On general Eulerian integral of certain products of two multivariable I-functions defined by Nambisan and a class of polynomials

$F.Y. \ AYANT^1$

1 Teacher in High School , France

ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Nambisan et al [2], a class of multivariable polynomials and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular cases concerning the multivariable H-function and the Srivastava-Daoust polynomial

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, Srivastava-Daoust polynomial

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1.Introduction

In this paper, we evaluate a general class of polynomials and the product of two multivariable I-functions defined by Nambisan et al [2]. This function is an extension of the multivariable H-function defined by Srivastava et al [8]. We will use the contracted form.
The I-function of $s-$variable is defined and represented in the following manner.

$$I(z'_1,\cdots,z'_s) = I^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s}_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}\left(\begin{array}{c|c} z'_1 & (a'_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(s)}_j;A'_j)_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (b'_j;\beta'^{(1)}_j,\cdots,\beta'^{(s)}_j;B'_j)_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{c} (c'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,\gamma'^{(s)}_j;C'^{(r)}_j)_{1,p'_r} \\ \\ (d'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)_{1,q'_r} \end{array}\right) \tag{1.1}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z'^{t_i}_i\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.2}$$

where $\phi(t_1,\cdots,t_s),\zeta_i(t_i), i=1,\cdots,s$ are given by :

$$\phi(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'}\Gamma^{A'_j}\left(1-a'_j+\sum_{i=1}^{s}\alpha'^{(i)}_jt_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A'_j}\left(a'_j-\sum_{i=1}^{s}\alpha'^{(i)}_jt_j\right)\prod_{j=m'+1}^{q'}\Gamma^{B'_j}\left(1-b'_j+\sum_{i=1}^{s}\beta^{(i)}_jt_j\right)} \tag{1.3}$$

$$\zeta_i(t_i) = \frac{\prod_{j=1}^{n'_i}\Gamma^{C'^{(i)}_j}\left(1-c'^{(i)}_j+\gamma'^{(i)}_j t_i\right)\prod_{j=1}^{m'_i}\Gamma^{D'^{(i)}_j}\left(d'^{(i)}_j-\delta'^{(i)}_j t_i\right)}{\prod_{j=n'_i+1}^{p'_i}\Gamma^{C'^{(i)}_j}\left(c'^{(i)}_j-\gamma'^{(i)}_j t_i\right)\prod_{j=m'_i+1}^{q'_i}\Gamma^{D'^{(i)}_j}\left(1-d'^{(i)}_j+\delta'^{(i)}_j t_i\right)} \tag{1.4}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if

$$U_i = \sum_{j=1}^{p'} A'_j \alpha'^{(i)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(i)}_j + \sum_{j=1}^{p'_i} C'^{(i)}_j \gamma'^{(i)}_j - \sum_{j=1}^{q'_i} D'^{(i)}_j \delta'^{(i)}_j \leqslant 0, i = 1, \cdots, s \tag{1.5}$$

The integral (2.1) converges absolutely if

where$|arg(z'_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, s$

$$\Delta_k = -\sum_{j=n'+1}^{p'} A'_j \alpha'^{(k)}_j - \sum_{j=1}^{q'} B'_j \beta^{(k)}_j + \sum_{j=1}^{m'_k} D'^{(k)}_j \delta'^{(k)}_j - \sum_{j=m'_k+1}^{q'_k} D'^{(k)}_j \delta'^{(k)}_j + \sum_{j=1}^{n'_k} C'^{(k)}_j \gamma'^{(k)}_j - \sum_{j=n'_k+1}^{p'_k} C'^{(k)}_j \gamma'^{(k)}_j > 0 \tag{1.6}$$

Consider the second multivariable I-function.

