}{(x-iw_1)} + \frac{u_2}{(x+iw_2)}$$

$$w_2(-r) = \frac{u_1}{(x-iw_1)} + \frac{u_2}{(x+iw_2)}$$

we obtain the solution in the form

$$(4.18)\qquad w_2(s)=\left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\left(\frac{\cosh(qs)}{\cos(qr)}\right)-\frac{R}{q^2}\left[\frac{\cosh(qs)}{\cosh(qr)}-1\right]$$

Using variable separable method, the solution of the problem (4.17) with the conditions

$$w_1(r,n)=0,\qquad w_1(-r,n)=0,\qquad w_1(s,0)=-w_2(s),\qquad w_1(s,1)=-w_2(s)$$

is obtained in the form

$$\begin{aligned}(4.19)\ w_1(s,n) &= 2\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{r}s\right)\left\{\left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\frac{r_1\pi}{r^2A^2\cosh(qr)}\right.\\ &- \left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\frac{(-1)^{r_1}r_1\pi}{r^2A^2}+\frac{R}{q^2}\frac{r_1\pi}{r^2A^2\cosh(qr)}\\ &- \left.\frac{R}{q^2}\frac{(-1)^{r_1}r_1\pi}{r^2A^2}-\frac{R}{q^2}\frac{[1-(-1)^{r_1}]}{r_1\pi}\right\}\left(\frac{[\sinh(A(n-1))-\sinh(An)]}{\sinh(A)}\right)\end{aligned}$$

where $A=\sqrt{\frac{q^2r^2+r_1^2\pi^2}{r^2}}$

Now by substituting (4.18) and (4.19) in (4.16) we have

$$\begin{aligned}U_b(s,n) &= \left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\left(\frac{\cosh(qs)}{\cosh(qr)}\right)-\frac{R}{q^2}\left[\frac{\cosh(qs)}{\cosh(qr)}-1\right]+\frac{2\pi}{r^2}\\ &\times \sum_{r_1=0}^{\infty}r_1\sin\left(\frac{r_1\pi}{r}s\right)\left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\left\{\frac{[\sinh(A(n-1))-\sinh(An)]}{A^2\cosh(qr)\sinh(A)}\right\}\\ &- \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)\left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\\ &\times \left\{\frac{[\sinh(A(n-1))-\sinh(An)]}{A^2\sinh(A)}\right\}+\frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}r_1\sin\left(\frac{r_1\pi}{r}s\right)\\ &\times \left\{\frac{R}{q^2}\frac{[\sinh(A(n-1))-\sinh(An)]}{A^2\sinh(A)\cosh(qr)}\right\}-\frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)\end{aligned}$$

$$\times \left\{\frac{R}{q^2}\frac{[\sinh(A(n-1))-\sinh(An)]}{A^2\sinh(A)}\right\} - \frac{2}{\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)^{r_1}]}{r_1}\sin\left(\frac{r_1\pi}{r}s\right)$$
$$\times \left\{\frac{R}{q^2}\frac{[\sinh(A(n-1))-\sinh(An)]}{\sinh(A)}\right\}$$

$$\begin{aligned}
V_b(s,n) &= \frac{1}{(1+x\omega)}\left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\left(\frac{\cosh(qs)}{\cosh(qr)}\right) - \frac{R}{q^2}\frac{1}{(1+x\omega)}\\
&\times \left[\frac{\cosh(qs)}{\cosh(qr)}-1\right] + \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty} r_1\sin\left(\frac{r_1\pi}{r}s\right)\left[\frac{u_1}{(x-iw_1)}+\frac{u_2}{(x+iw_2)}\right]\\
&\times \left\{\frac{[\sinh(A(n-1))-\sinh(An)]}{(1+x\omega)A^2\cosh(qr)\sinh(A)}\right\} - \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)\left[\frac{u_1}{(x-iw_1)}\right.\\
&+ \left.\frac{u_2}{(x+iw_2)}\right]\left\{\frac{[\sinh(A(n-1))-\sinh(An)]}{(1+x\omega)A^2\sinh(A)}\right\} + \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty} r_1\sin\left(\frac{r_1\pi}{r}s\right)\\
&\times \left\{\frac{R}{q^2}\frac{[\sinh(A(n-1))-\sinh(An)]}{(1+x\omega)A^2\sinh(A)\cosh(qr)}\right\} - \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)\\
&\times \left\{\frac{R}{q^2}\frac{[\sinh(A(n-1))-\sinh(An)]}{(1+x\omega)A^2\sinh(A)}\right\} - \frac{2}{\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)^{r_1}]}{r_1}\sin\left(\frac{r_1\pi}{r}s\right)\\
&\times \left\{\frac{R}{q^2}\frac{[\sinh(A(n-1))-\sinh(An)]}{(1+x\omega)\sinh(A)}\right\}
\end{aligned}$$

$$\begin{aligned}
u_b(s,n,t) &= \frac{u_1(\phi_1+i\phi_2)}{(C^2+D^2)} + \frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cos\left[\frac{(2r_2+1)\pi}{2r}s\right]\\
&\times \left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\omega)^2}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]}+\frac{e^{x_4t}(x_4+iw_1)(1+x_4\omega)^2}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right] + \frac{u_2(\phi_3-i\phi_4)}{(K^2+L^2)}\\
&+ \frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cos\left[\frac{(2r_2+1)\pi}{2r}s\right]\left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\omega)^2}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]}\right.\\
&+ \left.\frac{e^{x_4t}(x_4-iw_2)(1+x_4\omega)^2}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\right] - \frac{J}{\nu}\left\{\frac{1}{X^2\cosh(Xr)}\left[(p_0+p_1t)\Big(\cosh(Xs)\right.\right.\\
&- \left.cosh(Xr)\Big) + \frac{p_1(1+l)}{2\nu X^3\cosh(Xr)}\Big(s\sinh(Xs)-r\sinh(Xr)\Big)\right]\\
&- \frac{p_1(\cosh(Xs)-\cosh(Xr))}{X^4\cosh^2(Xr)}\left[\frac{(1+l)}{2\nu}\Big(Xr\sinh(Xr)+2\cosh(Xr)\Big)\right] - \frac{4\nu}{\pi}\\
&\times \sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\cos\left[\frac{(2r_2+1)\pi}{2r}s\right]\left[\frac{e^{x_3t}(1+x_3\omega)^2[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]}\right.
\end{aligned}$$

$$+ \quad \frac{e^{x_4 t}(1+x_4\omega)^2\left[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2\right]}{x_4^2[l+(1+x_4\omega)^2]}\Bigg]\Bigg\} + \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty} r_1 \sin\left(\frac{r_1\pi}{r}s\right)$$

$$\times \quad \Bigg\{\frac{u_1\left(\phi_5+i\phi_6\right)}{(O^2+P^2)(R_1'^2+R_2^2)(C^2+D^2)} + \frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)}{\beta^2}$$

$$\times \quad \frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\sinh(\beta)}\Bigg[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\omega)^2}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]}$$

$$+ \quad \frac{e^{x_4 t}(x_4+iw_1)(1+x_4\omega)^2}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\Bigg] + \frac{2u_1\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\frac{\sin(r_2 n\pi)}{\cos(\alpha_1 r)}$$

$$\times \quad \Bigg[\frac{e^{x_7 t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\Bigg]$$

$$+ \quad \frac{u_2\left(\phi_7-i\phi_8\right)}{(O'^2+P'^2)(R_3'^2+R_4^2)(K^2+L^2)} + \frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)}{\beta^2}$$

$$\times \quad \frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\sinh(\beta)}\Bigg[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\omega)^2}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]}$$

$$+ \quad \frac{e^{x_4 t}(x_4-iw_2)(1+x_4\omega)^2}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\Bigg] + \frac{2u_2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\frac{\sin(r_2 n\pi)}{\cos(\alpha_1 r)}$$

$$\times \quad \Bigg[\frac{e^{x_7 t}(x_7-iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8-iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\Bigg]\Bigg\} - \frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1$$

$$\times \quad \sin\left(\frac{r_1\pi}{r}s\right)\Bigg\{\frac{u_1\left(\phi_9-i\phi_{10}\right)}{(R_1'^2+R_2^2)(O^2+P^2)} + \frac{2u_1\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\sin(r_2 n\pi)$$

$$\times \quad \Bigg[\frac{e^{x_7 t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\Bigg] + \frac{u_2\left(\phi_{11}+i\phi_{12}\right)}{(R_3'^2+R_4^2)(O'^2+P'^2)}$$

$$+ \quad \frac{2u_2\nu}{\pi}\sum_{r_2=0}^{\infty}[1-(-1)^{r_2}]r_2\sin(r_2 n\pi)\Bigg[\frac{e^{x_7 t}(x_7-iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]}$$

$$+ \quad \frac{e^{x_8 t}(x_8-iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\Bigg]\Bigg\} + \frac{2\pi J}{r^2\nu}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{r}s\right)\Bigg\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}$$

$$\times \quad \Bigg[(p_0+p_1t)\Big(\sinh(Y(n-1))-\sinh(Yn)\Big) + \frac{p_1(1+l)}{2Y\nu}\Big((n-1)\cosh(Y(n-1))$$

$$- \quad n\cosh(Yn)\Big)\Bigg] - \frac{p_1(\sinh(Y(n-1))-\sinh(Yn))}{X^4Y^4\cosh^2(Xr)\sinh^2(Y)}\Bigg[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y)$$

$$+ \quad \frac{r(1+l)}{2\nu}XY^2\sinh(Xr)\sinh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\cosh(Xr)\sinh(Y)\Bigg] - \frac{4\nu}{\pi}$$

$$\times \quad \sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\beta^2\sinh(\beta)}$$

