No-Regret Optimal Control Characterization for an Ill-Posed Wave Equation

Abdelhak Hafdallah[#1], Abdelhamid Ayadi[*2], Chafia Laouar[#3]

[#]Department of mathematics, University of Tébessa, Rue de Constantine 12002, Algeria
[*]Laboratoire des systèmes dynamiques et control, University of OEB, Algeria.
[#] Laboratoire des systèmes dynamiques et contro, University of OEB, Algeria

Abstract — *In this paper, we give a characterization (optimality system) of a quadratic optimal control for an ill-posed wave equation without using the extra hypothesis of Slater (i.e . U_{ad} set of admissible controls has a non-empty interior). By using a parabolic regularization we get a missing data problem where we associate a no-regret control to obtain a singular optimality system, then we pass to limit and by a corrector of order zero we complete the information.*

Keywords — *Ill-posed wave equation, no-regret control, optimal control, parabolic regularization.*

I. INTRODUCTION

The optimal controls of ill-posed problems are not generally regular then its characterization may be different of standard optimal control problems. The aim of this paper is the optimal control characterization for an ill-posed wave problem (instable problem) without requiring Slater hypothesis i.e. the set of admissible controls U_{ad} has a non-empty interior (see [1] and [9]), which is a strong hypothesis for this we'll try to avoid this condition by making another approach it's the regularization i.e. approximate the hyperbolic ill-posed problem by a sequence of parabolic well-posed problems but with incomplete data and taking control u such that $J(u,g) \le J(0,g)\ \forall g$ in some Hilbert space it's the first idea which leads to the no-regret control definition (the no-regret control notion has been used by J.L.Lions in [3] after been introduced by Savage [7] in statistics) and approximate it by a sequence of low-regret controls. After backing by limit to no-regret control we miss information about the final data $y(T)$ and by a zero order corrector (see[4]) we recover her .

II. PRELIMINARIES

Consider an open domain $\Omega \subset \mathbb{R}^N$ with smooth boundary Γ, and denote by $Q = (0,T)\times\Omega$, and by $\Sigma = (0,T)\times\Gamma$, $v \in L^2(Q)$. It's well known that the following wave problem:

$$\begin{cases} y'' - \Delta y = v & \text{in } Q \\ y(0) = 0;\ y'(0) = 0 & \text{in} \Omega \\ y = 0 & \text{on } \Sigma \end{cases} \quad (1)$$

is well-posed with:

$$y \in L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right),$$
$$y'' \in L^2\left(0,T;H^{-1}(\Omega)\right)$$

(see [2] or [5]). If we substitute the condition $y(0) = 0$ by $y(T) = 0$ the above system has no solution (ill-posed).

Counter-example: Consider the one-dimensional wave equation:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = v & \text{in }]0,1[\times]0,T[\\ y(x,0) = 0;\ y(x,T) = 0 & \text{in}]0,1[\\ y(0,t) = 0;\ y(1,t) = 0 & \text{in}]0,T[\end{cases} \quad (2)$$

where $v \in L^2\left(0,T;L^2(0,1)\right)$, with

$$v(x,t) = \sqrt{\frac{2}{\pi}}\sum_{n\ge1} v_n \sin n\pi x$$

and $v_n \in \mathbb{R}$ for every $n \ge 1$. The solution $y(x,t)$ if it exists may be written

$$y(x,t) = \sum_{n\ge1} y_n(t)\, w_n(x)$$

with $w_n(x) = \sqrt{2/\pi}\sin n\pi x$. Then we obtain the second order ordinary differential equation for every $n \ge 1$:

$$\begin{cases} \frac{\partial^2 y_n}{\partial t^2} + n^2\pi^2 y_n = v_n \\ y_n(0) = 0; y_n(T) = 0 \end{cases} \quad (3)$$

And by variation of constants we get:

$$y_n(t) = \frac{2v_n}{n^2\pi^2} sin\frac{n\pi(t-T)}{2}\frac{\sin\frac{n\pi t}{2}}{\cos\frac{n\pi T}{2}}$$

but $\lim_{n\to+\infty}\frac{2v_n}{n^2\pi^2} sin\frac{n\pi(t-T)}{2}\frac{\sin\frac{n\pi t}{2}}{\cos\frac{n\pi T}{2}}$ does not exist, i.e. the series diverges and the solution does not exist.

Remark 1: The above problem has a unique solution but for v in some dense subset of $L^2(Q)$.

