Some Common Fixed Point Theorems for Expanding Mappings in Complete Cone Metric Space

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ABSTRACT: *The purpose of this paper, we prove some common fixed point theorems for Expanding onto self mappings in complete cone metric spaces. We are generalizing the well- known recent results [10].*

KEY WORD: *Complete cone metric space, common fixed point, expanding mapping*

I   INTRODUCTION

Very recently, Huang and Zhang [1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space. They prove some fixed point Theorems for contractive mappings using normality of the cone. The results in [1] were generalized by Sh. Rezapour and Hamlbarani [2] omitted the assumption of normality on the cone, which is a milestone in cone metric space. Later on many authors have generalized and extended Huang and Zhang [1] fixed point theorems (see, e.g. [2, 3, 4, 5, 6]). In 1984, the concept of expanding mappings was introduced by Wang et.al. [7]. In 1992, Daffer and Kaneko [8] defined expanding mappings for pair of mappings in complete metric spaces and proved some fixed point theorem. In 2012, X. Huang, Ch. Zhu and Xi Wen [9] proved some fixed point theorems for expanding mappings cone metric spaces and they have also extended the results of Daffer and Kaneko [8]. In 2015, K. Prudhvi [10] proved some fixed point theorems for expanding mappings cone metric spaces and they have also extended and improved the results of X. Huang, Ch. Zhu and Xi Wen [9]

In this manuscript, the known result [10] is extending to cone metric spaces where the existence of common fixed points for expanding mappings on cone metric spaces is investigated.

II PRELIMINARY NOTES

Definition 2.1[3] : Let E be a real Banach space and P, a subset of E. Then P is called a cone if and only if:

(i)   P is closed, non-empty and P $\neq$ {0} ;

(ii) $a, b \in R$, $a, b \geq 0$ $x, y \in \boldsymbol{P} \Rightarrow ax + by \in \boldsymbol{P}$;

(iii) $x \in$ P and $- x \in$ P => $x = 0$.

Given a cone P$\subseteq$E, we define a Partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in \boldsymbol{P}$. We shall write x $\ll$ y to denote $x \leq y$ but $x \neq y$ to denote $y - x \in p^0$, where p^0 stands for the interior of P.

Remark 2.2 [7]: $\lambda p^0 \subseteq p^0 \, for \; \lambda > 0$ and $p^0 + p^0 \subseteq p^0$

Definition 2.2 [3] : Let X be a non-empty set and $d : X \times X \to E$ a mapping such that

(d$_1$) $0 \leq$ d(x, y) for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(d$_2$) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d$_3$) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and $(X,$ d) is called a cone metric space.

Example 2.4 [3] : Let $E = R^{2}$, P {$(x, y) \in E: x, y \geq 0$} and $X = Y$ defined by $d(x, y) = (\alpha | x - y |, \beta | x - y |, \gamma | x - y)$ where $\alpha, \beta, \gamma \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.5 [3]: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

$(i)\{x_n\}_{n \geq 1}$ converges to x whenever to every $c \in E$ with $0 \ll c$ there is a natural number N such that

$d(x_n, x) \ll c$ for all n ≥ N.

$(ii)\{x_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \geq c$ for all $n, m \geq N$.

(iii) (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent in X.

Definition 2.6 [3]:
(X, d) be a cone metric space, P be a cone in real Banach space
$$E, if$$
(i) $a \in$ p and $a \ll c$ for some $k\epsilon[0,1]$ then

$$a = 0.$$

(ii) $a \in$ p and $a \ll c$ for some $k\epsilon[0,1]$ then

$$a = 0.$$

(iii) $u \leq v, v \ll w$, then $u \ll w$.

Lemma 2.7
*Let (x, d) be a cone metric space and P **be** a cone metric space in*

real Banach space E and , $\alpha_2, \alpha_3. \alpha_4 \alpha \geq 0$. If
$x_n \to x, y_n \to y, z_n \to$ z, an and
$\to$
p in x and $\alpha_1 d(x_n, x), +\alpha_2 d(y_n, y) +\alpha_3 d(z_n, z) +. \alpha_4 d(p_n, p)$.

Then $a = 0$.

III Main Result:

In this section, we prove common fixed point theorem for expanding mappings in complete cone metric spaces. The following theorem improved and extended the theorem 2.2 of [10].

Theorem 3.1: Let (X, d) be a complete cone metric space with respect to a cone P containing in a real Banach space E. Let T_1, T_2 be any two surjective self mappings of X satisf
$d(T_1 x, \ T_2 \ y) \geq \alpha_1 d(x, y) + \alpha_2 \ d(x, \ T_1 x) + \alpha_3 d(y, T_2 y) + \alpha_4 d(y, T_1 x)$………........ (3.1.1)
for each $x, y \in X$, $x \neq y$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$
with $\alpha_1 + \alpha_2 + \alpha_3, + 2\alpha_4 > 1$. Then T_1 and T_2 have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since T_1 and T_2 surjective mappings, there exist points $x_1 \in T_1^{-1}(x_0)$ and $x_2 \in T_2^{-1}(x_1)$ that is $T_1(x_1) = x_0$ and $T_2(x_2) = x_1$. In this way, we define the sequence $\{x_n\}$ with $x_{2n+1} \epsilon T_1^{-1}(x_{2n})$ and $x_{2n+2} \epsilon T_2^{-1}(x_{2n+1})$ i.e. $x_{2n} = T_1 \ x_{2n+1}$ for n= 0, 1 ,2,………….....................(3.1.2)

$$x_{2n+1} = T_2 x_{2n+2}$$

For n =0, 1, 2…………….(3.1.2)