$$I(z''_1, \cdots, z''_u) = I^{0,n'':m''_1,n''_1;\cdots;m''_u,n''_u}_{p'',q'':p''_1,q''_1;\cdots;p''_u,q''_u}\left(\begin{array}{c|c} z''_1 & (a''_j; \alpha''^{(1)}_j, \cdots, \alpha''^{(u)}_j; A''_j)_{1,p''} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z''_u & (b''_j; \beta''^{(1)}_j, \cdots, \beta''^{(u)}_j; B''_j)_{1,q''} : \end{array}\right.$$

$$\left.\begin{array}{c} (c''^{(1)}_j, \gamma''^{(1)}_j; C''^{(1)}_j)_{1,p''_1}; \cdots; (c''^{(u)}_j, \gamma''^{(u)}_j; C''^{(u)}_j)_{1,p''_u} \\ \\ (d''^{(1)}_j, \delta''^{(1)}_j; D''^{(1)}_j)_{1,q''_1}; \cdots; (d''^{(u)}_j, \delta''^{(u)}_j; D''^{(u)}_j)_{1,q''_u} \end{array}\right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^u}\int_{L''_1}\cdots\int_{L''_u}\psi(x_1,\cdots,x_u)\prod_{i=1}^{u}\xi_i(x_i) z''^{x_i}_i \mathrm{d}x_1\cdots\mathrm{d}x_u \tag{1.8}$$

where $\psi(x_1, \cdots, x_u), \xi_i(x_i), i = 1, \cdots, u$ are given by :

$$\psi(x_1,\cdots,x_u) = \frac{\prod_{j=1}^{n''}\Gamma^{A''_j}\left(1-a''_j+\sum_{i=1}^{u}\alpha''^{(i)}_j x_j\right)}{\prod_{j=n''+1}^{p''}\Gamma^{A''_j}\left(a''_j-\sum_{i=1}^{u}\alpha''^{(i)}_j x_j\right)\prod_{j=m'+1}^{q''}\Gamma^{B''_j}\left(1-b''_j+\sum_{i=1}^{u}\beta''^{(i)}_j x_j\right)} \tag{1.9}$$

$$\xi_i(x_i) = \frac{\prod_{j=1}^{n''_i}\Gamma^{C''^{(i)}_j}\left(1-c''^{(i)}_j+\gamma''^{(i)}_j x_i\right)\prod_{j=1}^{m''_i}\Gamma^{D''^{(i)}_j}\left(d''^{(i)}_j-\delta''^{(i)}_j x_i\right)}{\prod_{j=n''_i+1}^{p''_i}\Gamma^{C''^{(i)}_j}\left(c''^{(i)}_j-\gamma''^{(i)}_j x_i\right)\prod_{j=m''_i+1}^{q''_i}\Gamma^{D''^{(i)}_j}\left(1-d''^{(i)}_j+\delta''^{(i)}_j x_i\right)} \tag{1.10}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i' = \sum_{j=1}^{p''} A_j'' \alpha_j''^{(i)} - \sum_{j=1}^{q''} B_j' \beta_j''^{(i)} + \sum_{j=1}^{p_i''} C_j''^{(i)} \gamma_j''^{(i)} - \sum_{j=1}^{q_i''} D_j''^{(i)} \delta_j''^{(i)} \leqslant 0, i = 1, \cdots, u \tag{1.11}$$

The integral (2.1) converges absolutely if

where$|arg(z_k'')| < \frac{1}{2}\Delta_k''\pi, k = 1, \cdots, u$

$$\Delta_k' = -\sum_{j=n''+1}^{p''} A_j'' \alpha_j''^{(k)} - \sum_{j=1}^{q''} B_j'' \beta_j''^{(k)} + \sum_{j=1}^{m_k''} D_j''^{(k)} \delta_j''^{(k)} - \sum_{j=m_k''+1}^{q_k''} D_j''^{(k)} \delta_j''^{(k)} + \sum_{j=1}^{n_k''} C_j''^{(k)} \gamma_j''^{(k)} - \sum_{j=n_k''+1}^{p_k''} C_j''^{(k)} \gamma_j''^{(k)} > 0 \tag{1.12}$$

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \cdots, h_u}[z_1, \cdots, z_u] = \sum_{R_1, \cdots, R_u = 0}^{h_1 R_1 + \cdots h_u R_u \leqslant L} (-L)_{h_1 R_1 + \cdots + h_u R_u} B(E; R_1, \cdots, R_u) \frac{z_1^{R_1} \cdots z_u^{R_u}}{R_1! \cdots R_u!} \tag{1.13}$$

The coefficients are $B[E; R_1, \ldots, R_v]$ arbitrary constants, real or complex.

