Solving Initial Value Problems by using the Method of Laplace Transforms

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Abstract — *This paper is an overview of the Laplace transform and its applications to Solve initial value problem. We will present a general overview of the Laplace transform of derivatives and examples to illustrate the utility of this method in solving initial value problem.*
Keywords—*Initial value problem, Laplace transform, inverse Laplace transform, initial conditions, boundary conditions etc*

I. INTRODUCTION

The method of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations. While it might seem to be a somewhat cumber some method at times, it is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions. In this paper we see that the Laplace transform, when applied to a differential equation, would change derivatives into algebraic expressions in terms of s and (the transform of) the dependent variable itself. Thus, it can transform a differential equation into an algebraic equation In this introductory section, we discuss definitions, theorems, and use of the Laplace transform in solving initial value problem. Note that the proofs in this section are omitted, however if the reader is so inclined, the details are given in many standard texts on complex analysis and integral transforms.

II. DEFINITION

Let f (t) be defined for $t \geq 0$. The Laplace transform of f (t), denoted by F(s) or L {f (t)}, is an integral transform given by the Laplace integral:

$$L \{f'(t)\} = \int_0^\infty e^{-st} f(t) \, dt.$$

Provided that this (inadequate) integral exists, i.e. that the integral is convergent.
The Laplace transform is an action that transforms a function of t (i.e., a function of time domain), defined on $[0, \infty)$, to a function of s .F(s) is the Laplace transform, or simply transform, of f (t). Together the two functions f (t) and F(s) are called a Laplace transform pair.
For functions of t continuous on $[0, \infty)$, the above revamp to the frequency domain is one-to-one. That is, different continuous functions will have different transforms.

LAPLACE TRANSFORMS FORMULAE

III. TABLE I
FORMULAE

Sr no.	F(t)	$L\{F(t)\}=\phi(s)$
1.	1	$\frac{1}{s}$
2.	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
3.	$\sin(at)$	$\frac{a}{s^2+a^2}$
4.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$
5.	$\sinh(at)$	$\frac{a}{s^2-a^2}$
7.	$\cos(at)$	$\frac{s}{s^2+a^2}$
8.	e^{at}	$\frac{1}{s-a}$

Example: Let f (t) = 1, then $F(s)=\frac{1}{s}$, s>0 .
Sol[n]:

$$L \{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st} \, dt$$

The integral is divergent whenever $s \leq 0$. However, when $s > 0$, it converges to

$$= -\frac{1}{s}(0\text{-}e^0) \quad = -\frac{1}{s}(-1) = \frac{1}{s} = F(s).$$

Solution of Initial Value Problems
Theorem:[Laplace transform of derivatives]
Suppose f is of exponential order, and that f is continuous and f ′ is piecewise continuous on any interval $0 \leq t \leq A$. Then

$$L \{f'(t)\} = s \, L \{f(t)\} - f(0)$$

Applying the theorem multiple times incomes:

$$L \{f''(t)\} = s^2 \, L \{f(t)\} - s \, f(0) - f'(0),$$

$$L \{f'''(t)\} = s3 \, L \{f(t)\} - s2 \, f(0) - s \, f'(0) - f''(0).$$

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

& so on…..

This is an awfully useful part of the Laplace transform, that it changes differentiation with respect to t into multiplication by s .

We are now ready to see how the Laplace transform can be used to solve differentiation equations.

A. Solving initial value problems using the method of Laplace transforms

To solve a linear differential equation using Laplace transforms, there are only 3 basic steps:

Example 1 Solve the following IVP.

$$y' - 10y' + 9y = 5t, \qquad y(0) = -1 \quad y'(0) = 2$$

Solution

The first step in using Laplace transforms to solve an IVP is to take the transform of every term in the differential equation.

$$\mathcal{L}\{y'\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using the appropriate formulas from our **table of Laplace transforms** gives us the following.

$$s^2 Y(s) - sy(0) - y'(0) - 10\big(sY(s) - y(0)\big) + 9Y(s) = \frac{5}{s^2}$$

Plug in the initial conditions and collect all the terms that have a *Y(s)* in them.

$$\big(s^2 - 10s + 9\big)Y(s) + s - 12 = \frac{5}{s^2}$$

Solve
for *Y(s)*.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

At this point it's convenient to recall just what we're trying to do. We are trying to find the solution, *y(t)*, to an IVP. What we've managed to find at this point is not the solution, but its Laplace transform. So, in order to find the solution all that we need to do is to take the inverse transform.