$$
\begin{aligned}
\times & \left[\frac{e^{x_3 t}(1+x_3\omega)^2[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]}\right.\\
+ & \left.\frac{e^{x_4 t}(1+x_4\omega)^2[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\right] - \frac{2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\left[\frac{\sin(r_2 n\pi)}{\alpha_1^2\cos(\alpha_1 r)}\right]\\
\times & \left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]}\right.\\
+ & \left.\frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\Bigg\} - \frac{2\pi J}{\nu r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\\
\times & \sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\sinh(Y)}\left[(p_0+p_1t)\sinh(Y(n-1))-\sinh(Yn)\right.\right.\\
+ & \left.\frac{p_1(1+l)}{2\nu Y}\Big((n-1)\cosh(Y(n-1))-n\cosh Yn\Big)\right]\\
- & \frac{p_1(\sinh(Y(n-1))-\sinh(Yn))}{X^4Y^4\sinh^2 Y}\left[\frac{(1+l)}{2\nu}X^2Y\cosh(Y)+\frac{(1+l)}{\nu}\right.\\
\times & \left.(X^2+Y^2)\sinh(Y)\right] - \frac{2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\left[\frac{\sin(r_2 n\pi)}{\alpha_1^2}\right]\\
\times & \left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]}\right.\\
+ & \left.\frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\Bigg\} - \frac{2J}{\nu\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)^{r_1}]}{r_1}\\
\times & \sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2\sinh(Y)}\left[(p_0+p_1t)\Big(\sinh(Y(n-1))-\sinh(Yn)\Big)\right.\right.\\
+ & \left.\frac{p_1(1+l)}{2\nu Y}\Big((n-1)\cosh(Y(n-1))-n\cosh(Y)n\Big)\right]\\
- & \frac{p_1(\sinh(Y(n-1))-\sinh(Yn))}{X^3Y\sinh^2(Y)}\left[\frac{(1+l)}{2\nu}X\cosh(Y)+\frac{(1+l)}{\nu}\sinh(Y)\right]\\
+ & \frac{(\sinh(\alpha(n-1))-\sinh(\alpha n))}{(x_1-x_2)\sinh(\alpha)}\left[\frac{e^{x_1 t}[p_1+p_0x_1+(\rho g\sin\gamma)x_1^2]}{x_1^2}\right.\\
- & \left.\frac{e^{x_2 t}[p_1+p_0x_2+(\rho g\sin\gamma)x_2^2]}{x_2^2}\right] + 2\nu\pi\sum_{r_2=0}^{\infty}[1-(-1)^{r_1}]r_2\frac{\sin(r_2 n\pi)}{\alpha_1^2}\\
\times & \left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]}\right.\\
+ & \left.\frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
v_b(s,n,t) &= \frac{u_1\,(\phi_1'+i\phi_2')}{(C^2+D^2)(1+\omega w_1^2)}+\frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cos\left[\frac{(2r_2+1)\pi}{2r}s\right]\\
&\times \left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\omega)}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]}+\frac{e^{x_4t}(x_4+iw_1)(1+x_4\omega)}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right]\\
&+ \frac{u_2\,(\phi_3'-i\phi_4')}{(K^2+L^2)(1+\omega w_1^2)}+\frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cos\left[\frac{(2r_2+1)\pi}{2r}s\right]\\
&\times \left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\omega)}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]}+\frac{e^{x_4t}(x_4-iw_2)(1+x_4\omega)}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\right]\\
&- \frac{J}{\nu}\left\{\frac{1}{X^2\cosh(Xr)}\left[(p_0+p_1t)\Big(cosh(Xs)-cosh(Xr)\Big)+\frac{p_1(1+l)}{2\nu X^3\cosh(Xr)}\right.\right.\\
&\times \left.\Big(s\sinh(Xs)-r\sinh(Xr)\Big)\right]-\frac{p_1(\cosh(Xs)-\cosh(Xr))}{X^4\cosh^2(Xr)}\left[\frac{(1+l)}{2\nu}\right.\\
&\times \left.(Xr\sinh(Xr)+2\cosh(Xr))+\omega X^2\cosh(Xr)\right]-\frac{4\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\\
&\times \cos\left[\frac{(2r_2+1)\pi}{2r}s\right]\left[\frac{e^{x_3t}(1+x_3\omega)\,[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]}\right.\\
&+ \left.\left.\frac{e^{x_4t}(1+x_4\omega)\,[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\right]\right\}+\frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}r_1\sin\left(\frac{r_1\pi}{r}s\right)\\
&\times \left\{\frac{u_1\,(\phi_5'+i\phi_6')}{(O^2+P^2)({R_1'}^2+R_2^2)(C^2+D^2)(1+\omega w_2^2)}+\frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\right.\\
&\times \frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\beta^2\sinh(\beta)}\left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\omega)}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]}\right.\\
&+ \left.\frac{e^{x_4t}(x_4+iw_1)(1+x_4\omega)}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right]+\frac{2u_1\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\frac{\sin(r_2n\pi)}{\cos(\alpha_1r)}\\
&\times \left[\frac{e^{x_7t}(x_7+iw_1)(1+x_7\omega)}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]}+\frac{e^{x_8t}(x_8+iw_1)(1+x_8\omega)}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\right]\\
&+ \frac{u_2\,(\phi_7'-i\phi_8')}{(O'^2+P'^2)({R_3'}^2+R_4^2)(K^2+L^2)(1+\omega w_1^2)}+\frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\\
&\times \frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\beta^2\sinh(\beta)}\left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\omega)}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]}\right.\\
&+ \left.\frac{e^{x_4t}(x_4-iw_2)(1+x_4\omega)}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right]+\frac{2u_2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\frac{\sin(r_2n\pi)}{\cos(\alpha_1r)}\\
&\times \left.\left[\frac{e^{x_7t}(x_7-iw_2)(1+x_7\omega)}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]}+\frac{e^{x_8t}(x_8-iw_2)(1+x_8\omega)}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\right]\right\}-\frac{2\pi}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1
\end{aligned}
$$

$$\times \quad \sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{u_1\,(\phi_9' - i\phi_{10}')}{(R_1'^{\,2} + R_2^2)(O^2 + P^2)(1 + \omega w_1^2)} + \frac{2u_1\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1 - (-1)^{r_2}]}{r_2}\right.$$

$$\times \quad \sin(r_2 n\pi)\left[\frac{e^{x_7 t}(x_7 + iw_1)(1 + x_7\omega)}{(x_7^2 + w_1^2)[l + (1 + x_7\omega)^2]} + \frac{e^{x_8 t}(x_8 + iw_1)(1 + x_8\omega)}{(x_8^2 + w_1^2)[l + (1 + x_8\omega)^2]}\right]$$

$$+ \quad \frac{u_2\,(\phi_{11}' + i\phi_{12}')}{(R_3'^{\,2} + R_4^2)(O'^2 + P'^2)(1 + \omega w_1^2)} + \frac{2u_2\nu}{\pi}\sum_{r_2=0}^{\infty}[1 - (-1)^{r_2}]r_2\sin(r_2 n\pi)$$

$$\times \quad \left.\left[\frac{e^{x_7 t}(x_7 - iw_2)(1 + x_7\omega)}{(x_7^2 + w_2^2)[l + (1 + x_7\omega)^2]} + \frac{e^{x_8 t}(x_8 - iw_2)(1 + x_8\omega)}{(x_8^2 + w_2^2)[l + (1 + x_8\omega)^2]}\right]\right\}$$

$$+ \quad \frac{2\pi J}{r^2\nu}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}\left[(p_0 + p_1 t)\Big(\sinh(Y(n-1))\right.\right.$$

$$- \quad \left.\sinh(Yn)\Big) + \frac{p_1(1+l)}{2Y\nu}\Big((n-1)\cosh(Y(n-1)) - n\cosh(Yn)\Big)\right]$$

$$- \quad \frac{p_1[\sinh(Y(n-1)) - \sinh(Yn)]}{X^4Y^4\cosh^2(Xr)\sinh^2(Y)}\left[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y) + \frac{r(1+l)}{2\nu}XY^2\right.$$

$$\times \quad \sinh(Xr)\sinh(Y) + \frac{(1+l)}{\nu}(X^2 + Y^2)\cosh(Xr)\sinh(Y) + \omega X^2Y^2\sinh(Y)$$

$$\times \quad \left.\cosh(Xr)\right] - \frac{4\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[\sinh(\beta(n-1)) - \sinh(\beta n)]}{\beta^2\sinh(\beta)}$$

$$\times \quad \left[\frac{e^{x_3 t}(1 + x_3\omega)[p_1 + p_0x_3 + (\rho g\sin\gamma)x_3^2]}{x_3^2[l + (1 + x_3\omega)^2]} + \frac{e^{x_4 t}(1 + x_4\omega)[p_1 + p_0x_4 + (\rho g\sin\gamma)x_4^2]}{x_4^2[l + (1 + x_4\omega)^2]}\right]$$

$$- \quad \frac{2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1 - (-1)^{r_2}]}{r_2}\left[\frac{\sin(r_2 n\pi)}{\alpha_1^2\cos(\alpha_1 r)}\right]\left[\frac{e^{x_7 t}(1 + x_7\omega)[p_1 + p_0x_7 + (\rho g\sin\gamma)x_7^2]}{x_7^2[l + (1 + x_7\omega)^2]}\right.$$

$$+ \quad \left.\left.\frac{e^{x_8 t}(1 + x_8\omega)[p_1 + p_0x_8 + (\rho g\sin\gamma)x_8^2]}{x_8^2[l + (1 + x_8\omega)^2]}\right]\right\} - \frac{2\pi J}{\nu r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)$$

$$\times \quad \left\{\frac{1}{X^2Y^2\sinh(Y)}\left[(p_0 + p_1 t)(\sinh(Y(n-1)) - \sinh(Yn)) + \frac{p_1(1+l)}{2\nu Y}\right.\right.$$

$$\times \quad \left.\Big((n-1)\cosh(Y(n-1)) - n\cosh(Yn)\Big)\right] - \frac{p_1\,(\sinh(Y(n-1)) - \sinh(Yn))}{X^4Y^4\sinh^2(Yr)}$$

$$\times \quad \left[\frac{(1+l)}{2\nu}X^2Y\cosh(Y) + \frac{(1+l)}{\nu}(X^2 + Y^2)\sinh(Y) + \omega X^2Y^2\sinh(Y)\right] - \frac{2\nu}{\pi}$$