Note that, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$, then x_{2n} is fixed point of T_1 and T_2. Now putting x $= x_{2n+1}$ and $y = x_{2n+2}$ from(3.1.1), we have

$$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n+1}, T_2 x_{2n+2})$$

$$\geq \alpha_1 d(x_{2n+1}, x_{2n+2})$$

$$+\alpha_2 \ d(x_{2n+1}, T_1 x_{2n+1})$$

$$+\alpha_3 d(x_{2n+2}, T_2 x_{2n+2})$$

$$+\alpha_4 d(x_{2n+2}, x_{2n})$$

$$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \alpha_1 d(x_{2n+1}, x_{2n+2})$$

$$+ \alpha_2 \ d(x_{2n+1}, x_{2n})$$

$$+\alpha_3 d(x_{2n+2}, x_{2n+1})$$

$$+\alpha_4 d(x_{2n+2}, x_{2n})$$

$$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \alpha_1 d(x_{2n+1}, x_{2n+2})$$

$$+ \alpha_2 \ d(x_{2n+1}, x_{2n}) +\alpha_3 d(x_{2n+2}, x_{2n+1})$$

$$+\alpha_4 d(x_{2n+1} \ x_{2n}) +\alpha_4 d(x_{2n+1}, x_{2n+2})$$

$$\Rightarrow d(x_{2n}, x_{2n+1}) \geq (\alpha_1 + \alpha_3 + \alpha_4) d((x_{2n+1}, x_{2n+2}) + (\alpha_2 + \alpha_4) d(x_{2n+1}, x_{2n})$$

$$\Rightarrow d(x_{2n+1}, x_{2n+2})$$

$$\leq \frac{1-(\alpha_2 + \alpha_4)}{(\alpha_1 + \alpha_3 + \alpha_4)} \ d(x_{2n}, x_{2n+1})$$

Where h $= \frac{1-(\alpha_2 + \alpha_4)}{(\alpha_1 + \alpha_3 + \alpha_4)} < 1$

In general

$$d(x_{2n}, x_{2n+1}) \leq h \ d(x_{2n-1}, x_{2n})$$

$$\Rightarrow d(x_{2n}, x_{2n+1}) \leq h^{2n} d(x_{2n-1}, x_{2n})$$

$$......................................(3.1.5)$$

So for every positive integer p, we have

$$d(x_{2n}, x_{2n+p}) \leq d(x_{2n}, x_{2n+1})$$

$+ (x_{2n+1} \, x_{2n+2}) + \ldots\ldots$

$. + d(x_{2n+p-1}, x_{2n+p})$

$\leq (\quad h^{2n} + \quad h^{2n+1} + \ldots\ldots + h^{2n+p-1}) \, d\,(x_0, x_1)$

$. \qquad = \qquad h^{2n}(1 + h + h^2 + \ldots\ldots + h2n+p-1 dx0, x1$

$< \dfrac{h^{2n}}{1-h} d(x_0, x_1) \ldots\ldots\ldots\ldots\ldots.(3.1.6)$

Therefore $\{x_{2n}\}$ is a Cauchy sequence, which is complete space in X there exist $x^* \epsilon X$ such that $x_{2n \to} x^*$. Since R_1 is surjective map, there exist a point y in X such that

$y \epsilon R_1^{-1}(x^*)$. i.e. $x^* = T_1(y) \ldots\ldots\ldots.(3.1.7)$

Now consider

$d(x_{2n}, x^*) \ = \ d(T_1 x_{2n+1}, T_1 y)$

$\geq \qquad\qquad \alpha_1 d(x_{2n+1}, y)$
$+ \alpha_2 \, d(x_{2n+1}, T_1 \, x_{2n+1}) + \alpha_3 d(y, T_1 y)$

$+ \qquad \alpha_4 \, d(y, T_1 \, x_{2n+1}$
$\Rightarrow d(x^*, x^*) \ \geq \alpha_1 d(x^*, y) + \alpha_2 d(x^*, x^*)$

$+ \alpha_3 d(y, x^*) + \alpha_4 d(y, x^*)$

Letting $n \to \infty$, we get

$\Rightarrow \qquad 0 \geq (\alpha_1 + \alpha_3 + \alpha_4) \, d(x^*, y)$

$\Rightarrow \ d(x^* y) \quad = 0 \,$,

as $(\alpha_1 + \alpha_3 + \alpha_4) > 0$

$\Rightarrow x^* = y \ldots\ldots\ldots\ldots.(3.1.8)$

Hence x^* is a fixed point of T_1 as $T_1 y = x^* = y$. Now if z be another fixed point of T_1,

i.e. $T_1 z = z$. Then

$d(x^*, z) = d(T_1 x^*, T_1 z)$

$\geq \alpha_1 d(x^*, z) + \alpha_2 d(x^*, x^*) + \alpha_3 d(z, R_1 z)$
$+ \alpha_4 d(z, x^*)$

$\geq (\alpha_1 + \alpha_4) \, d(x^*, z)$

$\Rightarrow d(x^*, z) \leq \dfrac{1}{(\alpha_1 + \alpha_4)} \, d(x^*, z) \Rightarrow d(x^*, z) = 0$ as $(\alpha_1 + \alpha_4) > 0$ as and by proposition 2.6 (i).

$\Rightarrow x^* = z$. Therefore T_1 has a unique fixed point .Similarly it can be established that $T_2 x^* = x^*$. Hence $T_1 x^* = x^* = T_2 x^*$.Thus x^* is the unique common fixed point of T_1 and T_2. These completed the proof of the theorem.

IV CONCLUSION

In this paper, we have proved some common fixed point theorems for expanding mapping in complete Cone Metric Space. Our results improve and generalize the results given by Prudhvi, K.(10).

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