Solution of Fuzzy Multi-Objective Linear Programming Problems using Fuzzy Programming Techniques based on Exponential Membership Functions

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Abstract*—*In this paper a Fuzzy Multi Objective LPP is first reduced to crisp Multi Objective LPP using ranking function. The crisp Multi Objective LPP is then solved by Zimmerman Technique using exponential membership functions. The results are compared with those obtained using trapezoidal and hyperbolic membership functions in Zimmerman Technique.

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Keywords *- Multi Objective Linear Programming Problem, Fuzzy Multi-objective Linear Programming Problem, Exponential Membership Function.*

I. INTRODUCTION

Most of the real world problems are inherently characterized by multiple, conflicting and incommensurate aspect of evaluation. These areas of evolution are generally operationalized by objective functions to be optimized in the framework of multiple objective linear programming models. Furthermore, when addressing real world problems, frequently the parameters are imprecise numerical quantities. Fuzzy quantities are very adequate for modeling these situations. Bellmann and Zadeh [1]introduced the concept of fuzzy quantities and also the concept of fuzzy decision making.The most common approach to solve fuzzy linear programming problem is to change them into corresponding deterministic linear program. Some methods based on comparison of fuzzy numbers have been suggested by H.R.Maleki[10], A.Ebrahimnejad, S.H.Nasser[12], F.Roubens[7]. L.Campos[5], A.Munoz.Zimmermann[2,3] have introduced fuzzy programming approach to solve crisp multi objective linear programming problem. H.M.Nehiet.al[11]. used ranking function suggested by Delgodoet.al[9]. to solve fuzzy MOLPP.Leberling[4] used a special type non-linear (hyperbolic) membership function for the vector maximum linear programming problem. Dhingra and Moskowitz[6] defined other type of non-linear (exponential, quadratic and logarithmic) membership functions and applied them to an optimal design problem. Verma, Biswal and Biswas[8] used the fuzzy programming technique with some non-linear (hyperbolic and exponential) membership functions to solve a multi objective transportation problems. R.B. Dash and P.D.P Dash[13] introduced a method in which a fuzzy MOLLP is first reduced to crisp MOLLP using ranking function suggested by F. Roubens[7].Then he solved crisp MOLPP using Zimmerman technique based on trapezoidal membership function.

In this paper, following R.B.Dash[13] we reduce Fuzzy MOLPP to crisp MOLPP using Rouben'sRanking function. Then we solve the crisp problem applying exponential membership function. Finally we obtain the membership functions of Fuzzy MOLPP. These results are compared with those obtained using trapezoidal and Hyperbolic membership functions in Zimmerman's Technique.

II. MULTI OBJECTIVE LINEAR PROGRAMMING

The problem to optimize multiple conflicting objective functions simultaneously under given constraints is called multi objective linear programming problem and can be given as follows.

Max $f(x) = (f_1(x), f_2(x) \ldots f_k(x))^T$

s.t.$x \in X = \{ x \in R^n \mid g_j(x) \le 0 , j=1,2 ..m\}$ (2.1)

Where$f_1(x)$, $f_2(x)$ … $f_k(x)$ are k distinct nonlinear objective functions of the decision variables and X is the feasible set of constrained decision.

A. Definition 2.1

x^* is said to be a complete optimize solution for (2.1) if there exist $x^* \in X$

s. t. $f_i(x^*) \ge f_i(x)$, i= 1, 2, 3….k

for all $x \in X$.

III.EXPONENTIAL MEMBERSHIP FUNCTION FOR FUZZY NUMBERS

An exponential membership function is defined by

$$\mu^E Z_p(x) = \begin{cases} 1 & \text{if } Z_p \le L_p \\ \dfrac{e^{-s\psi_p(x)} - e^{-s}}{1-e^{-s}} & \text{if } L_p \le Z_p \le U_p \\ 0 & \text{if } Z_p \le U_p \end{cases} \quad (3.1)$$

Where $\psi_p(x) = \frac{Z_p(x) - L_p}{U_p - L_p}$

P = 1, 2, 3…p and s is a non-zero parameter prescribed by the decision maker

A fuzzy number A = (a, b, c) is said to be a triangular fuzzy number if its membership function is given by

$$\mu_A^E(x) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 1 & x = b \\ \frac{x-b}{b-c} & b \le x \le c \\ 0 & \text{Otherwise} \end{cases}$$

Assume that R: F → *R*. R is linear ordered function that maps each fuzzy number into the real number, in which F denotes the whole fuzzy numbers. Accordingly for any two fuzzy numbers $\tilde{a}$ and $\tilde{b}$we have.