2. Integral representation of Lauricella function of several variables

The Lauricella function $F_D^{(k)}$ is defined as , see Srivastava et al [6]

$$F_D^{(k)}[a, b_1, \cdots, b_k; c; x_1, \cdots, x_k] = \frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^k \Gamma(b_j)} \frac{1}{(2\pi\omega)^k} \int_{L_1} \cdots \int_{L_k} \frac{\Gamma\left(a + \sum_{j=1}^k \zeta_j\right)\Gamma(b_1 + \zeta_1), \cdots, \Gamma(b_k + \zeta_k)}{\Gamma\left(c + \sum_{j=1}^k \zeta_j\right)}$$

$$\prod_{j=1}^k \Gamma(-\zeta_j)(-x_j)^{\zeta_i} \, \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_k \tag{2.1}$$

where $max\left[|arg(-x_1)|, \cdots, |arg(-x_k)|\right] < \pi, c \neq 0, -1, -2, \cdots$.

In order to evaluate a number of integrals of multivariable I-functions, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k (f_j t + g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^k (af_j + g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha, -\sigma_1, \cdots, -\sigma_k; \alpha+\beta; -\frac{(b-a)f_1}{af_1+g_1}, \cdots, -\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.2}$$

where $a, b \in \mathbb{R}(a < b), \alpha, \beta, f_i, g_i, \sigma_i \in \mathbb{C}, (i = 1, \cdots, k)$; $min(Re(\alpha), Re(\beta)) > 0$ and

$$\max_{1 \leqslant j \leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i + g_i}\right|\right\} < 1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([7], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.3}$$

integrating term by term with the help of the integral given by Saigo and Saxena [3, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [7, page 60].

3. Eulerian integral

Let

$$X=m'_1,n'_1;\cdots;m'_s,n'_s;m''_1,n''_1;\cdots;m''_u,n''_u;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.1}$$

$$Y=p'_1,q'_1;\cdots;p'_s,q'_s;p''_1,q''_1;\cdots;p''_u,q''_u;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.2}$$

$$A=(a'_j;A'^{(1)}_j,\cdots,A'^{(s)}_j,0,\cdots,0,0,\cdots,0,0,\cdots,0;A'_j)_{1,p'} \tag{3.3}$$

$$B=(b'_j;B'^{(1)}_j,\cdots,B'^{(s)}_j,0\cdots,0,0\cdots,0,0,\cdots,0;B'_j)_{1,q'} \tag{3.4}$$

$$A'=(a''_j;0,\cdots,0,A''^{(1)}_j,\cdots,A''^{(u)}_j,0,\cdots,0,0,\cdots,0;A''_j)_{1,p''} \tag{3.5}$$

$$B'=(b''_j;0,\cdots,0,B''^{(1)}_j,\cdots,B''^{(u)}_j,0,\cdots,0,0,\cdots,0;B''_j)_{1,q''} \tag{3.6}$$

$$\mathrm{C}=(c'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,\gamma'^{(s)}_j;C'^{(s)}_j)_{1,p'_s};(c''^{(1)}_j,\gamma''^{(1)}_j;C''^{(1)}_j)_{1,p''_1};\cdots;$$

$$(c''^{(u)}_j,\gamma^{(u)}_j;C''^{(u)}_j)_{1,p''_u};(1,0;1);\cdots;(1,0;1);(1,0;1);\cdots;(1,0;1) \tag{3.7}$$

$$D=(d'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)_{1,q_s};(d''^{(1)}_j,\delta''^{(1)}_j;D''^{(1)}_j)_{1,q''_1};\cdots;$$