For example, consider

$$\hat{V} = \left\{ \begin{array}{c} w = \sum_{i=1}^N \lambda_i w_i\ such\ that: -\Delta w_i = \lambda_i w_i\ in\Omega \\ w_i = 0\ on\ \partial\Omega\ and\ w_i \in L^2(\Omega) \end{array} \right\}$$

there is $f \in L^2(0,T)$ and $w \in \hat{V}$ such that:

$$v(x,t) = \left(\sum_{i=1}^N \lambda_i w_i\right) f(t)$$

for given $v \in \hat{V}\otimes L^2(0,T)$ (which is dense in $L^2(Q)$). Just take y of the form $y(x,t) = \zeta(t)w(x)$ where $\zeta = (\zeta_1,\zeta_2,\dots,\zeta_N)$ solves the following system of ODEs:

$$\begin{cases} \dfrac{\partial^2 \zeta_i}{\partial t^2} + \lambda_i \zeta_i = f(t) & i = 1, \dots, N \\ \zeta_i(0) = 0 \, ; \, \zeta_i(T) = 0 \end{cases}$$

and (2) has a unique solution.■

For the rest of this paper we shall consider the following wave equation:

$$\begin{cases} y'' - \Delta y = v & \text{in } Q \\ y(0) = 0; \ y(T) = 0 & \text{in} \Omega \\ y = 0 & \text{on } \Sigma \end{cases} \quad (4)$$

III.THE OPTIMAL CONTROL PROBLEM

Let U_{ad} a non-empty closed convex subset of $L^2(Q)$ the space of controls, with the quadratic cost function:

$J(v,y) = \|y - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}$ (5)

where $\in U_{ad}$, y_d is a desired state in $L^2(Q)$, $N > 0$.

A pair $(u,z) \in U_{ad} \times L^2(Q)$ satisfying (4) is called a control-state feasible pair. We suppose in what follows that there exists at least an admissible pair, and we consider the optimal control problem:

$\inf_{(v,y)\in U_{ad}\times L^2(Q)} J(v,y)$ (6)

which has a unique solution (u,z) that we should characterize.

Lemma 2: The problem (6) has one only solution (u,z) called the optimal pair.

Proof: $J: L^2(Q) \times L^2(Q) \to \mathbb{R}$ is a lower semi-continuous function, strictly convex, and coercitive. Hence there is a unique admissible pair (u,z) solution to (6). A celebrate classical method to characterize the optimal control solution of (4)-(6) is the penalization method, which consists to approxim -ate (u,z) by the solution of some penalized problem. More precisely, for $\varepsilon > 0$ we define the penalized cost function:

$$J_\varepsilon(v,y) = J(v,y) + \frac{1}{2\varepsilon}\|y'' - \Delta y - v\|^2_{L^2(Q)}$$

The optimal pair $(u_\varepsilon, z_\varepsilon)$ then converges to (u,z) when $\varepsilon \to 0$.

The optimality conditions of Euler-Lagrange for $(u_\varepsilon, z_\varepsilon)$ are the following:

$$\frac{d}{dt} J_\varepsilon\big(u_\varepsilon, z_\varepsilon + t(y - z_\varepsilon)\big)\Big|_{t=0} = 0 \ \forall z \in L^2(Q) \quad (7)$$

and

$$\frac{d}{dt} J_\varepsilon\big(u_\varepsilon, z_\varepsilon + t(y - z_\varepsilon)\big)\Big|_{t=0} \geq 0 \ \forall v \in U_{ad} \quad (8)$$

■

Theorem 3: Under the Slater hypothesis

U_{ad} has a non-empty interior (9)

there is a unique $(u,z,p) \in U_{ad} \times L^2(Q) \times L^2(Q)$, solution to the optimal control problem (4) and (5) - (6). Moreover, this solution is characterized by the following singular optimality system (SOS):

$$\begin{cases} z'' - \Delta z = u & p'' - \Delta p = z - y_d & in\ Q \\ z(0) = 0; \ z(T) = 0, & p(0) = 0; \ p(T) = 0, & in \Omega \\ z = 0 & p = 0 & on\ \Sigma \end{cases} \quad (10)$$

and the variational inequality

$$\langle p + Nu, v - u \rangle_{L^2(Q)} \geq 0 \quad \forall v \in U_{ad} \quad (11)$$

Proof: Again we introduce the penalized cost function:

$$J_\varepsilon = J(v,y) + \frac{1}{2\varepsilon}\|y'' - \Delta y - v\|^2_{L^2(Q)}$$

for and $v \in U_{ad}$ and $y \in D(R)$ where:

$$D(R) = \left\{ \begin{matrix} \varphi : \varphi, \varphi'' - \Delta\varphi \in L^2(Q), \\ \varphi(0) = \varphi(T) = 0, \varphi = 0 \ on\ \Sigma \end{matrix} \right\} \quad (12)$$
$$R\varphi = \varphi'' - \Delta\varphi$$

Let $(u_\varepsilon, z_\varepsilon)$ be the solution of

$$\inf_{(v,y)\in U_{ad}\times D(R)} J_\varepsilon(v,y) = J_\varepsilon(u_\varepsilon, z_\varepsilon) \quad (13)$$

We have

$$z_\varepsilon'' - \Delta z_\varepsilon = u_\varepsilon + \sqrt{\varepsilon} f_\varepsilon, \qquad \|f_\varepsilon\|_{L^2(Q)} \leq C \quad (14)$$