$$\times \quad \sum_{r_2=0}^{\infty}\frac{[1 - (-1)^{r_2}]}{r_2}\left[\frac{\sin(r_2 n\pi)}{\alpha_1^2}\right]\left[\frac{e^{x_7 t}(1 + x_7\omega)[p_1 + p_0x_7 + (\rho g\sin\gamma)x_7^2]}{x_7^2[l + (1 + x_7\omega)^2]}\right.$$

$$+ \quad \left.\left.\frac{e^{x_8 t}(1 + x_8\omega)[p_1 + p_0x_8 + (\rho g\sin\gamma)x_8^2]}{x_8^2[l + (1 + x_8\omega)^2]}\right]\right\} - \frac{2J}{\nu\pi}\sum_{r_1=0}^{\infty}\frac{[1 - (-1)^{r_1}]}{r_1}\sin\left(\frac{r_1\pi}{r}s\right)$$

$$\times \left\{ \frac{1}{X^2 \sinh(Y)} \left[(p_0 + p_1 t) \Big(\sinh(Y(n-1)) - \sinh(Yn) \right. \right. + \frac{p_1(1+l)}{2\nu Y}$$

$$\times \left. \Big((n-1)\cosh(Y(n-1)) - n\cosh(Yn) \Big) \right] - \frac{p_1(\sinh(Y(n-1)) - \sinh(Yn))}{X^3 Y \sinh^2(Y)}$$

$$\times \left[\frac{(1+l)}{2\nu} X \cosh(Y) + \frac{(1+l)}{\nu} \sinh(Y) + \omega X^2 \sinh(Y) \right] + \frac{\sinh(\alpha(n-1)) - \sinh(\alpha n)}{(x_1 - x_2)\sinh(\alpha)}$$

$$\times \left[\frac{e^{x_1 t}[p_1 + p_0 x_1 + (\rho g \sin\gamma) x_1^2]}{x_1^2(1 + x_1\omega)} - \frac{e^{x_2 t}[p_1 + p_0 x_2 + (\rho g \sin\gamma) x_2^2]}{x_2^2(1 + x_2\omega)} \right]$$

$$+ \quad 2\nu\pi \sum_{r_2=0}^{\infty} [1 - (-1)^{r_1}] r_2 \frac{\sin(r_2 n\pi)}{\alpha_1^2} \Bigg] \left[\frac{e^{x_7 t}(1 + x_7\omega)[p_1 + p_0 x_7 + (\rho g \sin\gamma) x_7^2]}{x_7^2[l + (1 + x_7\omega)^2]} \right.$$

$$+ \left. \left. \frac{e^{x_8 t}(1 + x_8\omega)[p_1 + p_0 x_8 + (\rho g \sin\gamma) x_8^2]}{x_8^2[l + (1 + x_8\omega)^2]} \right] \right\}$$

Shearing Stress (Skin Friction):

The Shear stress at the boundaries $s = r$, $s = -r$ and $n = 0$, $n = 1$ are given by

$$D_{rn} = -\mu u_1 \frac{(A_1 + iA_2)}{(C^2 + D^2)} + \frac{u_1 \mu \nu \pi^2}{2r^3} \sum_{r_2=0}^{\infty} (2r_2 + 1)^2 \left[\frac{e^{x_3 t}(1 + x_3\omega)^2 (x_3 + iw_1)}{(x_3^2 + w_1^2)[l + (1 + x_3\omega)^2]} \right.$$

$$+ \left. \frac{e^{x_4 t}(1 + x_4\omega)^2 (x_4 + iw_1)}{(x_4^2 + w_1^2)[l + (1 + x_4\omega)^2]} \right] - \mu u_2 \frac{(A_3 - iA_4)}{(K^2 + L^2)} + \frac{u_2 \mu \nu \pi^2}{2r^3} \sum_{r_2=0}^{\infty} (2r_2 + 1)^2$$

$$\times \left[\frac{e^{x_3 t}(1 + x_3\omega)^2 (x_3 - iw_2)}{(x_3^2 + w_2^2)[l + (1 + x_3\omega)^2]} + \frac{e^{x_4 t}(1 + x_4\omega)^2 (x_4 - iw_2)}{(x_4^2 + w_2^2)[l + (1 + x_4\omega)^2]} \right] - \frac{\mu J}{\nu} \left\{ \frac{1}{X^2 \cosh(Xr)} \right.$$

$$\times \left[(p_0 + p_1 t) X \sinh(Xr) + \frac{p_1(1+l)}{2\nu X^3 \cosh(Xr)} \Big(Xr\cosh(Xr) + \sinh(Xr) \Big) \right]$$

$$- \frac{p_1 \sinh(Xr)}{X^3 \cosh^2(Xr)} \left[\frac{(1+l)}{2\nu} \Big(Xr\sinh(Xr) + 2\cosh(Xr) \Big) \right]$$

$$+ \frac{2\nu\mu}{r} \sum_{r_2=0}^{\infty} \left[\frac{e^{x_3 t}(1 + x_3\omega)^2 \, [p_1 + p_0 x_3 + (\rho g \sin\gamma) x_3^2]}{x_3^2[l + (1 + x_3\omega)^2]} \right.$$

$$+ \left. \left. \frac{e^{x_4 t}(1 + x_4\omega)^2 \, [p_1 + p_0 x_4 + (\rho g \sin\gamma) x_4^2]}{x_4^2[l + (1 + x_4\omega)^2]} \right] \right\} - \frac{2\mu\pi^2}{r^3} \sum_{r_1=0}^{\infty} (-1)^{r_2} {r_1}^2$$

$$\times \left\{ \frac{u_1(\phi_5 + i\phi_6)}{(O^2 + P^2)({R_1'}^2 + R_2^2)(C^2 + D^2)} + \frac{u_1 \nu \pi}{r^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} \frac{(2r_2 + 1)}{\beta^2} \right.$$

$$\times \frac{[\sinh(\beta(n-1)) - \sinh(\beta n)]}{\sinh(\beta)} \left[\frac{e^{x_3 t}(1 + x_3\omega)^2 (x_3 + iw_1)}{(x_3^2 + w_1^2)[l + (1 + x_3\omega)^2]} \right.$$

$$+ \quad \frac{e^{x_4 t}(1+x_4\omega)^2(x_4+iw_2)}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\Bigg] + \frac{2u_1\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\frac{\sin(r_2 n\pi)}{\cos(\alpha_1 r)}$$

$$\times \quad \left[\frac{e^{x_7 t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\right]$$

$$+ \quad \frac{u_2\,(\phi_7 - i\phi_8)}{(O'^2+P'^2)({R'_3}^2+R_4^2)(K^2+L^2)} + \frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)$$

$$\times \quad \frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\beta^2\sinh(\beta)}\Bigg[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\omega)^2}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]}$$

$$+ \quad \frac{e^{x_4 t}(x_4-iw_2)(1+x_4\omega)^2}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\Bigg] + \frac{2u_2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\frac{\sin(r_2 n\pi)}{\cos(\alpha_1 r)}$$

$$\times \quad \left[\frac{e^{x_7 t}(x_7-iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8-iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\right]\Bigg\}$$

$$+ \quad \frac{2\pi^2\mu}{r^3}\sum_{r_1=0}^{\infty}r_1^2\Bigg\{\frac{u_1\,(\phi_9 - i\phi_{10})}{({R'_1}^2+R_2^2)(O^2+P^2)} + \frac{2u_1\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\sin(r_2 n\pi)$$

$$\times \quad \left[\frac{e^{x_7 t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\right] + \frac{u_2\,(\phi_{11}+i\phi_{12})}{({R'_3}^2+R_4^2)(O'^2+P'^2)}$$

$$+ \quad \frac{2u_2\nu}{\pi}\sum_{r_2=0}^{\infty}[1-(-1)^{r_2}]r_2\sin(r_2 n\pi)\Bigg[\frac{e^{x_7 t}(x_7-iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]}$$

$$+ \quad \frac{e^{x_8 t}(x_8-iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\Bigg]\Bigg\} - \frac{2\pi^2\mu J}{r^2\nu}\sum_{r_1=0}^{\infty}(-1)^{r_2}r_1^2\Bigg\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}$$

$$\times \quad \Bigg[(p_0+p_1t)\Big(\sinh(Y(n-1))-\sinh(Yn)\Big) + \frac{p_1(1+l)}{2Y\nu}\Big((n-1)\cosh(Y(n-1))$$

$$- \quad n\cosh(Yn)\Big)\Bigg] - \frac{p_1(\sinh(Y(n-1))-\sinh(Yn))}{X^4Y^4\cosh^2(Xr)\sinh^2(Y)}\Bigg[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y)$$

$$+ \quad \frac{r(1+l)}{2\nu}XY^2\sinh(Xr)\sinh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\cosh(Xr)\sinh(Y)\Bigg] - \frac{4\nu}{\pi}$$

$$\times \quad \sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[\sinh(\beta(n-1))-\sinh(\beta n)]}{\beta^2\sinh(\beta)}\Bigg[\frac{e^{x_3 t}(1+x_3\omega)^2[p_1+p_0x_3+\rho g\sin\gamma x_3^2]}{x_3^2[l+(1+x_3\omega)^2]}$$

$$+ \quad \frac{e^{x_4 t}(1+x_4\omega)^2[p_1+p_0x_4+\rho g\sin\gamma x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\Bigg] - \frac{2\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{r_2}\left[\frac{\sin(r_2 n\pi)}{\alpha_1^2\cos(\alpha_1 r)}\right]$$

$$\times \quad \left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+\rho g\sin\gamma x_7^2]}{x_7^2[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+\rho g\sin\gamma x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\Bigg\}$$

$$+ \quad \frac{2\pi^2\mu J}{\nu r^3}\sum_{r_1=0}^{\infty}{r_1}^2\Bigg\{\frac{1}{X^2Y^2\sinh(Y)}\Bigg[(p_0+p_1t)\Big(\sinh(Y(n-1))-\sinh(Yn)\Big)$$