$$(d''^{(u)}_j,\delta''^{(u)}_j;D''^{(u)}_j)_{1,q''_u};(0,1;1);\cdots;(0,1;1);(0,1;1);\cdots;(0,1;1) \tag{3.8}$$

$$K_1=(1-\alpha-\sum_{i=1}^{r}R_i(\mu_i+\mu'_i);\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu'_u,1,\cdots,1,\upsilon_1,\cdots,\upsilon_l;1) \tag{3.9}$$

$$K_2=(1-\beta-\sum_{i=1}^{r}R_i(\rho_i+\rho'_i);\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,0,\cdots,0,\tau_1,\cdots,\tau_l;1) \tag{3.10}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1;1]_{1,P} \tag{3.11}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{r}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,\underset{j}{1},\cdots,0,\zeta'_j,\cdots,\zeta_j^{(l)};1]_{1,k} \tag{3.12}$$

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{r}R_i(\mu_i+\mu'_i+\rho_i+\rho'_i);\mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu'_1+\rho'_1,\cdots,\mu'_u+\rho'_u,$$

$$L_1 = (1-\alpha-\beta-\sum_{i=1}^{r} R_i(\mu_i+\mu'_i+\rho_i+\rho'_i);\mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu'_1+\rho'_1,\cdots,\mu'_u+\rho'_u,$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l;1) \quad (3.13)$$

$$L_Q = [1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1;1]_{1,Q} \quad (3.14)$$

$$L_j = [1+\sigma_j-\sum_{i=1}^{r} R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,0,\zeta'_j,\cdots,\zeta_j^{(l)};1]_{1,k} \quad (3.15)$$

$$B_r = \frac{(-L)_{h_1R_1+\cdots+h_rR_r}B(E;R_1,\cdots,R_r)}{R_1!\cdots R_r!} \quad (3.16)$$

$$\mathfrak{A} = A,A';\mathfrak{B} = B,B' \quad (3.17)$$

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_r}\begin{pmatrix} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t = (b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^r R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^k(af_j+g_j)^{\sigma_j-\sum_{i=1}^r R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}B_r z_1^{R_1}\cdots z_r^{R_r}$$

$$I^{0,n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\varsigma_j^{(1)}}} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\varsigma_j^{(l)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \end{array}\right) \quad (3.18)$$

We obtain the I-function of $s+u+k+l$ variables.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\rho_i,\mu'_j,\rho'_j\lambda_v^{(i)};\lambda_v'^{(i)}\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j\in\mathbb{C}\ \ (i=1,\cdots,s;j=1,\cdots;u;v=1,\cdots,k)$
$\zeta_j^{(i)}>0(i=1,\cdots,l;j=1,\cdots,k)$

(B) $m_j,n_j,p_j,q_j(j=1,\cdots,s),n',p',q'\in\mathbb{N}^*$ **;**$\delta_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,q_i;i=1,\cdots,s)$

$\alpha_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,p';i=1,\cdots,s),\beta_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,q';i=1,\cdots,s),\gamma_j'^{(i)}\in\mathbb{R}_+(j=1,\cdots,p'_i;i=1,\cdots,s)$

$a'_j(j=1,\cdots,p),b'_j(j=1,\cdots,q),c_j'^{(i)}(j=1,\cdots,p'_i,i=1,\cdots,s),d_j'^{(i)}(j=1,\cdots,q'_i,i=1,\cdots,s)\in\mathbb{C}$

The exposants

$A'_j(j=1,\cdots,p'),B'_j(j=1,\cdots,q'),C'^{(i)}_j(j=1,\cdots,p'_i;i=1,\cdots,s),D'^{(i)}_j(j=1,\cdots,q'_i;i=1,\cdots,s)$
of various gamma function involved in (1.10) and (1.11) may take non integer values.