Before doing that let's notice that in its present form we will have to do partial fractions twice. However, if we combine the two terms up we will only be doing partial fractions once. Not only that, but the denominator for the combined term will be identical to the denominator of the first term. This means that we are going to partial fraction up a term with that denominator no matter what so we might as well make the numerator slightly messier and then just partial fraction once.

This is one of those things where we are apparently making the problem messier, but in the process we are going to save ourselves a fair amount of work

Combining the two terms gives,

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

The partial fraction decomposition for this transform is,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerators equal gives,

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Picking appropriate values of *s* and solving for the constants gives,

$s = 0$	$5 = 9B$	$\Rightarrow$	$B = \dfrac{5}{9}$
$s = 1$	$16 = -8D$	$\Rightarrow$	$D = -2$
$s = 9$	$248 = 648C$	$\Rightarrow$	$C = \dfrac{31}{81}$
$s = 2$	$45 = -14A + \dfrac{4345}{81}$	$\Rightarrow$	$A = \dfrac{50}{81}$

Plugging in the constants gives,

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Finally taking the inverse transform gives us the solution to the IVP.

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t$$

B. Example 2 Solve the following IVP.

$$2y' + 3y' - 2y = te^{-2t}, \qquad y(0) = 0 \quad y'(0) = -2$$

Solution

As with the first example, let's first take the Laplace transform of all the terms in the differential equation. We'll the plug in the initial conditions to get,

$$2\left(s^2 Y(s)-sy(0)-y'(0)\right)+3\left(sY(s)-y(0)\right)-2Y(s)=\frac{1}{(s+2)^2}$$

$$\left(2s^2+3s-2\right)Y(s)+4=\frac{1}{(s+2)^2}$$

Now solve for *Y(s)*.

$$Y(s)=\frac{1}{(2s-1)(s+2)^3}-\frac{4}{(2s-1)(s+2)}$$

Now, as we did in the last example we'll go ahead and combine the two terms together as we will have to partial fraction up the first denominator anyway, so we may as well make the numerator a little more complex and just do a single partial fraction. This will give,

$$Y(s)=\frac{1-4(s+2)^2}{(2s-1)(s+2)^3}$$

$$=\frac{-4s^2-16s-15}{(2s-1)(s+2)^3}$$

The partial fraction decomposition is then,

$$Y(s)=\frac{A}{2s-1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)^3}$$

Setting numerator equal gives,

$$-4s^2-16s-15=A(s+2)^3+B(2s-1)(s+2)^2+C(2s-1)(s+2)+D(2s-1)$$

$$=(A+2B)s^3+(6A+7B+2C)s^2+(12A+4B+3C+2D)s$$
$$+8A-4B-2C-D$$

In this case it's probably easier to just set coefficients equal and solve the resulting system of equation rather than pick values of *s*. So, here is the system and its solution.

$$
\begin{aligned}
s^3: &\quad A+2B=0 \\
s^2: &\quad 6A+7B+2C=-4 \\
s^1: &\quad 12A+4B+3C+2D=-16 \\
s^0: &\quad 8A-4B-2C-D=-15
\end{aligned}
\Rightarrow
\begin{aligned}
A=-\frac{192}{125} \quad B=\frac{96}{125} \\[4pt]
C=-\frac{2}{25} \quad D=-\frac{1}{5}
\end{aligned}
$$

We will get a common denominator of 125 on all these coefficients and factor that out when we go to plug them back into the transform. Doing this gives,

$$Y(s)=\frac{1}{125}\left(\frac{-192}{2\left(s-\frac{1}{2}\right)}+\frac{96}{s+2}-\frac{10}{(s+2)^2}-\frac{25\frac{2!}{2!}}{(s+2)^3}\right)$$

Notice that we also had to factor a 2 out of the denominator of the first term and fix up the numerator of the last term in order to get them to match up to the correct entries in our table of transforms.

