Second Hankel Determinant for Analytic Functions Defined By Linear Operator

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Abstract: *Let S(λ , n, m) denote the class of analytic and univalent functions in the open unit disk, D = $\{z:|z|<1\}$ with normalized conditions. In the present article an upper bound for the Second Hankel determinant $|a_2a_4-a_3^2|$ is obtained for the analytic functions defined by linear operator.*

Keywords : *Univalent function, Starlike function, convex function, Hankel derminant, Linear Operator.*

1 INTRODUCTION, DEFINITION AND MOTIVATION

Let *D* be the unit disk $\{z:|z|<1\}$. *A* be the class of functions analytic in *D*, satisfying the conditions,

$$f(0) \quad and \quad f'(0)=1 \qquad (1.1)$$

then each function *f* in *A* has the Taylor s expansions,

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \qquad Z\in D \qquad (1.2)$$

The q th determinant for q ≥ 1 and n ≥ is stated by Noonan and Thomas [14] as,

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & & & \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (1.3)$$

This determinant has also been considered by several authors. For example, Noorin [13] determinant the rate of growth of H $_q$ (n) as n $\to\infty$ for function f (1.1) with bounded boundary. Ebrenbary in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were deicussed by Layman s article[9]. It is well known that [4] for f ∈ S and given by (1.2) the sharp inequality $|a_3a_2^2|\le 1$ holds. This corresponds to the hankel determinant with q = 2 and k = 1.

After that, Fekete-Szego further generalized the estimate $|a_3-\mu a_2^2|$ with real μ and f ∈ S for a given class of functions in A the sharp bound for the non linear function $|a_2a_4-a_3^2|$ is known as the Second Hankel Determinant.

This corresponds to the Hankel determinant q = 2 and k = 2. In particular sharp bounds of article [8][12][16][17] for different subclass of univalent function.

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=|a_2a_4-a_3^2| \qquad (1.4)$$

Motivated by the above mentioned results obtained by different authors in this direction. In this paper we consider a certain subclass of analytic functions and obtain an upper bound to the function $|a_2a_4-a_3^2|$ for the function f belonging to this class defined as follows,

Definition 1.1 *A function f ∈ A is said to be in the class,*

$$S^*=\{f(z)\in S;\Re\{\frac{zf'(z)}{f(z)}\}>0 \quad Z\in 0\} \qquad (1.5)$$

$$C=\{f(z)\in S;\Re\{1+\frac{zf'(z)}{f(z)}\}>0 \quad Z\in D\} \qquad (1.6)$$

for f $_j$ ∈ A given by,

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k \quad (j=1,2) \qquad (1.7)$$

the Hadamard product (on convolution) f $_1$ * f $_2$ of f $_1$ and f $_2$ is defined by,

$$(f_1*f_2)z=z+\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k(Z\in D) \qquad (1.8)$$

recall that a family of the Hurwitz-Lerch zeta function $\Phi_{\mu\nu}^{\rho\sigma}$ (z,S,a) [13] is defined by,

$$\Phi_{\mu,\nu}^{\rho,\sigma}(z,S,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho,n}z^n}{(\nu)_{\sigma,n}(n+a)^s} \qquad (1.9)$$

$(\mu\in C;\nu\in C\setminus Z_0^-;\rho,\sigma\in R^+;\rho<\sigma \quad and \quad s,z\in C;$

$\rho=\sigma \quad and \quad s\in C \quad when \quad |z|<1;$

$\rho=\sigma \quad and \quad \Re(s-\mu+\nu)>1 \quad when \quad |z|=1)$

Contains as its special case not only the Hurwitz-Lerch Zeta function,

$$\Phi^{\rho,\sigma}_{\mu,\nu}(z,S,a)=\Phi^{0,0}_{\mu,\nu}(z,S,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}$$

(1.10)

but also the following generalized Hurwitz-Zeta function introduced and studied earlier by Goyal and Laddha [22]

$$\Phi^{1,1}_{\mu,1}(z,S,a)=\Phi_{\mu}(z,S,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s}$$

(1.11)

which for convenience are called the Goyal-Laddha-Hurwitz-Lerch Zeta function. Here $(x)_k$ is Pochhammer symbol (or the shifted factorial, since $(1)_k$ = k!) and $(x)_k$ given in terms of the Gamma functions can be written as,

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}\begin{cases}1, & if\ \ k=0\ \ and\ \ x\in\mathbb{C}\setminus\{0\};\\ x(x+1)\ldots(x+k-1), & if\ \ k\in\mathbb{N}\ \ and\ \ x\in\mathbb{C}.\end{cases}$$