$m''_j,n''_j,p''_j,q'_j(j=1,\cdots,u),n'',p'',q''\in\mathbb{N}^*$ **;** $\delta''^{(i)}_j\in\mathbb{R}_+(j=1,\cdots,q''_i;i=1,\cdots,u)$

$\alpha''^{(i)}_j\in\mathbb{R}_+(j=1,\cdots,p'';i=1,\cdots,u),\beta''^{(i)}_j\in\mathbb{R}_+(j=1,\cdots,q'';i=1,\cdots,u),\gamma''^{(i)}_j\in\mathbb{R}_+(j=1,\cdots,p''_i;i=1,\cdots,u)$

$a''_j(j=1,\cdots,p''),b''_j(j=1,\cdots,q''),c''^{(i)}_j(j=1,\cdots,p''_i,i=1,\cdots,u),d''^{(i)}_j(j=1,\cdots,q''_i,i=1,\cdots,u)\in\mathbb{C}$

The exposants

$A''_j(j=1,\cdots,p''),B''_j(j=1,\cdots,q''),C''^{(i)}_j(j=1,\cdots,p''_i;i=1,\cdots,u),D''^{(i)}_j(j=1,\cdots,q''_i;i=1,\cdots,u)$

of various gamma function involved in (1.15) and (1.16) may take non integer values.

(C) $$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

(D) $$Re\left[\alpha+\sum_{i=1}^{s}\mu_i\min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{\delta'^{(i)}_j}+\sum_{i=1}^{u}\mu'_i\min_{1\leqslant j\leqslant m''_i}\frac{d''^{(i)}_j}{\delta''^{(i)}_j}\right]>0$$

$$Re\left[\beta+\sum_{i=1}^{s}\rho_i\min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{\delta'^{(i)}_j}+\sum_{i=1}^{u}\rho'_i\min_{1\leqslant j\leqslant m''_i}\frac{d''^{(i)}_j}{\delta''^{(i)}_j}\right]>0$$

(E) $$U_i=\sum_{j=1}^{p'}A'_j\alpha'^{(i)}_j-\sum_{j=1}^{q'}B'_j\beta'^{(i)}_j+\sum_{j=1}^{p'_i}C'^{(i)}_j\gamma'^{(i)}_j-\sum_{j=1}^{q'_i}D'^{(i)}_j\delta'^{(i)}_j\leqslant 0,i=1,\cdots,s$$

$$U'_i=\sum_{j=1}^{p''}A''_j\alpha''^{(i)}_j-\sum_{j=1}^{q''}B'_j\beta''^{(i)}_j+\sum_{j=1}^{p''_i}C''^{(i)}_j\gamma''^{(i)}_j-\sum_{j=1}^{q''_i}D''^{(i)}_j\delta''^{(i)}_j\leqslant 0,i=1,\cdots,u$$

(F) $$\Delta_k=-\sum_{j=n'+1}^{p'}A'_j\alpha'^{(k)}_j-\sum_{j=1}^{q'}B'_j\beta^{(k)}_j+\sum_{j=1}^{m'_k}D'^{(k)}_j\delta'^{(k)}_j-\sum_{j=m'_k+1}^{q'_k}D'^{(k)}_j\delta'^{(k)}_j+\sum_{j=1}^{n'_k}C'^{(k)}_j\gamma'^{(k)}_j-\sum_{j=n'_k+1}^{p'_k}C'^{(k)}_j\gamma'^{(k)}_j$$

$$-\mu_i-\rho_i-\sum_{l=1}^{k}\lambda^{(i)}_j>0\,;i=1,\cdots,s$$

$$\Delta'_k=-\sum_{j=n''+1}^{p''}A''_j\alpha''^{(k)}_j-\sum_{j=1}^{q''}B''_j\beta''^{(k)}_j+\sum_{j=1}^{m''_k}D''^{(k)}_j\delta''^{(k)}_j-\sum_{j=m''_k+1}^{q''_k}D''^{(k)}_j\delta''^{(k)}_j+\sum_{j=1}^{n''_k}C''^{(k)}_j\gamma''^{(k)}_j-\sum_{j=n''_k+1}^{p''_k}C''^{(k)}_j\gamma''^{(k)}_j$$

$$-\mu'_i-\rho'_i-\sum_{l=1}^{k}\lambda'^{(i)}_j>0\,;i=1,\cdots,u$$

(G) $$\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(i)}_j}\right)\right|<\frac{1}{2}\Delta_i\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,s)$$

$$\left|arg\left(z_i'\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right| < \frac{1}{2}\Delta_i'\pi \quad (a \leqslant t \leqslant b; i=1,\cdots,u)$$