Taking the inverse transform then gives,

$$y(t)=\frac{1}{125}\left(-96e^{\frac{t}{2}}+96e^{-2t}-10te^{-2t}-\frac{25}{2}t^2e^{-2t}\right)$$

C. Example1: *Solve the differential equation by laplace transform,*

$$y'' - 6y' + 5y = 0, \text{ with initial}$$
conditions $y(0) = 1$, $y'(0) = -3$

[Step 1] : Transform both sides

$$\text{L }\{y'' - 6y' + 5y\} = \text{L }\{0\}$$
$$(s^2 \text{ L }\{y\} - s\,y(0) - y'(0)) - 6(s \text{ L }\{y\} - y(0)) + 5\text{L }\{y\} = 0$$

[Step 2] : Simplify to find $Y(s) = \text{L }\{y\}$

$$(s2 \text{ L }\{y\} - s -(-3)) - 6(s \text{ L }\{y\} - 1) + 5\text{L }\{y\} = 0$$

$$(s2 - 6 s + 5) \text{ L }\{y\} - s + 9 = 0$$
$$(s2 - 6 s + 5) \text{ L }\{y\} = s - 9$$
$$\text{L }\{y\} = \frac{s-9}{s2-6s+5}$$

[Step 3] : Find the inverse transform y(t)

Use partial fractions to simplify,

$$\text{L }\{y\} = \frac{s-9}{s2-6s+5} = \frac{a}{s-1} + \frac{b}{s-5}$$
$$\frac{s-9}{s2-6s+5} = \frac{a(s-5)}{(s-1)(s-5)} + \frac{b(s-1)}{(s-5)(s-1)}$$

On equating the numerators on both sides , we get

$$s - 9 = a(s - 5) + b(s - 1)$$

Equating the corresponding coefficients:

$$1 = a + b \qquad a = 2 \;-9 = -5a - b$$
$$b = -1$$

Hence,

$$\text{L }\{y\} =. \frac{s-9}{s2-6s+5} = \frac{2}{s-1} - \frac{1}{s-5}$$

The last expression corresponds to the Laplace transform of $2e\,t - e\,5t$.

Therefore, it must be that

$$y(t) = 2e\,t - e\,5t.$$

Thism method can be used to solve linear differential equations of any order, rather than just second order equations as in the earlier example. The method will also solve a nonhomogeneous linear differential equation directly, using the exact same three basic steps, without having to discretely solve for the complementary and particular solutions.These points are illustrated in the next examples.

D. Example: *Solve   y″ − 3y′ + 2y = e3t, with initial conditions y(0) = 1,   y′(0) = 0*

[Step 1]: Transform both sides

$(s^2$ L {y} − s y(0) − y′(0)) − 3(sL {y} − y(0)) + 2L {y} = L {e3t}

[Step 2]: Simplify to find Y(s) = L {y}

$(s^2$ L {y} − s − 0) − 3(s L {y} − 1) + L {y} = 1 / (s − 3)

(s2 − 3 s + 2) L {y} − s + 3 = 1 / (s − 3)^2

(s2 − 3 s + 2) L {y} = s − 3 + $\frac{1}{s-3}$ = $\frac{(s-3)2 +1}{(s-3)}$

L {y} = $\frac{s2-6s+10}{(s2-3s+2)(s-3)}$ = $\frac{s2-6s+10}{(s-1)(s-2)(s-3)}$

[Step 3]: Find the inverse transform y(t)

By partial fractions,

L {y} = .$\frac{s2-6s+10}{(s-1)(s-2)(s-3)}$ = $\frac{5}{2}\frac{1}{s-1}$ - $2\frac{1}{s-2}$ + $\frac{1}{2}\frac{1}{s-3}$

Therefore,

$$y(t) = \frac{5}{2}e^t - 2e^t + \frac{1}{2}e^{3t}.$$

CONCLUSION:

In this paper, we applied Laplace transform & inverse Laplace transform method to find the exact solution of initial or boundary value problem with initial conditions. It provided a formula of a solution by just using a basic knowledge of Laplace transform. Some examples were given to show the effectiveness of a new method. It may be concluded that this technique is very dominant and capable in finding solutions. I hope that future students will also get the chance to work on such interesting problems.

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