$$
\begin{aligned}
&+ \frac{p_1(1+l)}{2\nu Y}\Big((n-1)\cosh(Y(n-1)) - n\cosh(Yn) \Big) \Big] \\
&- \frac{p_1(\sinh(Y(n-1)) - \sinh(Yn))}{X^4Y^4\sinh^2(Y)} \left[\frac{(1+l)}{2\nu} X^2 Y \cosh(Y) \right. \\
&+ \left. \frac{(1+l)}{\nu}(X^2+Y^2)\sinh(Y) \right] - \frac{2\nu}{\pi}\sum_{r_2=0}^{\infty} \frac{[1-(-1)^{r_2}]}{r_2}\left[\frac{\sin(r_2 n\pi)}{\alpha_1^2}\right. \\
&\times \left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]} \right. \\
&+ \left. \left. \frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]} \right] \right\} \\
&+ \frac{2\mu J}{\nu r}\sum_{r_1=0}^{\infty}[1-(-1)^{r_1}]\left\{ \frac{1}{X^2\sinh(Y)}\left[(p_0+p_1t)(\sinh(Y(n-1)) - \sinh(Yn))\right.\right. \\
&+ \frac{p_1(1+l)}{2\nu Y}\Big((n-1)\cosh(Y(n-1)) - n\cosh(Yn) \Big) \Big] \\
&- \frac{p_1(\sinh(Y(n-1)) - \sinh(Yn))}{X^3Y\sinh^2(Y)} \left[\frac{(1+l)}{2\nu} X\cosh(Y) + \frac{(1+l)}{\nu}\sinh(Y) \right] \\
&+ \frac{(\sinh(\alpha(n-1)) - \sinh(\alpha n))}{(x_1-x_2)\sinh(\alpha)} \left[\frac{e^{x_1 t}[p_1+p_0x_1+(\rho g\sin\gamma)x_1^2]}{x_1^2} \right. \\
&- \left. \frac{e^{x_2 t}[p_1+p_0x_2+(\rho g\sin\gamma)x_2^2]}{x_2^2} \right] + 2\nu\pi\sum_{r_2=0}^{\infty}[1-(-1)^{r_1}]r_2\frac{sin(r_2 n\pi)}{\alpha_1^2} \\
&\times \left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]} \right. \\
&+ \left. \frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]} \right]
\end{aligned}
$$

$$
D_{-rn} = -D_{rn}
$$

$$
\begin{aligned}
D_{s0} &= \frac{2\pi\mu}{r^2}\sum_{r_1=0}^{\infty} r_1 \sin\left(\frac{r_1\pi}{r}s\right)\left\{ \frac{u_1\,(A_5+iA_6)}{(O^2+P^2)({R'_1}^2+R_2^2)(C^2+D^2)} + 2u_1\nu\mu \right. \\
&\times \sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{\cos(\alpha_1 r)}\left[\frac{e^{x_7 t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]} \right] \\
&+ \frac{u_2\,(A_7-iA_8)}{(O'^2+P'^2)({R'_3}^2+R_4^2)(K^2+L^2)} + 2u_2\nu\mu\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{\cos(\alpha_1 r)}
\end{aligned}
$$

$$\times \left[\frac{e^{x_7 t}(x_7 - iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8 - iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\right]\Bigg\} - \frac{2\pi\mu}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1} r_1$$

$$\times \sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{u_1\left(A_9 - iA_{10}\right)}{({R'_1}^2+R_2^2)(O^2+P^2)} + 2u_1\nu\sum_{r_2=0}^{\infty}[1-(-1)^{r_2}]\right.$$

$$\times \left[\frac{e^{x_7 t}(x_7 + iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8 + iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\right] + \frac{u_2\left(A_{11}+iA_{12}\right)}{({R'_3}^2+R_4^2)(O'^2+P'^2)}$$

$$+ \left. 2u_2\nu\sum_{r_2=0}^{\infty}[1-(-1)^{r_2}]\left[\frac{e^{x_7 t}(x_7 - iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(x_8 - iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\right]\right\}$$

$$+ \frac{2\pi\mu J}{r^2\nu}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}\Big[(p_0+p_1t)\left(\cosh(Y)-1\right)\right.$$

$$+ \frac{p_1(1+l)}{2Y\nu}\sinh(Y)\Big] - \frac{p_1(\cosh(Y)-1)}{X^4Y^3\cosh^2(Xr)\sinh^2(Y)}\Big[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y)$$

$$+ \frac{r(1+l)}{2\nu}XY^2\sinh(Xr)\sinh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\cosh(Xr)\sinh(Y)\Big]$$

$$- \frac{4\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{(\cosh(\beta)-1)}{\beta\sinh(\beta)}\left[\frac{e^{x_3 t}(1+x_3\omega)^2[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]}\right.$$

$$+ \left.\frac{e^{x_4 t}(1+x_4\omega)^2[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\right] - 2\nu\sum_{r_2=0}^{\infty}[1-(-1)^{r_2}]\frac{1}{\alpha_1^2\cos(\alpha_1 r)}$$

$$\times \left.\left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+\rho g\sin\gamma x_7^2]}{x_7^2[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+\rho g\sin\gamma x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\right\}$$

$$- \frac{2\pi\mu J}{\nu r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1} r_1\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\sinh(Y)}\Big[(p_0+p_1t)Y(\cosh(Y)-1)\right.$$

$$- \frac{p_1(1+l)}{2\nu}\cosh(Y)\Big] - \frac{p_1(\cosh(Y)-1)}{X^4Y^3\sinh^2(Y)}\Big[\frac{(1+l)}{2\nu}X^2Y\cosh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)$$

$$\times \sinh(Y)\Big] - 2\nu\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_2}]}{\alpha_1^2}\left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]}\right.$$

$$+ \left.\left.\frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\right\} - \frac{2\mu J}{\nu\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)^{r_1}]}{r_1}\sin\left(\frac{r_1\pi}{r}s\right)$$

$$\times \left\{\frac{1}{X^2\sinh(Y)}\Big[(p_0+p_1t)Y(\cosh(Y)-1) + \frac{p_1(1+l)}{2\nu}\sinh(Y)\Big] - \frac{p_1(\cosh(Y)-1)}{X^3\sinh^2(Y)}\right.$$

$$\times \left[\frac{(1+l)}{2\nu}X\cosh(Y) + \frac{(1+l)}{\nu}\sinh(Y)\right] + \frac{\alpha(\cosh(\alpha)-1)}{(x_1-x_2)\sinh(\alpha)}$$

$$\times \left[\frac{e^{x_1 t}[p_1+p_0x_1+(\rho g\sin\gamma)x_1^2]}{x_1^2} - \frac{e^{x_2 t}[p_1+p_0x_2+(\rho g\sin\gamma)x_2^2]}{x_2^2}\right]$$

$$+ \quad 2\nu\pi^2 \sum_{r_2=0}^{\infty}[1-(-1)^{r_1}]\frac{r_2}{\alpha_1^2}\left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]}\right.$$

$$+ \quad \left.\frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right] - \mu u_1\frac{(A_1+iA_2)}{(C^2+D^2)} + \frac{u_1\mu\nu\pi^2}{2r^3}\sum_{r_2=0}^{\infty}(2r_2+1)^2$$

$$\times \quad \left[\frac{e^{x_3 t}(1+x_3\omega)^2(x_3+iw_1)}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2(x_4+iw_1)}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right] - \mu u_2\frac{(A_3-iA_4)}{(K^2+L^2)} + \frac{u_2\mu\nu\pi^2}{2r^3}$$

$$\times \quad \sum_{r_2=0}^{\infty}(2r_2+1)^2\left[\frac{e^{x_3 t}(1+x_3\omega)^2(x_3-iw_2)}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2(x_4-iw_2)}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\right] - \frac{\mu J}{\nu}$$

$$\times \quad \left\{\frac{1}{X^2\cosh(Xr)}\left[(p_0+p_1t)X\sinh(Xr) + \frac{p_1(1+l)}{2\nu X^3\cosh(Xr)}(Xr\cosh Xr+\sinh Xr)\right]\right.$$

$$- \quad \frac{p_1\sinh(Xr)}{X^3\cosh^2(Xr)}\left[\frac{(1+l)}{2\nu}\Big(Xr\sinh(Xr)+2\cosh(Xr)\right] + \frac{2\nu\mu}{r}\sum_{r_2=0}^{\infty}$$

$$\times \quad \left.\left[\frac{e^{x_3 t}(1+x_3\omega)^2\left[p_1+p_0x_3+\rho g\sin\gamma x_3^2\right]}{x_3^2[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2\left[p_1+p_0x_4+\rho g\sin\gamma x_4^2\right]}{x_4^2[l+(1+x_4\omega)^2]}\right]\right\}$$

$$- \quad \frac{2\mu\pi^2}{r^3}\sum_{r_1=0}^{\infty}{r_1}^2\cos(\frac{r_1\pi}{r}s)\left\{\frac{u_1(\phi_5+i\phi_6)}{(O^2+P^2)({R'_1}^2+R_2^2)(C^2+D^2)} - \frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\right.$$

$$\times \quad \frac{(2r_2+1)}{\beta^2}\left[\frac{e^{x_3 t}(1+x_3\omega)^2(x_3+iw_1)}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2(x_4+iw_2)}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right]$$

$$+ \quad \frac{u_2\left(\phi_7-i\phi_8\right)}{(O'^2+P'^2)({R'_3}^2+R_4^2)(K^2+L^2)} - \frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)}{\beta^2}$$

$$\times \quad \left[\frac{e^{x_3 t}(x_3+iw_2)(1+x_3\omega)^2}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(x_4+iw_2)(1+x_4\omega)^2}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\right] + \frac{2\mu\pi^2}{r^3}\sum_{r_1=0}^{\infty}(-1)^{r_2}{r_1}^2$$

$$\times \quad \cos(\frac{r_1\pi}{r})\left\{\frac{u_1(\phi_9-i\phi_{10})}{(O^2+P^2)({R'_1}^2+R_2^2)} + \frac{u_2(\phi_{11}+i\phi_{12})}{(O'^2+P'^2)({R'_3}^2+R_4^2)}\right\} - \frac{2\pi^2\mu J}{r^3\nu}\sum_{r_1=0}^{\infty}{r_1}^2$$

$$\times \quad \cos\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}\left[(p_0+p_1t)\left(\cosh(Y)-1\right) + \frac{p_1(1+l)}{2\nu}\right.\right.$$

$$\times \quad \left.\sinh(Y)\right] - \frac{p_1(\cosh(Y)-1)}{X^4Y^3\cosh^2(Xr)\sinh^2(Y)}\left[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y) + \frac{r(1+l)}{2\nu}\right.$$