An optimality system is obtained by taking

$$p_\varepsilon = -\frac{1}{\varepsilon}(z_\varepsilon'' - \Delta z_\varepsilon - u_\varepsilon)$$

then from (7)

$$\langle z_\varepsilon - y_d, y \rangle_{L^2(Q)} - \langle p_\varepsilon, Ry \rangle_{L^2(Q)} = 0 \quad \forall y \in D(R) \quad (15)$$

and the variational inequality

$$\langle p_\varepsilon + Nu_\varepsilon, v - u_\varepsilon \rangle_{L^2(Q)} \geq 0 \quad \forall v \in U_{ad}$$

Then, We shall get the result by passing to limit, if we prove that

$$\|p_\varepsilon\|_{L^2(Q)} \leq C$$

By (9) and Remark 1 we can find $v_0 \in U_{ad}$ and $r > 0$ such that

$if\ \|v - v_0\| \leq r\ then\ v \in U_{ad}$

and that

$there\ exists\ y_0 \in D(R)\ verifies\ Ry_0 = v_0$

We have

$$\langle p_\varepsilon + Nu_\varepsilon, v - u_\varepsilon \rangle_{L^2(Q)} = X_\varepsilon + \langle p_\varepsilon, v - u_\varepsilon \rangle_{L^2(Q)}$$

With

$$X_\varepsilon = \langle p_\varepsilon + Nu_\varepsilon, v_0 - u_\varepsilon \rangle_{L^2(Q)} + \langle Nu_\varepsilon, v - v_0 \rangle_{L^2(Q)}$$

But with taking $y = y_0$ in (15), this yields

$$\langle p_\varepsilon, v_0 \rangle_{L^2(Q)} = \langle z_\varepsilon - y_d, y_0 \rangle_{L^2(Q)}$$

Both taking $y = z_\varepsilon$ and using (14) leads to

$$\langle p_\varepsilon, u_\varepsilon \rangle_{L^2(Q)} = \langle z_\varepsilon - y_d, z_\varepsilon \rangle_{L^2(Q)} + \|f_\varepsilon\|^2_{L^2(Q)}$$

Therefore

$$|X_\varepsilon| \leq C$$

Thus

$$\langle p_\varepsilon, v - v_0 \rangle_{L^2(Q)} \geq -C\ for\ every\ \ v \in U_{ad}\ with\ \|v - v_0\| \leq r$$

Whence

$$\|p_\varepsilon\|_{L^2(Q)} \leq \frac{C}{r}$$

■

The last theorem required to use the hypothesis of Slater, but some sets like $(L^2)^+ = \{f \in L^2(Q), f \geq 0\}$ that has an empty interior can be used as a feasible set of controls v, for this reason we'll give another approach.

IV. APPROXIMATION BY A PARABOLIC EQUATION WITH INCOMPLETE DATA

Let's approximate the problem (4) by:

$$\begin{cases} y_\varepsilon'' - \Delta y_\varepsilon - \varepsilon \Delta y_\varepsilon' = v & in\ Q \\ y_\varepsilon = 0 & on\ \Sigma \\ y_\varepsilon(0) = y_\varepsilon(T) = 0, y_\varepsilon'(0) = g & in \Omega \end{cases} \quad (16)$$

where g is an unknown function in $L^2(\Omega)$, ε > 0. For any data (v, g) there exists a unique solution $y_\varepsilon = y_\varepsilon(v, g)$ for the parabolic equation (16) (see [4]).

Consider the quadratic cost function:

$$J_\varepsilon(v, g) = \|y_\varepsilon(v, g) - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)} \quad (17)$$

We want

$$\inf_{v \in U_{ad}} J_\varepsilon(v, g) \ \textit{for every}\ g \in L^2(\Omega)$$

If $G \subset L^2(\Omega)$ is finite (17) has a sense, else it has no sense.

A natural idea is to take

$$\inf_{v \in U_{ad}} \left(\sup_{g \in L^2(\Omega)} J_\varepsilon(v, g) \right)$$

We could get $\sup_{g \in L^2(\Omega)} J_\varepsilon(v, g) = +\infty$. Another idea put forward by J.L.Lions : looking for controls such that:

$$J_\varepsilon(v, g) \le J_\varepsilon(0, g) \quad \forall g \in L^2(\Omega)$$

In other word

$$J_\varepsilon(v, g) - J_\varepsilon(0, g) \le 0 \quad \forall g \in L^2(\Omega)$$

Definition 4: We say that $v \in U_{ad}$ 2 is a no-regret control for (16) (17) if u is the solution of:

$$\inf_{v \in U_{ad}} \left(\sup_{g \in L^2(\Omega)} \left(J_\varepsilon(v, g) - J_\varepsilon(0, g) \right) \right) \quad (1$$

They called no-regret controls because they are doing better than $v = 0$ (doing nothing).