$\tilde{a} \underset{R}{\geq} \tilde{b}$ iff $R(\tilde{a}) \geq R(\tilde{b})$

$\tilde{a} \underset{R}{>} \tilde{b}$ iff $R(\tilde{a}) > R(\tilde{b})$

$\tilde{a} \underset{R}{=} \tilde{b}$ iff $R(\tilde{a}) = R(\tilde{b})$

We restrict our attention to linear ranking function, that is a ranking function *R* such that

$R(k\tilde{a} + \tilde{b}) = k R(\tilde{a}) + R(\tilde{b})$

For any $\tilde{a}$ and $\tilde{b}$ in F and any k ∈ R.

A. Rouben's ranking function

The ranking function suggested by F. Rouben is defined by

$$R(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf\ \tilde{a}_\alpha + \sup\ \tilde{a}_\alpha) d\alpha$$

This reduces to

$R(\tilde{a}) = \frac{1}{2}\ (a^L + a^U + \frac{1}{2}(\beta - \alpha))$

for a trapezoidal number

$\tilde{a} = (a^L - a, a^L, a^v, a^v + \beta)$

B. Solving Fuzzy multi objective Linear Programming Problem

A fuzzy multi objective linear programming problem is defined as followed

$$\begin{aligned} &\text{Max } \tilde{Z}_p = \sum_j \tilde{c}_{pj} x_j \quad p=1,2...q \\ &\text{s.t. } \sum_j a_{ij} x_j \le \tilde{b} \quad i = 1, 2...m \qquad (3.2) \\ &\text{where } x_j \ge 0 \end{aligned}$$

$\tilde{a}_{ij}$ and $\tilde{c}_{pj}$ are in the above relation are in trapezoidal form as

$\tilde{a}_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4)$

$\tilde{c}_{pj} = (c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4)$

C. Definition 3.2

$x \in X$ is said to be feasible solution to the FMOLP problem (3.2) if it satisfies constraints of (3.2).

D. Definition 3.3

$x^* \in X$ is said to be an optimal solution to this FMOLP problem (3.2) if there does not exist another x ϵ X such that $\tilde{z}_i(x) \ge \tilde{z}_i(x^*)$ for all i =1, 2…q. Now the FMOLP can be transformed to a classic form of a MOLP by applying ranking function *R* as follows.

$$\begin{aligned} &\text{Max } R(\tilde{z}_p) = \textstyle\sum_j R(\tilde{c}_{pj}) x_j \quad p = 1, 2…q \\ &\text{s.t. } \textstyle\sum_j R(\tilde{a}_{ij}) x_j \le R(\tilde{b}_i) \quad i = 1, 2…m \\ &\quad xj \ge 0 \end{aligned}$$

So we have

$$\begin{aligned} &\text{Max } z_p' = \textstyle\sum_j c_{pj}' x_j \quad p=1, 2…q \\ &\text{s.t. } \textstyle\sum_j a_{ij}' x_j \le b_i' \quad i=1, 2…m \qquad (3.3) \\ &\quad xj \ge 0 \end{aligned}$$

Where a_{ij}' , b_i' , c_j' are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ respectively which are obtained by applying the ranking function *R*.

E. Lemma 3.4

The optimum solution of (3.2) and (3.3) are equivalent.

Proof: Let M_1, M_2 be set of all feasible solutions of (3.2) and (3.3) respectively.

Then x ϵ M_1 iff $\sum_j (\tilde{a}_{ij}) x_j \le (\tilde{b}_i)$ i=1, 2…m

By applying ranking function we have

$\sum_j R(\tilde{a}_{ij}) x_j \le R(\tilde{b}_i)$ i=1, 2…m

$\Rightarrow \sum_j a_{ij}' x_j \le b_i'$

Hence x ϵ M_2

Thus $M_1 = M_2$

Let x* ϵ X be the complete optimal solution of (3.2).

Then $\tilde{z}_p(x^*) \ge \tilde{z}_p(x)$ for all x ϵ X

where 'X' is a feasible set of solutions.

Thus

$R\left(\tilde{z}_p(x^*)\right) \geq R\left(\tilde{z}_p(x)\right)$

$\Rightarrow R(\sum \tilde{c}_{pj} x_j^*) \geq R(\sum \tilde{c}_{pj} x_j)$

$\Rightarrow \sum R(\tilde{c}_{pj}) x_j^* \geq \sum R(\tilde{c}_{pj}) x_j \qquad j=1, 2..q$

$\Rightarrow \sum c'_{pj} x_j^* \geq \sum c'_{pj} x_j$

$\Rightarrow z'_p(x^*) \geq z'_p(x) \qquad \forall\ x$

IV. FUZZY PROGRAMMING TECHNIQUE

To solve MOLLP

$$\text{Max } z'_p = \sum_j c'_{pj} x_j \qquad p=1, 2,\ldots q$$

$$\text{s.t. } \sum_j a'_{ij} x_j \leq b'_i \qquad i=1, 2,\ldots n$$

$$x_i \geq 0$$

We use fuzzy programming technique suggested by Zimmermann. The method is presented briefly in the following steps.