$$\times \quad \left.XY^2\sinh(Xr)\sinh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\cosh(Xr)\sinh Y\right] + \frac{4\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)\beta^2}$$

$$\times \quad \left.\left[\frac{e^{x_3 t}(1+x_3\omega)^2[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2[p_1+p_0x_4+\rho g\sin\gamma x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\right\}\right.$$

$$- \quad \frac{2\pi^2\mu J}{\nu r^3}\sum_{r_1=0}^{\infty}(-1)^{r_1}{r_1}^2\cos\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\sinh(Y)}\left[(p_0+p_1t)\sinh(Y) + \frac{p_1(1+l)}{2\nu Y}\right.\right.$$

$$\times \quad \cosh(Y)] - \frac{p_1}{X^4Y^4}\left[\frac{(1+l)}{2\nu}X^2Y\cosh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\sinh(Y)\right] - \frac{2\mu J}{\nu r}$$
$$\times \quad \sum_{r_2=0}^{\infty}[1-(-1)^{r_1}]\cos\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2\sinh(Y)}\left[(p_0+p_1t)\sinh(Y) + \frac{p_1(1+l)}{2\nu Y}\cosh(Y)\right]\right.$$
$$+ \quad \frac{p_1}{X^3Y\sinh(Y)}\left[\frac{(1+l)}{2\nu}X\cosh(Y) + \frac{(1+l)}{\nu}\sinh(Y)\right] - \frac{1}{(x_1-x_2)}$$
$$\times \quad \left.\left[\frac{e^{x_1t}[p_1+p_0x_1+(\rho g\sin\gamma)x_1^2]}{x_1^2} - \frac{e^{x_2t}[p_1+p_0x_2+(\rho g\sin\gamma)x_2^2]}{x_2^2}\right]\right\}$$

$$D_{s1} = \frac{2\pi\mu}{r^2}\sum_{r_1=0}^{\infty}r_1\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{u_1\,(A5+iA6)}{(O^2+P^2)({R'_1}^2+R_2^2)(C^2+D^2)} + \frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\right.$$
$$\times \quad (2r_2+1)\frac{(1-\cosh(\beta))}{\beta\sinh(\beta)}\left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\omega)^2}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4t}(x_4+iw_1)(1+x_4\omega)^2}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right]$$
$$+ \quad 2u_1\nu\sum_{r_2=0}^{\infty}[(-1)^{r_2}-1]\frac{1}{\cos(\alpha_1 r)}\left[\frac{e^{x_7t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]}\right.$$
$$+ \quad \left.\frac{e^{x_8t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\right] + \frac{u_2\,(A7-iA8)}{(O'^2+P'^2)({R'_3}^2+R_4^2)(K^2+L^2)} + \frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}$$
$$\times \quad (2r_2+1)\frac{1-\cosh(\beta)}{\beta\sinh(\beta)}\left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\omega)^2}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4t}(x_4-iw_2)(1+x_4\omega)^2}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\right]$$
$$+ \quad 2u_2\nu\sum_{r_2=0}^{\infty}[(-1)^{r_2}-1]\frac{1}{\cos(\alpha_1 r)}\left[\frac{e^{x_7t}(x_7-iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]}\right.$$
$$+ \quad \left.\left.\frac{e^{x_8t}(x_8-iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\right]\right\} - \frac{2\pi\mu}{r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{u_1\,(A9-iA10)}{{R'_1}^2+R_2^2(O^2+P^2)}\right.$$
$$+ \quad 2u_1\nu\sum_{r_2=0}^{\infty}[(-1)^{r_2}-1]\left[\frac{e^{x_7t}(x_7+iw_1)(1+x_7\omega)^2}{(x_7^2+w_1^2)[l+(1+x_7\omega)^2]} + \frac{e^{x_8t}(x_8+iw_1)(1+x_8\omega)^2}{(x_8^2+w_1^2)[l+(1+x_8\omega)^2]}\right]$$
$$+ \quad \frac{u_2\,(A11+iA12)}{({R'_3}^2+R_4^2)(O'^2+P'^2)} + 2u_2\nu\sum_{r_2=0}^{\infty}[(-1)^{r_2}-1]\left[\frac{e^{x_7t}(x_7-iw_2)(1+x_7\omega)^2}{(x_7^2+w_2^2)[l+(1+x_7\omega)^2]}\right.$$
$$+ \quad \left.\left.\frac{e^{x_8t}(x_8-iw_2)(1+x_8\omega)^2}{(x_8^2+w_2^2)[l+(1+x_8\omega)^2]}\right]\right\} + \frac{2\pi\mu J}{r^2\nu}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}\right.$$
$$\times \quad \left[(p_0+p_1t)Y(1-\cosh(Y)) - \frac{p_1(1+l)}{2\nu}\sinh(Y)\right] - \frac{p_1(1-\cosh(Y))}{X^4Y^3\cosh^2(Xr)\sinh^2(Y)}$$
$$\times \quad \left[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y) + \frac{r(1+l)}{2\nu}XY^2\sinh(Xr)\sinh(Y) + \frac{(1+l)}{\nu}\right.$$

$$\times \quad (X^2+Y^2)\cosh(Xr)\sinh(Y)\Big] - \frac{4\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{(1-\cosh(\beta))}{\beta\sinh(\beta)}$$

$$\times \quad \left[\frac{e^{x_3 t}(1+x_3\omega)^2[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\right]$$

$$- \quad 2\nu\sum_{r_2=0}^{\infty}[(-1)^{r_2}-1]\left[\frac{1}{\alpha_1^2\cos(\alpha_1 r)}\right]\left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]}\right.$$

$$+ \quad \left.\frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\Bigg\} - \frac{2\pi\mu J}{\nu r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)$$

$$\times \quad \left\{\frac{1}{X^2Y^2\sinh(Y)}\left[(p_0+p_1t)Y(1-\cosh(Y)) - \frac{p_1(1+l)}{2\nu}\cosh(Y)\right] - \frac{p_1(1-\cosh(Y))}{X^4Y^3\sinh^2(Y)}\right.$$

$$\times \quad \left[\frac{(1+l)}{2\nu}X^2Y\cosh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\sinh(Y)\right] - 2\nu\sum_{r_2=0}^{\infty}\frac{[(-1)^{r_2}-1]}{\alpha_1^2}$$

$$\times \quad \left.\left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+(\rho g\sin\gamma)x_7^2]}{x_7^2[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+(\rho g\sin\gamma)x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\right\}$$

$$- \quad \frac{2\mu J}{\nu\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)^{r_1}]}{r_1}\sin\left(\frac{r_1\pi}{r}s\right)\left\{\frac{1}{X^2\sinh(Y)}\left[(p_0+p_1t)Y(1-\cosh(Y)) - \frac{p_1(1+l)}{2\nu}\right.\right.$$

$$\times \quad \left.\sinh(Y)\right] - \frac{p_1(1-\cosh(Y))}{X^3\sinh^2(Y)}\left[\frac{(1+l)}{2\nu}X\cosh(Y) + \frac{(1+l)}{\nu}\sinh(Y)\right] + \frac{\alpha(1-\cosh(\alpha))}{(x_1-x_2)\sinh\alpha}$$

$$\times \quad \left[\frac{e^{x_1 t}[p_1+p_0x_1+\rho g\sin\gamma x_1^2]}{x_1^2} - \frac{e^{x_2 t}[p_1+p_0x_2+\rho g\sin\gamma x_2^2]}{x_2^2}\right] + 2\nu\pi^2\sum_{r_2=0}^{\infty}[(-1)^{r_1}-1]r_2^2$$

$$\times \quad \left.\frac{1}{\alpha_1^2}\left[\frac{e^{x_7 t}(1+x_7\omega)^2[p_1+p_0x_7+\rho g\sin\gamma x_7^2]}{x_7^2[l+(1+x_7\omega)^2]} + \frac{e^{x_8 t}(1+x_8\omega)^2[p_1+p_0x_8+\rho g\sin\gamma x_8^2]}{x_8^2[l+(1+x_8\omega)^2]}\right]\right\}$$

$$- \quad \frac{u_1\mu\,(A_1+iA_2)}{(C^2+D^2)} + \frac{u_1\nu\pi^2\mu}{2r^3}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)^2\sin\left[\frac{(2r_2+1)\pi}{2r}s\right]$$

$$\times \quad \left[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\omega)^2}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(x_4+iw_1)(1+x_4\omega)^2}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right] - \frac{\mu u_2\,(A_3-iA_4)}{(K^2+L^2)}$$

$$+ \quad \frac{u_2\nu\pi^2\mu}{2r^3}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)^2\sin\left[\frac{(2r_2+1)\pi}{2r}s\right]\left[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\omega)^2}{(x_3^2+w_2^2)[l+(1+x_3\omega)^2]}\right.$$

$$+ \quad \left.\frac{e^{x_4 t}(x_4-iw_2)(1+x_4\omega)^2}{(x_4^2+w_2^2)[l+(1+x_4\omega)^2]}\right] - \frac{J\mu}{\nu}\left\{\frac{1}{X^2\cosh(Xr)}\left[(p_0+p_1t)X\sinh(Xs) + \frac{p_1(1+l)}{2\nu X^3}\right.\right.$$

$$\times \quad \left.\left[\frac{Xs\cosh(Xs)+\sinh(Xs)}{\cosh(Xr)}\right] - \frac{p_1\cosh(Xs)}{X^3\cosh^2(Xr)}\left[\frac{(1+l)}{2\nu}\left(Xr\sinh(Xr)+2\cosh(Xr)\right)\right]$$

$$+ \quad \frac{2\nu\mu}{r}\sum_{r_2=0}^{\infty}(-1)^{r_2}\sin\left[\frac{(2r_2+1)\pi}{2r}s\right]\left[\frac{e^{x_3 t}(1+x_3\omega)^2\,[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]}\right.$$