(H) $P \leqslant Q+1$. The equality holds, when , in addition,

either $P > Q$ and $\left|z_i''\left(\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right)\right|^{\frac{1}{Q-P}} <1 \qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z_i''\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right)\right|\right] < 1 \qquad (a \leqslant t \leqslant b)$

(I) The multiple series occuring on the right-hand side of (3.18) is absolutely and uniformly convergent.

Proof

First expressing the class of multivariable polynomials defined by Srivastava et al [5] in serie with the help of (1.13), expressing the I-function of s-variables and u-variables defined by Nambisan et al [2] by the Mellin-Barnes contour integral with the help of the equation (1.2) and (1.8) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of (f_jt+g_j) with $j=1,\cdots,k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable I-function of Nambisan, we obtain the desired result.

4.Particular case

a) if $A_j'=B_j'=C_j'^{(i)}=D_j'^{(i)}=A_j''=B_j''=C_j''^{(i)}=D_j''^{(i)}=1$, The multivariable I-functions defined by Nambisan reduces to multivariable H-function defined by Srivastava et al [8]. We have.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_r}\left(\begin{array}{c} \mathrm{x}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_r(t-a)^{\mu_u+\mu_u'}(b-t)^{\rho_r+\rho_r'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{array}\right)$$

$$H\begin{pmatrix} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^{r}R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j-\sum_{i=1}^{r}R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}B_rz_1^{R_1}\cdots z_r^{R_r}$$

$$H_{p'+p''+l+k+2,q'+q''+l+k+1;Y}^{0,n'+n''+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_u'(b-a)^{\mu_u'+\rho_u'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.18) with $A_j'=B_j'=C_j'^{(i)}=D_j'^{(i)}=A_j''=B_j''=C_j''^{(i)}=D_j''^{(i)}=1$

b) If $B(L;R_1,\cdots,R_r) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{R_r\phi_j^{(r)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{R_r\delta_j^{(r)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_r}[z_1,\cdots,z_r]$ reduces to generalized Lauricella function defined by Srivastava et al [4]. We have

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(r)}}^{1+\bar{A}:B';\cdots;B^{(r)}}\left(\begin{array}{c|}\mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}}\end{array}\right.$$

$$\left.\begin{array}{c}[(\text{-L});\mathrm{R}_1,\cdots,R_r][(a);\theta',\cdots,\theta^{(r)}]:[(b');\phi'];\cdots;[(b^{(r)});\phi^{(r)}]\\ [(\mathrm{c});\psi',\cdots,\psi^{(r)}]:[(d');\delta'];\cdots;[(d^{(r)});\delta^{(r)}]\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}}\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z'}_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}}\end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^r R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^k(af_j+g_j)^{\sigma_j-\sum_{i=1}^r R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}B_r' z_1^{R_1}\cdots z_r^{R_r}$$

$$I_{p'+p''+l+k+2,q'+q''+l+k+1;Y}^{0,n'+n''+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_u'(b-a)^{\mu_u'+\rho_u'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \end{array}\right) \quad (4.3)$$

under the same notations and conditions that (3.18
)

where $B_r' = \dfrac{(-L)_{h_1R_1+\cdots+h_rR_r}B(E;R_1,\cdots,R_r)}{R_1!\cdots R_r!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions and a class of multivariable polynomials defined by Srivastava et al [5].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions, defined by Nambisan et al [2], a class of multivariable polynomials defined by Srivastava et al [5] and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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Personal adress : 411 Avenue Joseph Raynaud
Le parc Fleuri , Bat B
83140 , Six-Fours les plages
Tel : 06-83-12-49-68
Department : VAR
Country : FRANCE