Step-1

Solve the multi objective linear programming problem by considering one objective at a time and ignoring all others. Repeat the process 'q' times for 'q' different objective functions.

Let X_1, X_2,….,X_q be the ideal situations for respective functions.

Step-2

Using all the above q ideal solutions in the step-1 construct a pay-off matrix of size q by q. Then from pay-off matrix find the lower bound(Lp) and upper bound(Up) for the objective function.

z'_p as: $L_p \leq z'_p \leq U_p \qquad p=1, 2,\ldots q$

Step-3

If we use the exponential membership function as defined (3.1) then an equivalent crisp model for the fuzzy model can be formulated as follows.

Min λ

$$\lambda \leq \frac{e^{-s\psi_p(x)} - e^{-s}}{1-e^{-s}} \qquad p=1, 2...q$$

$$\text{s.t. } \sum c'_{pj} x_j + (U_p - L_p)\lambda \geq U_p \qquad p=1, 2\ldots q$$

$$\sum a'_{ij} x_j \leq b'_i \qquad i=1, 2\ldots m$$

$$\lambda \geq 0,\ x_j \geq 0, \qquad j=1, 2\ldots n$$

The above problem can be further simplified as:

Min x_4

s.t

$$s\{1-\psi_p(x)\} \geq x_4 \qquad p=1, 2\ldots q$$

$$\sum c'_{pj} x_j + (U_p - L_p)x_4 \geq U_p \qquad p=1, 2\ldots q$$

$$\sum a'_{ij} x_j \leq b'_i \qquad i=1, 2...m$$

Solving (5.2) and (5.4) we get

$z_1 = \frac{5123}{669}$, $z_2 = \frac{14198}{2007}$

$$x_4 \geq 0,\ x_j \geq 0, \qquad j=1, 2\ldots n$$

Step-4

Solve crisp model to find the optimal compromise solutions. Evaluate the values of objective functions at the compromise solutions.

A. Numerical example

$$\text{Max}: \tilde{z}_1(x) = \widetilde{10}x_1 + \widetilde{11}x_2 + \widetilde{15}x_3$$

$$\text{Max}: \tilde{z}_2(x) = \tilde{5}x_1 + \tilde{4}x_2 + \tilde{9}x_3$$

$$\text{s.t } \tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}x_3 \leq \widetilde{15}$$

$$\tilde{7}x_1 + \tilde{5}x_2 + \tilde{3}x_3 \leq \widetilde{30}$$

$$\tilde{3}x_1 + \tilde{4}x_2 + \widetilde{10}x_3 \leq \widetilde{100}$$

$$x_1, x_2, x_3 \geq 0 \qquad (5.1)$$

Where

$\tilde{1} = (0.94, 1, 1.1)$

$\tilde{1} = (0.98, 1, 1.06)$

$\tilde{1} = (0.96, 1, 1.12)$

$\tilde{7} = (6.2, 6.4, 7.4, 7.6)$

$\tilde{5} = (4.3, 4.4, 5.2, 5.7)$

$\tilde{3} = (2.2, 2.3, 3.3, 3.8)$

$\tilde{3} = (2.3, 2.5, 3.3, 3.5)$

$\tilde{4} = (3.2, 3.4, 4.2, 4.8)$

$\widetilde{10} = (9.3, 10, 10.3)$

$\widetilde{15} = (14.4, 14.6, 15.2, 15.6)$

$\widetilde{80} = (79.2, 79.3, 80.3, 80.8)$

$\widetilde{100} = (99.3, 99.4, 100.2, 100.7)$

$\widetilde{10} = (9.2, 9.4, 10.2, 10.4)$

$\widetilde{11} = (10.3, 10.6, 11.2, 11.5)$

$\widetilde{15} = (14.4, 14.5, 15.1, 15.6)$

$\tilde{5} = (4.9, 5, 5.5)$

$\tilde{4} = (3.2, 4, 4.4)$

$\tilde{9} = (8.6, 9, 9.6)$

Using ranking function suggested by Rouben [7] the problem reduces to

$$\max z'_1(x) = R(\widetilde{10})x_1 + R(\widetilde{11})x_2 + R(\widetilde{15})x_3$$