(1.12)

It follows that the Aabed Mohammed et al. [1] introduced the linear operator, $\mathcal{L}^{\lambda,n}_m$ f(z) as the following. For α = 1, in (1.11), we consider the function,

$$G(z)=z\Phi_{\mu}(z,S,1)=z+\sum_{k=2}^{\infty}\frac{(\mu)_{k-1}}{(k-1)!}\frac{z^k}{k^s}\qquad(1.13)$$

Thus,

$$G(z)*G(z)^{-1}=\frac{z}{(1-z)^{\lambda+1}},\quad \lambda>-1\qquad(1.14)$$

$$=z+\sum_{k=2}^{\infty}\frac{(\lambda+1)_{k-1}}{(k-1)!}z^k$$

For s = n, λ ∈ $\mathbb{N}_0$ and μ = m, we define the linear operator

$$\mathcal{L}^{\lambda,n}_m f(z)=z+\sum_{k=2}^{\infty}\frac{(\lambda+1)_{k-1}}{(m)_{k-1}}k^n a_k z^k\qquad(1.15)$$

Definition 1.2 *The function f(z) ∈ A is said to be in the class S(λ, n, m)if it satisfies the inequality,*

$$\Re\{\frac{z(\mathcal{L}^{\lambda,n}_m f(z))'}{\mathcal{L}^{\lambda,n}_m f(z)}\}>0\quad \forall Z\in D\qquad(1.16)$$

2 Preliminaries & Notations

Lemma 2.1 *If the function p ∈ P is given by the series,*

$$p(z)=1+C_1z+C_2z^2+\ldots$$

then the following sharp estimate holds,

$$|p_k|\le 2\qquad k=1,2,\ldots$$

Lemma 2.2 *If the function p ∈ p is given by the series then,*

$$2p_2=p_1^2+x(4-p_1^2)$$

$$4p_3=p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z$$

for some x,z, $|x|\le 1$, $|z|\le 1$.

Lemma 2.3 *The power series of p given in (2.1) converges in δ in to function p if and only if the Toeplitz determinant*

$$D_n=\begin{vmatrix}2 & C_1 & C_2 & \ldots & C_n\\ C-1 & 2 & C_1 & \ldots & C_{n-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ C-n & C-n+1 & \ldots & \ldots & 2\end{vmatrix}$$

Where, $n=1,2,3,\ldots$ & $C_k=\overline{C_k}$ $\forall$ $non-negative$

They are strictly positive except for

$$p(z)=\sum_{k=1}^{m}p_k p_0(e^{it_k z})$$

$p_k>0$, t_k real and $t_k\ne t_j$ for k ≠ j in this case $D_n>0$ for n < m - 1 & D_n =0 for n ≥ m

(1.12)

3 Main Result

Theorem 3.1

$$|a_2a_4-a_3^2|\le\frac{m^2(m+1)^2}{3^{2n}(\lambda+1)^2(\lambda+2)^2}\qquad(3.1)$$

Proof. f ∈ S(λ, n, m) there exist on analytic function p ∈ P in the unit disk D with p(0) = 1 and $\Re\{p(z)\}>0$, such that

(1.13)

$$z\{\frac{(\mathcal{L}^{\lambda,n}_m f(z))'}{\mathcal{L}^{\lambda,n}_m f(z)}\}=P(z)=1+p_1z+p_2z^2+\ldots$$

(3.2)

$$\therefore z\{1+\sum_{k=2}^{\infty}\frac{(\lambda+1)_{k-1}}{(m)_{k-1}}k^{n+1}a_kz^{k-1}\}=p(z)[z+\sum_{k=2}^{\infty}\frac{(\lambda+1)_{k-1}}{(m)_{k-1}}k^na_kz^k]$$

(3.3)

Equating the Coefficients,

$$a_2=\frac{p_1m}{(1+\lambda)^{2n}}\qquad(3.4)$$

$$a_3=\frac{(p_2+p_1^2)m(m+1)}{2\times3^n(\lambda+1)(\lambda+2)}\qquad(3.5)$$

$$a_4=\frac{(2p_3+3p_1p_2+p_1^3)m(m+1)(m+2)}{3\times4^n(\lambda+1)(\lambda+2)(\lambda+3)}$$

(3.6)

using it we get,

$$|a_2a_4-a_3^2|=A(m\lambda)|2p_1p_3+3p_1^2p_2+p_1^4-\beta(m\lambda)(p_2+p_1^2)^2|$$