$$
\begin{aligned}
&+ \quad \frac{e^{x_4 t}(1+x_4\omega)^2\left[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2\right]}{x_4^2[l+(1+x_4\omega)^2]}\Bigg]\Bigg\} - \frac{2\mu\pi^2}{r^3}\sum_{r_1=0}^{\infty} r_1^2\cos\left(\frac{r_1\pi}{r}s\right) \\
&\times \quad \Bigg\{\frac{u_1\left(\phi_5+i\phi_6\right)}{(O^2+P^2)({R'_1}^2+R_2^2)(C^2+D^2)} + \frac{u_1\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)}{\beta^2} \\
&\times \quad \left[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\omega)^2}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(x_4+iw_1)(1+x_4\omega)^2}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right] \\
&+ \quad \frac{u_2\left(\phi_7-i\phi_8\right)}{(O'^2+P'^2)({R'_3}^2+R_4^2)(K^2+L^2)} + \frac{u_2\nu\pi}{r^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)}{\beta^2} \\
&\times \quad \left[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\omega)^2}{(x_3^2+w_1^2)[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(x_4+iw_1)(1+x_4\omega)^2}{(x_4^2+w_1^2)[l+(1+x_4\omega)^2]}\right]\Bigg\} + \frac{2\pi^2\mu}{r^3}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1^2 \\
&\times \quad \cos\left(\frac{r_1\pi}{r}s\right)\Bigg\{\frac{u_1\left(\phi_9-i\phi_{10}\right)}{({R'_1}^2+R_2^2)(O^2+P^2)} + \frac{u_2\left(\phi_{11}+i\phi_{12}\right)}{({R'_3}^2+R_4^2)(O'^2+P'^2)} + \frac{2\pi^2\mu J}{r^3\nu}\sum_{r_1=0}^{\infty}\cos\left(\frac{r_1\pi}{r}s\right) \\
&\times \quad \Bigg\{\frac{1}{X^2Y^2\cosh(Xr)\sinh(Y)}\left[(p_0+p_1t)\sinh(Y) + \frac{p_1(1+l)}{2Y\nu}\cosh(Y)\right] \\
&- \quad \frac{p_1\sinh(Y)}{X^4Y^4\cosh^2(Xr)\sinh^2(Y)}\Bigg[\frac{(1+l)}{2\nu}X^2Y\cosh(Xr)\cosh(Y) + \frac{r(1+l)}{2\nu}XY^2\sinh(Xr) \\
&\times \quad \sinh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\cosh(Xr)\sinh(Y)\Bigg] + \frac{4\nu}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)\beta^2} \\
&\times \quad \left[\frac{e^{x_3 t}(1+x_3\omega)^2[p_1+p_0x_3+(\rho g\sin\gamma)x_3^2]}{x_3^2[l+(1+x_3\omega)^2]} + \frac{e^{x_4 t}(1+x_4\omega)^2[p_1+p_0x_4+(\rho g\sin\gamma)x_4^2]}{x_4^2[l+(1+x_4\omega)^2]}\right] \\
&+ \quad \frac{2\pi\mu J}{\nu r^2}\sum_{r_1=0}^{\infty}(-1)^{r_1}r_1\sin\left(\frac{r_1\pi}{r}s\right)\Bigg\{\frac{1}{X^2Y^2\sinh(Y)}\Bigg[(p_0+p_1t)\sinh(Y) + \frac{p_1(1+l)}{2\nu Y} \\
&\times \quad \cosh(Y)\Bigg] - \frac{p_1}{X^4Y^4\sinh(Y)}\left[\frac{(1+l)}{2\nu}X^2Y\cosh(Y) + \frac{(1+l)}{\nu}(X^2+Y^2)\sinh(Y)\right] \\
&+ \quad \frac{2\mu J}{\nu\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)^{r_1}]}{r_1}\sin\left(\frac{r_1\pi}{r}s\right)\Bigg\{\frac{1}{X^2\sinh(Y)}\Bigg[(p_0+p_1t)\sinh(Y) + \frac{p_1(1+l)}{2\nu Y} \\
&\times \quad \cosh(Y)\Bigg] - \frac{p_1}{X^3Y\sinh(Y)}\left[\frac{(1+l)}{2\nu}X\cosh(Y) + \frac{(1+l)}{\nu}\sinh(Y)\right] + \frac{1}{(x_1-x_2)} \\
&\times \quad \left[\frac{e^{x_1 t}[p_1+p_0x_1+(\rho g\sin\gamma)x_1^2]}{x_1^2} - \frac{e^{x_2 t}[p_1+p_0x_2+(\rho g\sin\gamma)x_2^2]}{x_2^2}\right]\Bigg\}
\end{aligned}
$$

where

$$
J = \frac{M^2h^2}{U_0}, \quad \phi = \frac{\omega\rho h^2}{\nu}, \quad a_1 = 4r^2\omega, \quad a_2 = \omega, \quad a_3 = \omega r^2, \quad a_4 = \omega r^2
$$