Lemma 5: For every $v \in U_{ad}$ and $g \in L^2(\Omega)$ we have:

$$J_\varepsilon(v, g) - J_\varepsilon(0, g) = J_\varepsilon(v, 0) - J_\varepsilon(0,0) + 2\langle \xi_\varepsilon(0), g\rangle_{L^2(\Omega)} \quad (19)$$

With

$$\begin{cases} \xi_\varepsilon'' - \Delta\xi_\varepsilon + \varepsilon\Delta\xi_\varepsilon' = y_\varepsilon(v, g) & \textit{in } Q \\ \xi_\varepsilon = 0 & \textit{on } \Sigma \\ \xi_\varepsilon(T) = 0, \xi_\varepsilon'(T) = 0 & \textit{in } \Omega \end{cases} \quad (20)$$

Proof: By a simple calculus we find:

$$J_\varepsilon(v, g) - J_\varepsilon(0, g) = J_\varepsilon(v, g) - J_\varepsilon(0, g) + 2\langle y_\varepsilon(v, 0), y_\varepsilon(0, g)\rangle_{L^2(\Omega)}$$

and by using Green formula we get

$$\langle y_\varepsilon(v, 0), y_\varepsilon(0, g)\rangle_{L^2(\Omega)} = \langle \xi_\varepsilon(0), g\rangle_{L^2(\Omega)}$$

where ξ_ε is given by (20). ∎

The problem (18) is defined only for $v \in U_{ad}$ that verify

$$\sup_{g \in L^2(\Omega)} \left(J_\varepsilon(v, g) - J_\varepsilon(0, g) \right) \textit{ exists}$$

i.e. if and only if $v \in K$ where

$$K = \{w \in U_{ad} : \langle \xi_\varepsilon(0; w), g\rangle_{L^2(\Omega)} = 0, \forall g \in L^2(\Omega)\}$$

The main difficulty arises here is that this set is hard to characterize, to eliminate this difficulty we relax our problem and we define the low-regret control.

Definition 6: We say that $u \in U_{ad}$ is a low-regret control for (16)- (17) if u is a solution of:

$$\inf_{v \in U_{ad}} \left(\sup_{g \in L^2(\Omega)} \left(J_\varepsilon(v, g) - J_\varepsilon(0, g) - \gamma\|g\|^2_{L^2(\Omega)} \right) \right) \quad (21)$$

where $\gamma > 0$.

Use (19) to remark that:

$$\sup_{g \in L^2(\Omega)} \left(J_\varepsilon(v, g) - J_\varepsilon(0, g) - \gamma\|g\|^2_{L^2(\Omega)} \right) = J_\varepsilon(v, 0) - J_\varepsilon(0,0) + \sup_{g \in L^2(\Omega)} \left(2\langle \xi_\varepsilon(0), g\rangle_{L^2(\Omega)} - \gamma\|g\|^2_{L^2(\Omega)} \right)$$

$$= J_\varepsilon(v, 0) - J_\varepsilon(0,0) + \frac{1}{\gamma}\|\xi_\varepsilon(0)\|^2_{L^2(\Omega)}$$

to get a classical control problem :

$$\inf_{v \in U_{ad}} J_\varepsilon^\gamma(v) \quad (2$$

with

$$J_\varepsilon^\gamma(v) = J_\varepsilon(v, 0) - J_\varepsilon(0,0) + \frac{1}{\gamma}\|\xi_\varepsilon(0)\|^2_{L^2(\Omega)} \quad (23)$$

We see that (22) - (23) is a standard control problem. We replace (21) by (22)-(23) for the low-regret control problem.

Lemma 7: The problem (16)-(22)-(23) has a unique solution u_ε^γ called the approximate low-regret control.

Proof: We have $J_\varepsilon^\gamma(v) \ge -J_\varepsilon(0,0) = -\|y_d\|^2_{L^2(Q)}$ for every $v \in U_{ad}$, this means $d = \inf_{v \in U_{ad}} J_\varepsilon^\gamma(v)$ exists. Let (v_n) be a minimizing sequence with $d = \lim_{n \to +\infty} J_\varepsilon^\gamma(v_n)$ then

$$-\|y_d\|^2_{L^2(Q)} \le J_\varepsilon^\gamma(v_n) = J_\varepsilon(v_n, 0) - J_\varepsilon(0,0) + \frac{1}{\gamma}\|\xi_\varepsilon(0; v_n)\|^2_{L^2(\Omega)} \le d + 1$$

which gives the bounds

$$\|v_n\|_{L^2(Q)} \le C, \frac{1}{\sqrt{\gamma}}\|\xi_\varepsilon(0; v_n)\|_{L^2(\Omega)} \le C,$$

$$\|y_\varepsilon(v_n, 0) - y_d\|_{L^2(Q)} \le C$$

where C is a constant independent of n:

There exists u_ε^γ such that $v_n \rightharpoonup u_\varepsilon^\gamma$ weakly in U_{ad} (closed), also $y_\varepsilon(v_n, 0) \rightharpoonup y_\varepsilon(u_\varepsilon^\gamma, 0)$ because of continuity w.r.t. the data. Since $J_\varepsilon^\gamma(v)$ is strictly convex u_ε^γ unique. ∎