$$\max z'_2(x) = R(\tilde{5})x_1 + R(\tilde{4})x_2 + R(\tilde{9})x_3$$

s.t

$$R(\tilde{1})x_1 + R(\tilde{1})x_2 + R(\tilde{1})x_3 \leq R(\widetilde{15})$$

$$R(\tilde{7})x_1 + R(5)x_2 + R(\tilde{3})x_3 \leq R(\widetilde{80})$$

$$R(\tilde{3})x_1 + R(\tilde{4})x_2 + R(\widetilde{10})x_3 \leq R(\widetilde{100})$$

$$x_1, x_2, x_3 \geq 0$$

$$\Rightarrow \max z'_1(x) = 9.8x_1 + 10.9x_2 + 14.9x_3 \qquad (5.2)$$

$$\max z'_2(x) = 5.1x_1 + 3.9x_2 + 9.1x_3 \qquad (5.3)$$

s.t

$$1.01x_1 + 1.01x_2 + 1.02x_3 \leq 14.95$$

$$6.9x_1 + 4.9x_2 + 2.9x_3 \leq 79.9 \qquad (5.4)$$

$$2.9x_1 + 3.9x_2 + 9.9x_3 \leq 99.9$$

$$x_1, x_2, x_3 \geq 0$$

Solving (5.3) and (5.4) we get

$z_1 = \frac{15369}{2347}$, $z_2 = \frac{57544}{7041}$

Function	LB	UB
z_1'	185.94	188.87
z_2'	94.24	107.77

If we use exponential membership function with the parameter s=1, an equation crisp model can be formulated as

$$
\begin{aligned}
&\text{Min } x_4 \\
&\text{s.t} \\
&s[z_1(x)] + x_4(U_1 - L_1) \geq s(U_1) \\
&s[z_2(x)] + x_4(U_2 - L_2) \geq s(U_2) \\
&\sum a_{ij} x_j \leq b_i' \qquad i=1,2\ldots m \\
&x_4 \geq 0, x_j \geq 0, \qquad j=1,2\ldots n
\end{aligned}
$$

Using exponential function the problem reduces to

$$
\begin{aligned}
&\text{Min } X_4 \\
&9.8x_1 + 10.9x_2 + 14.9x_3 + 2.93x_4 \geq 188.87 \\
&5.1x_1 + 3.9x_2 + 9.1x_3 + 13.53x_4 \geq 107.77 \\
&1.01x_1 + 1.01x_2 + 1.02x_3 \leq 14.95 \qquad (5.4) \\
&6.9x_1 + 4.9x_2 + 2.9x_3 \leq 79.9 \\
&2.9x_1 + 3.9x_2 + 9.9x_3 \leq 99.9 \\
&x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Solving we get

$X_1 = 3.28119$
$X_2 = 3.82066$
$X_3 = 7.62465$
$X_4 = 0.49897$

Now the optimal value of the objective functions of FMOLPP(4.3) becomes

$$
\begin{aligned}
\tilde{z}_1^* &= \widetilde{10}x_1^* + \widetilde{11}x_2^* + \widetilde{15}x_3^* \\
&= (9.2, 9.4, 10.2, 10.4)x_1^* + (10.3, 10.6, 11.2, 11.5)x_2^* + (14.4, 14.5, 15.1, 15.6)x_3^* \\
&= (179.334706, 181.899607, 191.391745, 197.006506)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{z}_2^* &= \tilde{5}x_1^* + \tilde{4}x_2^* + \tilde{9}x_3^* \\
&= (4.9, 5, 5.5)x_1^* + (3.2, 4, 4.4)x_2^* + (8.6, 9, 9.8)x_3^* \\
&= (93.875933, 99.96399, 109.579019)
\end{aligned}
$$

The membership functions corresponding to the fuzzy objective functions are as follows.

$$
\mu_{\tilde{Z}_1}^{E}(x) = \begin{cases}
0 & x \leq 179.334706 \\
\dfrac{x - 179.334706}{2.564901} & 179.334706 < x \leq 181.899607 \\
1 & 181.899607 < x \leq 191.391745 \\
\dfrac{197.006506 - x}{5.614761} & 191.391745 < x \leq 197.006506 \\
0 & x \geq 197.006506
\end{cases}
$$

$$
\mu_{\tilde{Z}_2}^{E}(x) = \begin{cases}
0 & x \leq 93.875933 \\
\dfrac{x - 93.875933}{6.088057} & 93.875933 < x \leq 99.96399 \\
\dfrac{99.96399 - x}{9.615029} & 99.96399 < x \leq 109.579019 \\
0 & x \geq 109.579019
\end{cases}
$$

V. Conclusions

It is observed that the result obtained in this paper is very close to those obtained using trapezoidal membership function (as in [13]) and those using hyperbolic membership function (as in [14]) in the Zimmerman's algorithm..

Thus this is an alternative solution to the Fuzzy MOLPP.

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