$$
\begin{aligned}
b_1 &= \left[(C_r+M^2)\nu\omega+l+1\right]4r^2+(2r_2+1)^2\pi^2\nu\omega, \quad b_2=(1+l)+\nu\omega(C_r+M^2)\\
b_3 &= \left[(C_r+M^2)\nu\omega+r_2^2\pi^2\nu\omega+l+1\right]r^2+r_1^2\pi^2\nu\omega\\
b_4 &= (1+l)r^2+\nu\omega r^2(C_r+M^2)+r_1^2\pi^2\nu\omega, \quad c_1=(C_r+M^2)4\nu r^2+(2r_2+1)^2\pi^2\nu\\
c_2 &= C_r+M^2, \quad c_3=(C_r+M^2)\nu r^2+r_2^2\pi^2\nu r^2+r_1^2\pi^2, \quad c_4=\nu r^2(C_r+M^2)+r_1^2\pi^2\nu\\
x_1 &= \frac{-b_2+\sqrt{b_2^2-4a_2c_2}}{2a_2}, \quad x_2=\frac{-b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2}, \quad x_3=\frac{-b_1+\sqrt{b_1^2-4a_1c_1}}{2a_1}\\
x_4 &= \frac{-b_1-\sqrt{b_1^2-4a_1c_1}}{2a_1}, \quad x_5=\frac{-b_4+\sqrt{b_4^2-4a_4c_4}}{2a_4}, \quad x_6=\frac{-b_4-\sqrt{b_4^2-4a_4c_4}}{2a_4}\\
x_7 &= \frac{-b_3+\sqrt{b_3^2-4a_3c_3}}{2a_3}, \quad x_8=\frac{-b_3-\sqrt{b_3^2-4a_3c_3}}{2a_3}, \quad \beta^2=\frac{4r_1^2\pi^2-(2r_2+1)^2\pi^2}{4r^2}\\
\alpha^2 &= \frac{r_1^2\pi^2}{r^2}, \quad \alpha_1^2=r_2^2\pi^2+\frac{r_1^2\pi^2}{r^2}, \qquad X^2=C_r+M^2, \quad Y^2=X^2+\alpha^2\\
R_1 &= C_r+M^2+\frac{l\omega w_1^2}{\nu(1+\omega^2w_1^2)}, \quad R_2=\frac{w_1}{\nu}+\frac{lw_1}{\nu(1+\omega^2w_1^2)}, \quad R_4=\frac{w_2}{\nu}+\frac{lw_2}{\nu(1+\omega^2w_2^2)}\\
R_3 &= C_r+M^2+\frac{l\omega w_2^2}{\nu(1+\omega^2w_2^2)}, \quad R_1'=C_r+M^2+\frac{l\omega w_1^2}{\nu(1+\omega^2w_1^2)}+\frac{r_1^2\pi^2}{r^2}\\
R_3' &= C_r+M^2+\frac{l\omega w_2^2}{\nu(1+\omega^2w_2^2)}+\frac{r_1^2\pi^2}{r^2}, \quad \alpha_2=\sqrt{\frac{R_1+\sqrt{R_1^2+R_2^2}}{2}}\\
\beta_2 &= \sqrt{\frac{-R_1+\sqrt{R_1^2+R_2^2}}{2}}, \quad \alpha_3=\sqrt{\frac{R_3+\sqrt{R_3^2+R_4^2}}{2}}, \quad \beta_3=\sqrt{\frac{-R_3+\sqrt{R_3^2+R_4^2}}{2}}\\
\alpha_2' &= \sqrt{\frac{R_1'+\sqrt{{R_1'}^2+R_2^2}}{2}}, \quad \beta_2'=\sqrt{\frac{-R_1'+\sqrt{{R_1'}^2+R_2^2}}{2}}, \quad \alpha_3'=\sqrt{\frac{R_3'+\sqrt{{R_3'}^2+R_4^2}}{2}}\\
\beta_3' &= \sqrt{\frac{-R_3'+\sqrt{{R_3'}^2+R_4^2}}{2}}, \quad E=\cosh(\alpha_2 s)\cos(\beta_2 s), \quad F=\sinh(\alpha_2 s)\sin(\beta_2 s)\\
G &= \cosh(\alpha_3 s)\cos(\beta_3 s), \quad H=\sinh(\alpha_3 s)\sin(\beta_3 s), \quad K=\cosh(\alpha_3 r)\cos(\beta_3 r)\\
L &= \sinh(\alpha_3 r)\sin(\beta_3 r), \quad C=\cosh(\alpha_2 r)\cos(\beta_2 r), \quad D=\sinh(\alpha_2 r)\sin(\beta_2 r)\\
\psi_1 &= EC+FD, \qquad \psi_2=ED-FC, \quad \psi_3=GK+HL, \qquad \psi_4=GL-HK\\
\psi_5 &= \sigma_1R_1'-\sigma_2R_2, \qquad \psi_6=\sigma_2R_1'+\sigma_1R_2, \quad \psi_7=\sigma_3R_3'-\sigma_4R_4, \qquad \psi_8=\sigma_4R_3'+\sigma_3R_4\\
O &= \sinh(\alpha_2')\cos(\beta_2'), \quad P=\cosh(\alpha_2')\sin(\beta_2'), \quad O'=\sinh(\alpha_3')\cos(\beta_3')\\
P' &= \cosh(\alpha_3')\sin(\beta_3'), \quad U_2=\sinh(\alpha_2'(n-1))\cos(\beta_2'(n-1)), \quad S_2=\sinh(\alpha_2'n)\cos(\beta_2'n)
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= \cosh(\alpha_2'(n-1))\sin(\beta_2'(n-1)), \ \ T_2 = \cosh(\alpha_2' n)\sin(\beta_2' n), \ \ S_3 = \sinh(\alpha_3' n)\cos(\beta_3' n) \\
U_3 &= \sinh(\alpha_3'(n-1))\cos(\beta_3'(n-1)), \ \ V_3 = \cosh(\alpha_3'(n-1))\sin(\beta_3'(n-1)) \\
T_3 &= \cosh(\alpha_3' n)\sin(\beta_3' n), \ \ \sigma_1 = O(U_2-S_2)+P(V_2-T_2), \ \ \sigma_2 = P(U_2-S_2)-O(V_2-T_2) \\
\sigma_3 &= O'(U_3-S_3)+P'(V_3-T_3), \ \ \sigma_4 = P'(U_3-S_3)-O'(V_3-T_3), \ \ \xi_5 = C\psi_5 + D\psi_6 \\
\xi_6 &= D\psi_5 - C\psi_6, \quad \xi_7 = K\psi_7 + L\psi_8, \quad \xi_8 = L\psi_7 - K\psi_8 \\
\phi_1 &= \psi_1\cos(w_1t) - \psi_2\sin(w_1t), \ \ \phi_2 = \psi_2\cos(w_1t)+\psi_1\sin(w_1t) \\
\phi_3 &= \psi_3\cos(w_2t) - \psi_4\sin(w_2t), \ \ \phi_4 = \psi_4\cos(w_2t)+\psi_3\sin(w_2t) \\
\phi_5 &= \xi_5\cos(w_1t) - \xi_6\sin(w_1t), \ \ \phi_6 = \xi_6\cos(w_1t)+\xi_5\sin(w_1t) \\
\phi_7 &= \xi_7\cos(w_2t) - \xi_8\sin(w_2t), \ \ \phi_8 = \xi_8\cos(w_2t)+\xi_7\sin(w_2t) \\
\phi_9 &= \psi_5\cos(w_1t) + \psi_6\sin(w_1t), \ \ \phi_{10} = \psi_6\cos(w_1t)-\psi_5\sin(w_1t) \\
\phi_{11} &= \psi_7\cos(w_2t) + \psi_8\sin(w_2t), \ \ \phi_{12} = \psi_8\cos(w_2t)-\psi_7\sin(w_2t) \\
\psi_1' &= \psi_1 + \psi_2\omega w_1, \quad \psi_2' = \psi_2 - \psi_1\omega w_1 \quad \psi_3' = \psi_3 + \psi_4\omega w_2, \quad \psi_4' = \psi_4 - \psi_3\omega w_2 \\
\psi_5' &= \xi_5 + \xi_6\omega w_1, \quad \psi_6' = \xi_6 - \xi_5\omega w_1 \quad \psi_7' = \xi_7 + \xi_8\omega w_2, \quad \psi_8' = \xi_8 - \xi_7\omega w_2 \\
\psi_9' &= \psi_5 - \psi_6\omega w_1, \quad \psi_{10}' = \psi_6 + \psi_5\omega w_1, \quad \psi_{11}' = \psi_7 - \psi_8\omega w_2, \quad \psi_{12}' = \psi_8 + \psi_7\omega w_2 \\
\phi_1' &= \psi_1'\cos(w_1t) - \psi_2'\sin(w_1t), \quad \phi_2' = \psi_2'\cos(w_1t)+\psi_1'\sin(w_1t) \\
\phi_3' &= \psi_3'\cos(w_2t) - \psi_4'\sin(w_2t), \quad \phi_4' = \psi_4'\cos(w_2t)+\psi_3'\sin(w_2t) \\
\phi_5' &= \psi_5'\cos(w_1t) - \psi_6'\sin(w_1t), \quad \phi_6' = \psi_6'\cos(w_1t)+\psi_5'\sin(w_1t) \\
\phi_7' &= \psi_7'\cos(w_2t) - \psi_8'\sin(w_2t), \quad \phi_8' = \psi_8'\cos(w_2t)+\psi_7'\sin(w_2t) \\
\phi_9' &= \psi_9'\cos(w_1t) + \psi_{10}'\sin(w_1t), \quad \phi_{10}' = \psi_{10}'\cos(w_1t)-\psi_9'\sin(w_1t) \\
\phi_{11}' &= \psi_{11}'\cos(w_2t) + \psi_{12}'\sin(w_2t), \quad \phi_{12}' = \psi_{12}'\cos(w_2t)-\psi_{11}'\sin(w_2t) \\
E' &= \alpha_2\sinh(\alpha_2 r)\cos(\beta_2 r) - \beta_2\cosh(\alpha_2 r)\sin(\beta_2 r) \\
F' &= \alpha_2\cosh(\alpha_2 r)\sin(\beta_2 r) + \beta_2\sinh(\alpha_2 r)\cos(\beta_2 r) \\
A_1 &= (CE'+DF')\cos(w_1t) - (DE'-CF')\sin(w_1t)
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= (DE' - CF')\cos(w_1 t) + (CE' + DF')\sin(w_1 t) \\
G' &= \alpha_3 \sinh(\alpha_3 r)\cos(\beta_3 r) - \beta_3 \cosh(\alpha_3 r)\sin(\beta_3 r) \\
H' &= \alpha_3 \cosh(\alpha_3 r)\sin(\beta_3 r) + \beta_3 \sinh(\alpha_3 r)\cos(\beta_3 r) \\
A_3 &= (KG' + LH')\cos(w_2 t) - (LG' - KH')\sin(w_2 t) \\
A_4 &= (LG' - KH')\cos(w_2 t) + (KG' + LH')\sin(w_2 t) \\
\chi_5 &= [R_1'(O\alpha_2' + P\beta_2') - R_2(P\alpha_2' + O\beta_2')]\cosh(\alpha_2'(n-1))\cos(\beta_2'(n-1)) \\
&+ [R_1'(P\alpha_2' - O\beta_2') - R_2(O\alpha_2' - P\beta_2')]\sinh(\alpha_2'(n-1))\sin(\beta_2'(n-1)) \\
&+ [R_1'(P\alpha_2' + O\beta_2') - R_2(O\alpha_2' + P\beta_2')]\sinh(\alpha_2' n)\sin(\beta_2' n) \\
&+ [R_1'(O\alpha_2' - P\beta_2') - R_2(P\alpha_2' - O\beta_2')]\cosh(\alpha_2' n)\cos(\beta_2' n) \\
\chi_6 &= [R_2(O\alpha_2' + P\beta_2') + R_1'(P\alpha_2' + O\beta_2')]\cosh(\alpha_2'(n-1))\cos(\beta_2'(n-1)) \\
&+ [R_2(P\alpha_2' - O\beta_2') + R_1'(O\alpha_2' - P\beta_2')]\sinh(\alpha_2'(n-1))\sin(\beta_2'(n-1)) \\
&+ [R_2(P\alpha_2' + O\beta_2') - R_1'(O\alpha_2' + P\beta_2')]\sinh(\alpha_2' n)\sin(\beta_2' n) \\
&+ [R_2(O\alpha_2' - P\beta_2') + R_1'(P\alpha_2' - O\beta_2')]\cosh(\alpha_2' n)\cos(\beta_2' n) \\
A_5 &= (C\chi_5 + D\chi_6)\cos(w_1 t) - (D\chi_5 - C\chi_6)\sin(w_1 t) \\
A_6 &= (D\chi_5 - C\chi_6)\cos(w_1 t) + (C\chi_5 + D\chi_6)\sin(w_1 t) \\
\chi_7 &= [R_3'(O'\alpha_3' + P'\beta_3') - R_4(P'\alpha_3' + O'\beta_3')]\cosh(\alpha_3'(n-1))\cos(\beta_3'(n-1)) \\
&+ [R_3'(P'\alpha_3' - O'\beta_3') + R_4(O'\alpha_3' + P'\beta_3')]\sinh(\alpha_3'(n-1))\sin(\beta_3'(n-1)) \\
&- [R_3'(P'\alpha_3' - O'\beta_3') + R_4(O'\alpha_3' + P'\beta_3')]\sinh(\alpha_3' n)\sin(\beta_3' n) \\
&- [R_3'(O'\alpha_3' + P'\beta_3') - R_4(P'\alpha_3' - O'\beta_3')]\cosh(\alpha_3' n)\cos(\beta_3' n) \\
\chi_8 &= [R_4(O'\alpha_3' + P'\beta_3') + R_3'(P'\alpha_3' - O'\beta_3')]\cosh(\alpha_3'(n-1))\cos(\beta_3'(n-1)) \\
&+ [R_4(P'\alpha_3' - O'\beta_3') - R_3'(O'\alpha_3' + P'\beta_3')]\sinh(\alpha_3'(n-1))\sin(\beta_3'(n-1)) \\
&- [R_3'(P'\alpha_3' - O'\beta_3') + R_4(O'\alpha_3' + P'\beta_3')]\sinh(\alpha_3' n)\sin(\beta_3' n) \\
&+ [R_3'(O'\alpha_3' + P'\beta_2') + R_4'(P'\alpha_3' - O'\beta_3')]\cosh(\alpha_3' n)\cos(\beta_3' n)
\end{aligned}
$$

$$A_7 = (K\chi_7 + L\chi_8)\cos(w_2 t) + (L\chi_7 + K\chi_8)\sin(w_2 t)$$

$$A_8 = (L\chi_7 - K\chi_8)\cos(w_2 t) + (K\chi_7 + L\chi_8)\sin(w_2 t)$$

$$A_9 = \chi_5\cos(w_1 t) + \chi_6\sin(w_1 t), \qquad A_{10} = \chi_6\cos(w_1 t) - \chi_5\sin(w_1 t)$$

$$A_{11} = \chi_7\cos(w_2 t) + \chi_8\sin(w_2 t), \qquad A_{12} = \chi_8\cos(w_2 t) - \chi_7\sin(w_2 t)$$

Conclusion

The effect of an inclined angle ($\gamma = 30$ and $\gamma = 90$), time parameter ($t = 0.2$ and $t = 0.8$) and number density of dust particles ($N = 1$ and $N = 2.5$) on velocities of fluid and dust phase with respect to s and n are shown graphically in figures 3 to 8. From these one can observed the paraboloid nature of both fluid and dust phase velocities. Further, we can see that the flow of fluid particles is parallel to that of dust. Also we made following observations.