Proposition 8: The approximate low-regret control u_ε^γ is characterized by the unique $\{y_\varepsilon^\gamma, \xi_\varepsilon^\gamma, \rho_\varepsilon^\gamma, p_\varepsilon^\gamma\}$ defined by:

$$\begin{cases} y_\varepsilon^{\gamma''} - \Delta y_\varepsilon^\gamma - \varepsilon\Delta y_\varepsilon^{\gamma'} = u_\varepsilon^\gamma \\ \xi_\varepsilon^{\gamma''} - \Delta\xi_\varepsilon^\gamma + \varepsilon\Delta\xi_\varepsilon^{\gamma'} = y_\varepsilon^\gamma \\ \rho_\varepsilon^{\gamma''} - \Delta\rho_\varepsilon^\gamma - \varepsilon\Delta\rho_\varepsilon^{\gamma'} = 0 \\ p_\varepsilon^{\gamma''} - \Delta p_\varepsilon^\gamma + \varepsilon\Delta p_\varepsilon^{\gamma'} = y_\varepsilon^\gamma - y_d + \rho_\varepsilon^\gamma \text{ in } Q \\ y_\varepsilon^\gamma = 0, \xi_\varepsilon^\gamma = 0, \rho_\varepsilon^\gamma = 0, \ p_\varepsilon^\gamma = 0 \text{ on } \Sigma \\ y_\varepsilon^\gamma(0) = y_\varepsilon^{\gamma'}(0) = 0 \\ \xi_\varepsilon^\gamma(T) = \xi_\varepsilon^{\gamma'}(T) = 0 \\ \rho_\varepsilon^\gamma(0) = 0, \rho_\varepsilon^{\gamma'}(0) = \frac{1}{\gamma}\xi_\varepsilon^\gamma(0) \\ p_\varepsilon^\gamma(T) = p_\varepsilon^{\gamma'}(T) = 0 \ \text{ in } \Omega \end{cases}$$

and the variational inequality:

$$\langle p_\varepsilon^\gamma + N u_\varepsilon^\gamma, v - u_\varepsilon^\gamma\rangle_{L^2(Q)} \ge 0 \quad \forall v \in U_{ad}$$

Proof: First order Euler condition for (22) and (23) gives:

$$\langle -y_d, y_\varepsilon(w, 0)\rangle_{L^2(Q)} + \langle N u_\varepsilon^\gamma, w\rangle_{L^2(Q)} + \langle \frac{1}{\gamma}\xi_\varepsilon^\gamma(0), \xi_\varepsilon(w, 0)\rangle_{L^2(\Omega)} \ge 0$$

for every $v \in U_{ad}$

with $y_\varepsilon^\gamma = y_\varepsilon(u_\varepsilon^\gamma, 0)$, and $\xi_\varepsilon^\gamma = \xi_\varepsilon(u_\varepsilon^\gamma, 0)$. Let $\rho_\varepsilon^\gamma = \rho_\varepsilon(u_\varepsilon^\gamma, 0)$ be the solution of:

$$\begin{cases} \rho_\varepsilon^{\gamma''} - \Delta\rho_\varepsilon^\gamma - \varepsilon\Delta\rho_\varepsilon^{\gamma'} = 0 \text{ in } Q \\ \rho_\varepsilon^\gamma = 0 \ \text{ on } \Sigma \\ \rho_\varepsilon^\gamma(0) = 0, \rho_\varepsilon^{\gamma'}(0) = \frac{1}{\gamma}\xi_\varepsilon^\gamma(0) \text{ in } \Omega \end{cases}$$

Again by using Green formula:

$$\langle \rho_\varepsilon^\gamma, y_\varepsilon(w,0)\rangle_{L^2(Q)} = \langle \rho_\varepsilon^{\gamma'}(0), \xi_\varepsilon(0)\rangle_{L^2(\Omega)} = \langle \frac{1}{\gamma}\xi_\varepsilon^\gamma(0), \xi_\varepsilon(w,0)\rangle_{L^2(\Omega)}$$

Introduce $p_\varepsilon^\gamma = p_\varepsilon(u_\varepsilon^\gamma, 0)$ with:

$$\begin{cases} p_\varepsilon^{\gamma''} - \Delta p_\varepsilon^\gamma + \varepsilon\Delta p_\varepsilon^{\gamma'} = y_\varepsilon^\gamma - y_d + \rho_\varepsilon^\gamma \text{ in } Q \\ p_\varepsilon^\gamma = 0 \ \text{ on } \Sigma \\ p_\varepsilon^\gamma(T) = p_\varepsilon^{\gamma'}(0) = 0 \text{ in } \Omega \end{cases}$$

This gives

$$\langle y_\varepsilon^\gamma - y_d + \rho_\varepsilon^\gamma, y_\varepsilon(w,0)\rangle_{L^2(Q)} = \langle p_\varepsilon^\gamma, w\rangle_{L^2(Q)}$$

Finally:

$$\langle p_\varepsilon^\gamma + N u_\varepsilon^\gamma, w\rangle_{L^2(Q)} \geq 0 \quad \forall w \in U_{ad} \quad ∎$$

Now we give a singular optimality system for the approximate no-regret control solution of (16) (17) (18). Before doing this we give some a priori estimates as follows:

Proposition 9: The following a priori estimates hold for some $C > 0$

$$\|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C, \|y_\varepsilon^\gamma\|_{L^2(Q)} \leq C, \frac{1}{\sqrt{\gamma}}\|\xi_\varepsilon^\gamma(0)\|_{L^2(\Omega)} \leq C \quad (24)$$

$$\|y_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_\varepsilon^\gamma\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$
$$\varepsilon\|y_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C \quad (25)$$

$$\|\xi_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\xi_\varepsilon^\gamma\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$
$$\varepsilon\|\xi_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C \quad (26)$$

$$\|\rho_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon^\gamma\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$
$$\varepsilon\|\rho_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C \quad (27)$$

$$\|p_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|p_\varepsilon^\gamma\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$
$$\varepsilon\|p_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C \quad (28)$$

Proof: u_ε^γ is the approximate low-regret control then

$$J_\varepsilon^\gamma(u_\varepsilon^\gamma) \leq J_\varepsilon^\gamma(v) \quad \forall v \in U_{ad}$$

in particular when $v = 0$

$$J_\varepsilon(u_\varepsilon^\gamma, 0) - J_\varepsilon(0,0) + \frac{1}{\gamma}\|\xi_\varepsilon^\gamma(0)\|^2_{L^2(\Omega)} \leq \frac{1}{\gamma}\|\xi_\varepsilon(0,0)\|^2_{L^2(\Omega)}$$

but $\xi_\varepsilon(0,0)$ in $\bar{\Omega} \times [0,T]$ so

$$\|y_\varepsilon(u_\varepsilon^\gamma, 0) - y_d\|^2_{L^2(Q)} + N\|u_\varepsilon^\gamma\|^2_{L^2(Q)} + \frac{1}{\gamma}\|\xi_\varepsilon^\gamma(0)\|^2_{L^2(\Omega)} \leq \|y_d\|^2_{L^2(Q)} = Constant, \text{ for every } v \in U_{ad}$$

we obtain (24).

For (25), multiply by $y_\varepsilon^{\gamma'}$ and integrate over $(0,t) \times \Omega$ to obtain:

$$\frac{1}{2}\left\|y_\varepsilon^{\gamma'}(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla y_\varepsilon^\gamma(t)\|^2_{L^2(\Omega)} \leq \int_0^t \langle u_\varepsilon^\gamma, y_\varepsilon^{\gamma'}\rangle_{L^2(\Omega)} d\sigma$$

$$\Rightarrow \left\|y_\varepsilon^{\gamma'}(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|y_\varepsilon^\gamma(t)\|^2_{H_0^1(\Omega)} \leq \int_0^t \left[\left\|y_\varepsilon^{\gamma'}(s)\right\|^2_{L^2(\Omega)} + \|u_\varepsilon^\gamma(s)\|^2_{L^2(\Omega)}\right] ds$$

and by using Gronwall lemma we obtain the first part of (25).

By integrating over Ω we obtain

$$2\varepsilon\left\|\nabla y_\varepsilon^{\gamma'}(t)\right\|^2_{L^2(\Omega)} \leq \|u_\varepsilon^\gamma(t)\|^2_{L^2(\Omega)} + \left\|y_\varepsilon^{\gamma'}(t)\right\|^2_{L^2(\Omega)}$$

$$\Rightarrow \varepsilon\left\|y_\varepsilon^{\gamma'}(t)\right\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$

The estimates (26),(27) and (28) are the same. ∎

Theorem 10: The approximate no-regret control $u_\varepsilon = \lim_{\gamma\to 0} u_\varepsilon^\gamma$ for the ill-posed wave equation (4) is characterized by the unique $\{y_\varepsilon, \xi_\varepsilon, \rho_\varepsilon, p_\varepsilon\}$ given by:

$$\begin{cases} y_\varepsilon'' - \Delta y_\varepsilon - \varepsilon\Delta y_\varepsilon' = u_\varepsilon \\ \xi_\varepsilon'' - \Delta\xi_\varepsilon + \varepsilon\Delta\xi_\varepsilon' = y_\varepsilon \\ \rho_\varepsilon'' - \Delta\rho_\varepsilon - \varepsilon\Delta\rho_\varepsilon' = 0 \\ p_\varepsilon'' - \Delta p_\varepsilon + \varepsilon\Delta p_\varepsilon' = y_\varepsilon - y_d + \rho_\varepsilon \text{ in } Q \\ y_\varepsilon = 0, \xi_\varepsilon = 0, \rho_\varepsilon = 0, p_\varepsilon = 0 \ \text{ on } \Sigma \\ y_\varepsilon(0) = y_\varepsilon'(0) = 0 \\ \xi_\varepsilon(T) = \xi_\varepsilon'(T) = 0 \\ \rho_\varepsilon(0) = 0, \rho_\varepsilon'(0) = \lambda_\varepsilon(0) \\ p_\varepsilon(T) = p_\varepsilon'(T) = 0 \ \text{ in } \Omega \end{cases}$$

with the following limits:

$$u_\varepsilon = \lim_{\gamma\to 0} u_\varepsilon^\gamma, y_\varepsilon = \lim_{\gamma\to 0} y_\varepsilon^\gamma, \xi_\varepsilon = \lim_{\gamma\to 0}\xi_\varepsilon^\gamma,$$
$$\rho_\varepsilon = \lim_{\gamma\to 0}\rho_\varepsilon^\gamma, \qquad p_\varepsilon = \lim_{\gamma\to 0} p_\varepsilon^\gamma$$

and the variational inequality:

$$\langle p_\varepsilon + N u_\varepsilon, v - u_\varepsilon\rangle_{L^2(Q)} \geq 0 \quad \forall v \in U_{ad}$$

with

$$u_\varepsilon, y_\varepsilon, \xi_\varepsilon, \rho_\varepsilon, p_\varepsilon \in L^2(Q), \lambda_\varepsilon(0) \in L^2(\Omega)$$