(3.7)

where,

$$A(m,\lambda)=\frac{m^2(m+1)(m+2)}{3\times2^n\times4^n(\lambda+1)^2(\lambda+2)(\lambda+3)}$$

$$\beta(m,\lambda)=\frac{(m+1)(\lambda+3)\times3\times2^n\times4^n}{4\times3^{2n}(\lambda+2)(m+2)}$$

$$= A(m,\lambda)\{|[(1-\beta(m\lambda)]p_1^4 + (3-2\beta(m\lambda))p_1^2 p_2 + 2p_1p_3 - \beta(m\lambda)p_2^2 |\} \quad (3.8)$$

by applying Lemma (2.2) and (2.3),

$$= A(m,\lambda)\{|[(1-\beta(m\lambda)]p_1^4 + (3-2\beta(m\lambda))p_1^2[\frac{p_1^2x(4-p_1^2)}{2}] - \beta(m,\lambda)[\frac{p_1^2x(4-p_1^2)}{2}]^2 + 2p_1[\frac{p_1^3}{4} + \frac{(4-p_1^2)x}{2} - \frac{p_1(4-p_1^2)x}{2} - \frac{p_1(4-p_1^2)x^2}{4} + \frac{(4-p_1^2(1-|x|^2 z)}{2}]|\} \quad (3.9)$$

Since the function p($e^{i\theta_2}$), ($\theta \in R$) is also in the class p, we assume that without loss of generality that $p_1 > 0$. For convenience of notation we take $p_1 = p$, $p \in [0,2]$. Applying the triangle inequality with the assumptions $p_1 = p \in [0,2]$, $|x| = \rho$ and $|z| \leq 1$ it is obtained that,

$$|a_2a_4 - a_3^2| \leq A(m,\lambda)\{[(3-\frac{9}{4}\beta(m,\lambda))]p^4 + p^2(4-p^2)\rho [\frac{-3}{2} + \frac{3}{2}\beta(m,\lambda)] + \frac{p(4-p^2)\rho^2}{2} + \frac{\beta(m,\lambda)\rho^2(4-p^2)^2}{4} p(4-p^2)(1-\rho^2)\} = F(\rho)$$

We now maximize the function F(ρ) on the closed square [0,2] × [0,1] since,

$$F'(\rho) = A(m,\lambda)\{p^2(4-p^2)[\frac{-3}{2} + \frac{3}{2}\beta(m,\lambda)] + \frac{2\rho(4-p^2)}{2} + \beta(m,\lambda)2\rho(4-p^2) + p(4-p^2)(-2\rho)\}$$

with the elementary calculus we can shows that p '(ρ) > 0 for ρ > 0 implies that F is an increasing function. Let F(1) = G(p),

$$\therefore G(p) = A(m,\lambda)\{(3-\frac{9}{4}\beta(m,\lambda))p^4 + p^2(4-p^2)[\frac{-3}{2} + \frac{3}{2}\beta(m,\lambda)] + \frac{p(4-p^2)}{2} + \frac{\beta(m,\lambda)}{4}(4-p^2)^2\} \quad (3.10)$$

$$\therefore G'(p) = A(m,\lambda)\{(12-9\beta(m,\lambda))p^3 + p^2(4-p^2)[\frac{-3}{2} + \frac{3}{2}\beta(m,\lambda)] + [(4-p^2)2p + p^2(-2p)] + \frac{(4-p^2)}{2} + \frac{(-2p)p}{2} + \frac{\beta(m,\lambda)}{4}2(4-p^2)(-2p)\} \quad (3.11)$$

upper bound for (3.3), G(p) < G(0) for ρ = 1,

$$\therefore |a_2a_4 - a_3^2| \leq A(m,\lambda) \times \frac{\beta(m,\lambda)}{4} \times 16$$

$$\leq \frac{m^2(m+1)^2}{3^{2n}(\lambda+1)^2(\lambda+2)^2} \quad (3.12)$$

Corollary 3.1 *If m = 1, λ = 0, n = 0 we get,*

$$|a_2a_4 - a_3^2| \leq 1$$

This one coincides with the result in the Janteng et al.

Corollary 3.2 *After necessary calculation it is obtained that,*

$$G'(0) = 0, \quad G'(1) = 0 \quad and \quad G''(0) > 0, \quad G''(1) < 0$$

If G(p) has maximum at p = 1, hence we get,

$$|a_2a_4 - a_3^2| \leq \frac{9}{2}\beta(m,\lambda)A(m,\lambda)$$

If λ = 0, m = 1, n = 1.

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}$$

which also stated in Janteng et al.

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