- The favorable effect of an inclined angle on the velocity field, i.e, the velocity profiles for both fluid and dust particles increases as an inclined angle γ increases.
- The velocity fields of both fluid and dust particles increases as time parameter t increases.
- The velocities of both fluid and dust particles increases as number density of dust particles N decreases.

As a particular case if $p_1 = 0$, then the results coincides with the flow of a dusty fluid with constant pressure gradient and if the inclined angle $\gamma = 0$, then the results coincides with [5].

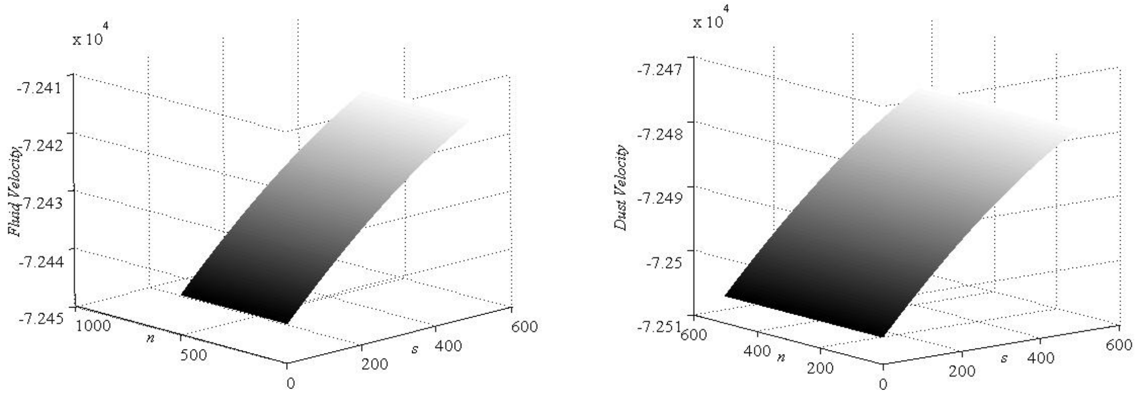

Figure-3: Variation of fluid velocity with s and n (for $\gamma = 30$ & $\gamma = 90$)

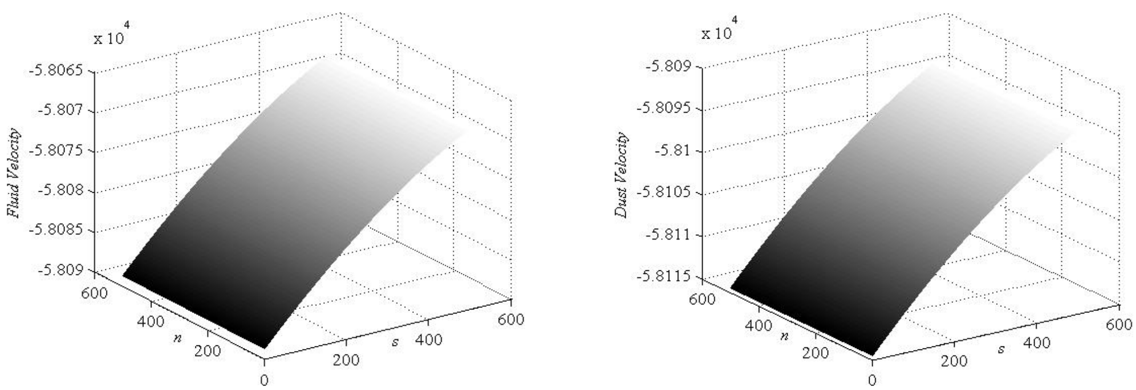

Figure-4: Variation of dust velocity with s and n (for $\gamma = 30$ & $\gamma = 90$)

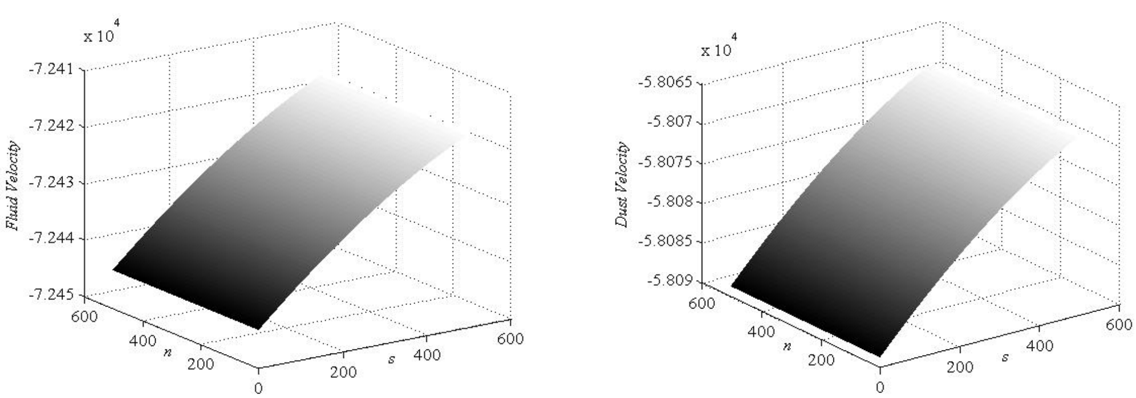

Figure-5: Variation of fluid velocity with s and n (for $t = 0.2$ & $t = 0.8$)

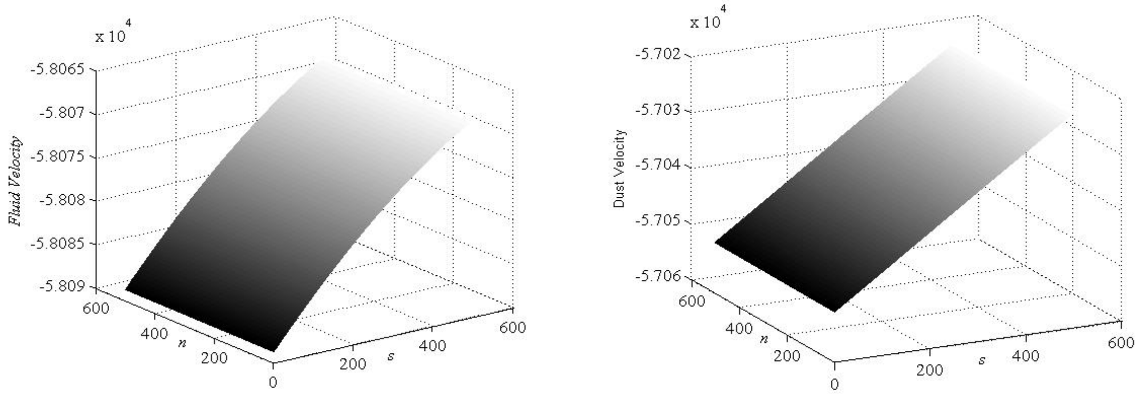

Figure-6: Variation of dust velocity with s and n (for $t = 0.2$ & $t = 0.8$)

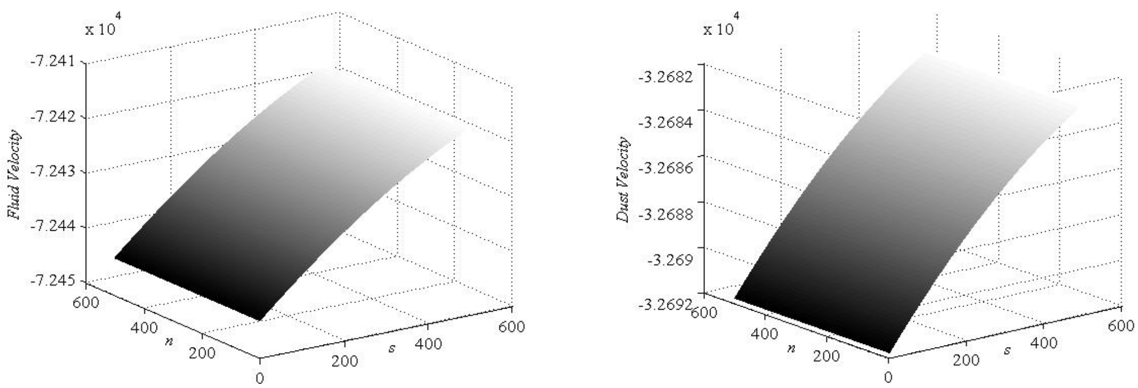

Figure-7: Variation of fluid velocity with s and n (for $N = 1$ & $N = 2.5$)

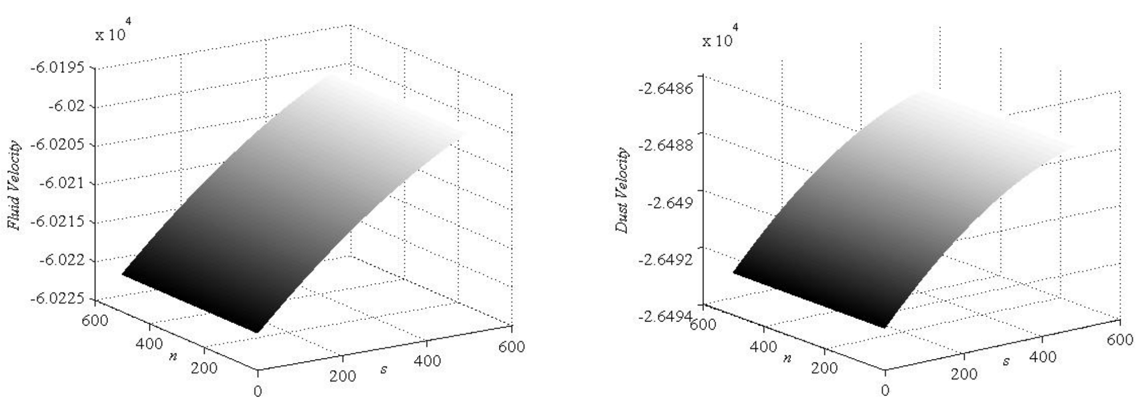

Figure-8: Variation of dust velocity with s and n (for $N = 1$ & $N = 2.5$)

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