Proof: By the above proposition $\|y_\varepsilon^\gamma\|_{L^2(Q)} \leq C$ and from (26) we see that ξ_ε^γ (resp. p_ε^γ) satisfies

$$\|\xi_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\xi_\varepsilon^\gamma\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$

respectively

$$\|p_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|p_\varepsilon^\gamma\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$

then $\xi_\varepsilon^\gamma \to \xi_\varepsilon$ (respectively $p_\varepsilon^\gamma \to p_\varepsilon$) weakly in $L^2(0,T;;H_0^1(\Omega))$ by compactness $\xi_\varepsilon^\gamma \to \xi_\varepsilon$ (respectively $p_\varepsilon^\gamma \to p_\varepsilon$) strongly in $L^2(0,T;L^2(\Omega))$.

Also $p_\varepsilon^\gamma \to p_\varepsilon$ strongly in $L^2(0,T;L^2(\Omega))$.

Finally it follows $\frac{1}{\gamma}\xi_\varepsilon^\gamma(0) \to \lambda_\varepsilon(0)$ in $L^2(\Omega)$. ∎

Finally, all those results leads to the following theorem that characterize the no-regret control for (4) -(6).

Theorem 11: The no-regret control $u = \lim_{\varepsilon\to 0} u_\varepsilon$ for the ill-posed wave equation (4) is characterized by the unique $\{y, \xi, \rho, p\}$ given by:

$$\begin{cases} y'' - \Delta y = u \\ \xi'' - \Delta \xi = y \\ \rho'' - \Delta \rho = 0 \\ p'' - \Delta p = y - y_d + \rho \text{ in } Q \\ y = 0, \xi = 0, \rho = 0, p = 0 \ \text{ on } \Sigma \\ y(0) = y'(0) = 0 \\ \xi(T) = \xi'(T) = 0 \\ \rho(0) = 0, \rho'(0) = \lambda(0) \\ p(T) = p'(T) = 0 \ \text{ in } \Omega \end{cases}$$

with the following limits:

$y = \lim_{\varepsilon \to 0} y_\varepsilon, \xi = \lim_{\varepsilon \to 0} \xi_\varepsilon, \rho = \lim_{\varepsilon \to 0} \rho_\varepsilon, p = \lim_{\varepsilon \to 0} p_\varepsilon$

and the variational inequality:

$\langle p + Nu, v - u \rangle_{L^2(Q)} \geq 0 \quad \forall v \in U_{ad}$

where

$u, y, \xi, \rho, p \in L^2(Q), \lambda(0) \in L^2(\Omega)$

Proof: See [4] . ∎

V. CORRECTOR OF ORDER 0

We remark that the passage to limit in the last theorem gives no information about $y(T)$, to recover this value we shall use the notion of zero order corrector which is introduced by Lions in [4].

First, we make the regularity hypothesis

$y, y' \in L^2(0, T; H_0^1(\Omega)) \quad (29)$

Definition 12: We say that is a zero order corrector if

$$\begin{cases} \langle \theta_\varepsilon'', \varphi \rangle_{L^2(\Omega)} + \varepsilon \langle \nabla \theta_\varepsilon', \nabla \varphi \rangle_{L^2(\Omega)} + \langle \nabla \theta_\varepsilon, \nabla \varphi \rangle_{L^2(\Omega)} \\ \quad = \langle \varepsilon f_{\varepsilon 1} + \sqrt{\varepsilon} f_{\varepsilon 2}, \varphi \rangle_{H_0^1(\Omega)} \ \forall \varphi \in H_0^1(\Omega) \\ \theta_\varepsilon + y_\varepsilon \in H_0^1(\Omega) \\ \theta_\varepsilon(0) = 0, \ \theta_\varepsilon(T) + \theta_\varepsilon(T) = 0 \end{cases}$$

with

$$\begin{cases} \|f_{\varepsilon 1}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \\ \|f_{\varepsilon 2}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \end{cases}$$

Remark 13: If is a zero order corrector then m is also a zero order corrector, then we'll take the corrector with

$$m = \begin{cases} 1 \text{ in the neighborhood of } t = T \\ 0 \text{in the neighborhood of } t = 0 \end{cases}$$

Theorem 14: Let θ_ε be a corrector of order 0 defined by (30) and (31), then

$\|y_\varepsilon - (\theta_\varepsilon + y)\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \|\theta'_\varepsilon - (\theta'_\varepsilon + y')\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\varepsilon} \ (32)$

$\theta'_\varepsilon - (\theta'_\varepsilon + y') \to 0 \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \ (33)$

Proof: Let $w_\varepsilon = y_\varepsilon - (\theta_\varepsilon + y)$ then for every $\varphi \in H_0^1(\Omega)$

$\langle w_\varepsilon'', \varphi \rangle_{L^2(\Omega)} + \varepsilon \langle \nabla w_\varepsilon', \nabla \varphi \rangle_{L^2(\Omega)} + \langle \nabla w_\varepsilon, \nabla \varphi \rangle_{L^2(\Omega)} = -\varepsilon \langle \nabla y_\varepsilon', \nabla \varphi \rangle_{L^2(\Omega)} - \langle \varepsilon f_{\varepsilon 1} + \sqrt{\varepsilon} f_{\varepsilon 2}, \varphi \rangle_{H_0^1(\Omega)}$

put $\varphi = w_\varepsilon'$, then

$\frac{1}{2}\frac{d}{dt}\|w_\varepsilon'\|^2_{L^2(\Omega)} + \varepsilon \|w_\varepsilon'\|^2_{H_0^1(\Omega)} + \frac{1}{2}\frac{d}{dt}\|w_\varepsilon\|^2_{H_0^1(\Omega)} = -\varepsilon \langle \nabla y_\varepsilon', \nabla w_\varepsilon' \rangle_{L^2(\Omega)} - \langle \varepsilon f_{\varepsilon 1} + \sqrt{\varepsilon} f_{\varepsilon 2}, w_\varepsilon' \rangle_{H_0^1(\Omega)}$

by integration over $(0, t)$

$$\|w_\varepsilon'(t)\|^2_{L^2(\Omega)} + \|w_\varepsilon(t)\|^2_{H_0^1(\Omega)} + 2\varepsilon \int_0^t \|w_\varepsilon'(s)\|^2_{H_0^1(\Omega)} + \leq C\varepsilon \left(\int_0^t \|w_\varepsilon'(s)\|^2_{H_0^1(\Omega)} \right)^{\frac{1}{2}} + C\sqrt{\varepsilon} \left(\int_0^t \|w_\varepsilon(s)\|^2_{H_0^1(\Omega)} \right)^{\frac{1}{2}}$$

by taking sup on $[0, T]$

$\|w_\varepsilon'\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w_\varepsilon\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \varepsilon \|w_\varepsilon'\|^2_{L^2(0,T;H_0^1(\Omega))} + \leq C\sqrt{\varepsilon} \left[\sqrt{\varepsilon} \|w_\varepsilon'\| \right.$

which gives (32) and

$\|w_\varepsilon'\|_{L^2(0,T;H_0^1(\Omega))} \leq C$

we deduce (33), with

$\|w_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C$

∎

By using zero order corrector with (32) and (33) we can complete information about y and announce the next theorem:

Theorem 15: The quadruplet $\{y, \xi, \rho, p\}$ satisfies by the mean of zero order corrector:

$$\begin{cases} y'' - \Delta y = u \\ \xi'' - \Delta \xi = y \\ \rho'' - \Delta \rho = 0 \\ p'' - \Delta p = y - y_d + \rho \text{ in } Q \\ y = 0, \xi = 0, \rho = 0, p = 0 \ \text{ on } \Sigma \\ y(0) = y'(0) = 0 \\ \xi(T) = \xi'(T) = 0 \\ \rho(0) = 0, \rho'(0) = \lambda(0) \\ p(T) = p'(T) = 0 \ \text{ in } \Omega \end{cases}$$

and the variational inequality:

$\langle p + Nu, v - u \rangle_{L^2(Q)} \geq 0 \quad \forall v \in U_{ad}$

VI. CONCLUSIONS

In this work we avoided requiring the Slater hypothesis (9) to characterize the optimal control of (4)-(5) by using the regularization technique that gives another approach for studying an optimal control of singular distributed problem.

REFERENCES

[1] F. Troltzsch, . *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, graduate Studies in mathematics, American Mathematical Society, Providence, RI, 112.(2010)

[2] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext. Springer, NewYork. (2011)

[3] J. L. Lions, *Contrôle à moindres regrets des systèmes distribués*. C. R. Acad. Sci. Paris Ser. I Math., Vol. 315, pp 1253-1257(1992).

[4] J. L. Lions, *Control of distributed singular systems*, Gauthier-villars 1985.

[5] J.L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Dunod, Paris, 1968.

[6] J. L. Lions, *Perturbations Singulieres dans les Problemes aux Limites et en Contrôle optimal*, Lecture Notes in Mathematics, Springer (1973).

[7] L.J. Savage, *The foundations of Statistics*, 2nd edition, Dover (1972).

[8] R. Dorville, O. Nakoulima and A. Omrane, *On the control of ill-posed distributed parameter systems*, EDP Sciences , April 2007, Vol.17, 50-66

[9] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, *Optimization with PDE constraints*. Springer, 2009.